REMARK ON SUBOBJECTS IN A TOPOS

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Abstract: In this paper, we study the partially subobjects in a topos, and the equivalence between \( S + T \) and \( S \cup T \) is obtained.

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1. Introduction and Preliminaries

Recall a topos \( \mathcal{E} \) is a category which has finite limits and every object of \( \mathcal{E} \) has a power object. For a fixed object \( A \) of category \( \mathcal{E} \), the power object of \( A \) is an object \( PA \) which represents \( \text{Sub}(\ - \times A) \), so that \( \text{Hom}_{\mathcal{E}}(\ - , PA) \simeq \text{Sub}(\ - \times A) \) naturally. It says precisely that for any arrow \( B' \xrightarrow{f} B \), the following diagram commutes, where \( \varphi \) is the natural isomorphism.

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{E}}(B, PA) & \xrightarrow{\varphi(A,B)} & \text{Sub}(B \times A) \\
\downarrow \text{Hom}_{\mathcal{E}}(f, PA) & & \downarrow \text{Sub}(f \times A) \\
\text{Hom}_{\mathcal{E}}(B', PA) & \xrightarrow{\varphi(A,B')} & \text{Sub}(B' \times A)
\end{array}
\]

Figure 1

As a matter of fact, the category of sheaves of sets on a topological space is a topos. In particular, the category of sets is a topos. For details of the treatment of toposes and sheaves please see Johnstone [1], Mac and Moerdijk [2], Joyal.
and Tierney [3], Johnstone and Joyal [4]. For a general background on category theory please refer to [5], [6].

In [2], Lattice and Heyting Algebra objects in a topos are well defined. In [xxx] the Partially ordered objects are well defined too. In this paper, we want to give some properties about subobjects of partially ordered objects.

More details about lattice and locale please see [7], [8], [9], [10].

2. Main Results

Throughout this paper, we work with a fixed topos $\mathcal{E}$. All objects mentioned belong to the topos $\mathcal{E}$.

In a topos $\mathcal{E}$, given two subobjects $S \hookrightarrow A$ and $T \hookrightarrow A$ we can define the greatest lower bound in the partially ordered set $\text{Sub}A$ of subjects of $A$ by taking the following pullback as following:

\[
\begin{array}{ccc}
S \cap T & \rightarrow & T \\
\downarrow & & \downarrow \\
S & \rightarrow & A
\end{array}
\]

And then we denote $S + T$ as the coproduct of the subobjects $S \hookrightarrow A$ and $T \hookrightarrow A$, $M$ is the image of $S + T$, as shown in the following diagram:

\[
\begin{array}{ccc}
S + T & \leftarrow & T \\
\downarrow & \Downarrow M & \downarrow \\
S & \rightarrow & A
\end{array}
\]

then, $M \rightarrow A$ belongs to $\text{Sub}A$ which contains $S$ and $T$ and so $M$ is the least upper bound of $S$ and $T$ in $\text{Sub}A$.

**Lemma 1.** [1] In a topos, every morphism $f$ has an image $m$ and factors as $f = me$, with $e$ epi.

**Proposition 2.** For any partially ordered object $B$ in a topos $\mathcal{E}$, the inclusion $i : \text{Sub}A \rightarrow \mathcal{E}/B$ has a left adjoint $k$ which sends each $f : A \rightarrow B$ to its image.
Proof. For each object $f : A \to B$ of the slice category $\mathcal{E}/B$ take $kf$ to be the image $m : M \to B$, regarded as a subobject of $B$. And if $f$ factors through a monic $h : C' \to B$, so does $m$ by Lemma 1. This shows that $\text{Hom}_{\mathcal{E}/B}(f, h) \cong \text{Hom}_{\text{Sub}(B)}(m, h)$. Hence $k$ defined by $kf = m$ is the required left adjoint to the inclusion $i$.

Theorem 3. Let $\mathcal{E}$ be a topos and $B$ a partially ordered object. If $S$ and $T$ are disjoint subobjects of $B$, then the join $S \cup T$ in $B$ is isomorphic to the coproduct $S + T$.

Proof. Let $h : S \to B$ and $k : T \to B$ be the conclusions of the given two subobjects. Their coproduct $S + T$ in $\mathcal{E}$ is also their coproduct in $\mathcal{E}/B$. The pullback functor $k^* : \mathcal{E}/B \to \mathcal{E}/T$ is a left adjoint, hence preserves the coproduct $\langle h, k \rangle : S + T \to B$ in $\mathcal{E}/B$ as the second square shows in the following diagram.

Now the pullback $k^*k$ is the identity, hence the pullback of $S + T \to B$ along $k$ is the identity $T \to T$ as the third square shows in the above diagram.

Similarly, this means the pullback along $\langle h, k \rangle$ turns $k$ into the inclusion $i_2 : T \to S + T$ and turns $h$ into $i_1 : S \to S + T$ as the first and the second square show in the above diagram.

Since the pullback functor $\langle h, k \rangle^*$ preserves coproducts, the pullback of $S + T \to B$ along itself is the identity as the second square shows. This in turn shows $S + T \to B$ is mono. Hence, $S + T$ is a subobject of $B$. So, by the definition of the join $S \cup T$, the join is $S + T$.

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References


