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A UNIFIED METHOD FOR EVALUATING RIEMANN ZETA FUNCTIONS, DIRICHLET SERIES, ASSOCIATED CLAUSEN FUNCTIONS, OTHER ALLIED SERIES, AND NEW CLASSES OF INFINITE SERIES

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Abstract: We have shown here for the first time that the completeness relation provides a simple unified theoretical framework for deriving different kinds of new recurