International Journal of Pure and Applied Mathematics
**Volume 92 No. 4 2014, 591-597**
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
**url:** http://www.ijpam.eu
**doi:** http://dx.doi.org/10.12732/ijpam.v92i4.14

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# NEIGHBORHOOD OF A CERTAIN FAMILY OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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**Abstract:** By the use of the integration fractional operator introduced by authors [11],we introduce and investigate two new subclasses of $p$-valently analytic functions of complex order.we obtain various results for each of these classes including coefficient inequalities and inclusion relationship involving the $(n, \delta)$-neighborhood of $p$-valenty analytic functions.

**Key Words:** analytic functions, $p$-valent functions, coefficient inequalities, inclusion relations, neighborhood properties, $(n, \delta)$-neighborhood

## 1. Introduction

Let $A_p(n)$ denote the class of all analytic functions $f(z)$ normalized by

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, (a_k \geq 0; n, p \in N := \{1, 2, 3, ...\}), \tag{1}$$

which are analytic and $p$-valent in the open unit disk, $U = \{z : |z| < 1\}$. Now

Received: February 26, 2014

for $f(z) \in A_p(n)$,we have

$$J_{\eta,\lambda} f(z) = \phi(p) z^p - \sum_{k=n+p}^{\infty} \phi(k) a_k z^k \tag{2}$$

Here $J_{\eta,\lambda} f(z)$, is the integration operator defined by authors, see [1],

$$\phi(k) = \frac{(\Gamma(k+1))^2 \Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(k+\eta-\lambda+1)\Gamma(k-\eta+1)},$$

$$\phi(p) = \frac{(\Gamma(p+1))^2 \Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(p+\eta-\lambda+1)\Gamma(p-\eta+1)},$$

$(n, p \in N, \eta - 1 < \lambda \leq \eta < 2)$. Making use of the function $J_{\eta,\lambda} f(z)$ given by (1.2),we introduce the following subclasses of $p$-valently analytic function class $A_p(n)$.

**Definition 1.1.** A function $f(z)$ defined by (1.1) is said to be in the class $J^P_{\eta,\lambda}(\gamma, b)$, if and only if:

$$\left| \frac{1}{b} \left( \frac{z J'_{\eta,\lambda} f(z) + \gamma z^2 J''_{\eta,\lambda} f(z)}{\gamma z J'_{\eta,\lambda} f(z) + (1-\gamma) J_{\eta,\lambda} f(z)} - p \right) \right| < 1 \tag{3}$$

$(n, p \in N, \eta - 1 < \lambda \leq \eta < 2, b \in C, 0 \leq \gamma \leq 1)$.

**Definition 1.2.** A function $f(z)$ defined by (1.1) is said to be in the class $G^P_{\eta,\lambda}(\gamma, b)$, if and only if:

$$\left| \frac{1}{b} \left( J'_{\eta,\lambda} f(z) + \gamma z J''_{\eta,\lambda} f(z) - p \right) \right| < p \tag{4}$$

$(n, p \in N, \eta - 1 < \lambda \leq \eta < 2, b \in C, 0 \leq \gamma \leq 1)$. Next, we define the $(n, \delta)$-neighborhood of a function $f(z) \in A_p(n)$ see [2]

$$N_{n,\delta}(f) := \left\{ g : g \in A_p(n), g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k and \sum_{k=n+p}^{\infty} k|a_k - b_k| \leq \delta \right\}. \tag{5}$$

It follows from (1.5) that, if

$$h(z) = z^p, (p \in N), \tag{6}$$

Then

$$N_{n,\delta}(h) := \left\{ g : g \in A_p(n), g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k and \sum_{k=n+p}^{\infty} k|b_k| \le \delta \right\}. \quad (7)$$

The main object of the present work is to investigate the various properties and characteristics of analytic $p$-valent functions belonging to the subclasses $J^P_{\eta,\lambda}(\gamma, b)$, and $G^P_{\eta,\lambda}(\gamma, b)$ which are introduced by making use of the operator $J_{\eta,\lambda}f(z)$ on the normalized $p$-valently analytic functions in $U$. Apart from deriving a set of coefficient inequalities for each of these two function classes, we establish some inclusion relationships involving the $(n, \delta)$-neighborhoods of analytic $p$-valent functions belonging to each of these classes.

## 2. Coefficient Inequalities

In this section, we prove the following results which yield the coefficient inequalities for functions in the subclasses $J^P_{\eta,\lambda}(\gamma, b)$, and $G^P_{\eta,\lambda}(\gamma, b)$.

**Theorem 2.1.** *Let $f(z) \in A_p(n)$ be given by (1.1). Then $f(z) \in J^P_{\eta,\lambda}(\gamma, b)$ if and only if*

$$\sum_{k=n+p}^{\infty} (k + |b| - p) \left[\gamma(k-1) + 1\right] \phi(k) a_k \le |b| \left[\gamma(p-1) + 1\right] \phi(p) \quad (8)$$

$$\phi(k) = \frac{(\Gamma(k+1))^2 \Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(k+\eta-\lambda+1)\Gamma(k-\eta+1)},$$

$$\phi(p) = \frac{(\Gamma(p+1))^2 \Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(p+\eta-\lambda+1)\Gamma(p-\eta+1)}.$$

*Proof.* Let a function $f(z)$ of the form (1.1) belong to the class $J^P_{\eta,\lambda}(\gamma, b)$. Then, in view of (1.2) and (1.3),we obtain the following inequality:

$$\Re \left( \frac{\sum\limits_{k=n+p}^{\infty} (p-k) \left[\gamma(k-1)+1\right] \phi(k) a_k z^k}{\left[\gamma(p-1)+1\right] \phi(p) z^p - \sum\limits_{k=n+p}^{\infty} \left[\gamma(k-1)+1\right] \phi(k) a_k z^k} \right) > -|b|. \quad (9)$$

Setting $z = r (0 \le r < 1)$ in (2.2), we observe that the expression in the denominator on the left-hand side of (2.2) is positive for $r = 0$ and also for all $(0 < r < 1)$. Thus by letting $z \to 1^-$ through real values,(2.2) leads us to desired assertion (2.1) of theorem 2.1. Conversely, by applying (2.1)and setting $|z| = 1$, we find from (1.2) that

$$\left|\frac{zJ'_{\eta,\lambda}f(z) + \gamma z^2 J''_{\eta,\lambda}f(z)}{\gamma z J'_{\eta,\lambda}f(z) + (1-\gamma)J_{\eta,\lambda}f(z)} - p\right|$$

$$= \left|\frac{\sum\limits_{k=n+p}^{\infty}(p-k)\left[\gamma(k-1)+1\right]\phi(k)a_k z^k}{\left[\gamma(p-1)+1\right]\phi(p)z^p - \sum\limits_{k=n+p}^{\infty}\left[\gamma(k-1)+1\right]\phi(k)a_k z^k}\right|$$

$$\leq \frac{|b|\left\{\left[\gamma(p-1)+1\right]\phi(p) - \sum\limits_{k=n+p}^{\infty}\left[\gamma(k-1)+1\right]\phi(k)a_k\right\}}{\left[\gamma(p-1)+1\right]\phi(p) - \sum\limits_{k=n+p}^{\infty}\left[\gamma(k-1)+1\right]\phi(k)a_k} = |b|.$$

Hence,by the maximum modulus principle, we infer that $f(z) \in J^P_{\eta,\lambda}(\gamma, b)$, which completes the proof of Theorem 2.1. □

Similarly, we can prove the following theorem.

**Theorem 2.2.** *Let $f(z) \in A_p(n)$ be given by (1.1). Then $f(z) \in G^P_{\eta,\lambda}(\gamma, b)$ if and only if*

$$\sum_{k=n+p}^{\infty}\left[\gamma(k-1)+1\right]k\phi(k)a_k \leq p(|b|-1) + [\gamma(p-1)+1]p\phi(p). \tag{10}$$

## 3. Inclusion Relations Involving the $(n, \delta)$-Neighborhoods

In this section, we establish several inclusion relations for the normalized $p$-valently analytic function classes $J^P_{\eta,\lambda}(\gamma, b)$ and $G^P_{\eta,\lambda}(\gamma, b)$ , involving the $(n, \delta)$-neighborhood defined in (1.7).

**Theorem 3.1.** *If*

$$\delta := \frac{|b|[\gamma(p-1)+1](n+p)\phi(p)}{(n+|b|)[\gamma(n+p-1)]\phi(n+p)}, (p > |b|), \tag{11}$$

*then*

$$J^P_{\eta,\lambda}(\gamma, b) \subset N_{n,\delta}(h). \tag{12}$$

*Proof.* Let $f(z) \in J^{P}_{\eta,\lambda}(\gamma, b)$. Then in view of Theorem 2.1,we have

$$(n+|b|)[\gamma(n+p-1)+1]\phi(n+p)\sum_{k=n+p}^{\infty} a_k \leq |b|[\gamma(p-1)+1]\phi(p) \tag{13}$$

which yields

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{|b|[\gamma(p-1)+1]\phi(p)}{(n+|b|)[\gamma(n+p-1)+1]\phi(n+p)} \tag{14}$$

Now,using (2.1) and (3.4),we get $[\gamma(n+p-1)+1]\phi(n+p)\sum_{k=n+p}^{\infty} ka_k \leq |b[\gamma(p-1)+1]\phi(p) + (p-|b|)[\gamma(n+p-1)+1]\phi(n+p)\sum_{k=n+p}^{\infty} a_k \leq |b|[\gamma(p-1)+1]\phi(p) + (p-|b|)[\gamma(n+p-1)+1]\phi(n+p)\left(\frac{|b|[\gamma(p-1)+1]\phi(p)}{(n+|b|)[\gamma(n+p-1)+1]\phi(n+p)}\right) = |b|[\gamma(p-1)+1]\phi(p) + \frac{|b|(p-|b|)[\gamma(p-1)+1]\phi(p)}{(n+|b|)} = \frac{|b|[\gamma(p-1)+1]\phi(p)(n+|b|)+|b|[\gamma(p-1)+1]\phi(p)(p-|b|)}{n+|b|}$ $= \frac{|b|[\gamma(p-1)+1]\phi(p)(n+p)}{n+|b|}$ Hence $\sum_{k=n+p}^{\infty} ka_k \leq \frac{|b|[\gamma(p-1)+1](n+p)\phi(p)}{(n+|b|)[\gamma(n+p-1)]\phi(n+p)} =: \delta, (p > |b|)$ which completes the proof of Theorem 3.1. □

In a similar manner,by applying the assertion (2.4) of Theorem 2.2 to functions in the class $G^{P}_{\eta,\lambda}(\gamma, b)$,we can prove the following inclusion relationship.

**Theorem 3.2.** *If*

$$\delta := \frac{p(|b|-1)+[\gamma(p-1)+1]p\phi(p)}{[\gamma(n+p-1)+1]\phi(n+p)} \tag{15}$$

*then*

$$G^{P}_{\eta,\lambda}(\gamma, b) \subset N_{n,\delta}(h). \tag{16}$$

## 4. Further Neighborhood Properties

In the last section, we determine the neighborhood properties for each of the following function classes: $J^{p,\alpha}_{\eta,\lambda}(\gamma, b)$, and $G^{p,\alpha}_{\eta,\lambda}(\gamma, b)$ Here the class $J^{p,\alpha}_{\eta,\lambda}(\gamma, b)$

consists of $f(z) \in A_p(n)$ for which there exists another function $g(z) \in J^p_{\eta,\lambda}(\gamma, b)$ such that:

$$\left|\frac{f(z)}{g(z)} - 1\right| < p - \alpha, (z \in U; 0 \leq \alpha < p). \tag{17}$$

Similarly, the class $G^{p,\alpha}_{\eta,\lambda}(\gamma, b)$ consists of $f(z) \in A_p(n)$ for which there exists another function $g(z) \in G^p_{\eta,\lambda}(\gamma, b)$ satisfying (4.1). Now, using (1.5),(4.1) and Theorem 2.1 ; then (1.5),(4.1) and Theorem 2.2 we can respectively prove the following results:

**Theorem 4.1.** *Let $g(z) \in J^p_{\eta,\lambda}(\gamma, b)$. Suppose that*

$$\alpha := p - \frac{\delta(n + |b|)[\gamma(n + p - 1) + 1]\phi(n + p)}{(n + p)\,[(n + |b|)[\gamma(n + p - 1) + 1]\phi(n + p) - |b|[\gamma(p - 1) + 1]\phi(p)]}. \tag{18}$$

*Then $N_{n,\delta}(g) \subset J^{p,\alpha}_{\eta,\lambda}(\gamma, b)$.*

**Theorem 4.2.** *Let $g(z) \in G^p_{\eta,\lambda}(\gamma, b)$. Suppose that*

$$\alpha := p - \frac{\delta[\gamma(n + p - 1) + 1]\phi(n + p)}{(n + p)\phi(n + p)[\gamma(n + p - 1) + 1] - p(|b| - 1) - [\gamma(p - 1) + 1]p\phi(p)}. \tag{19}$$

*Then $N_{n,\delta}(g) \subset G^{p,\alpha}_{\eta,\lambda}(\gamma, b)$.*

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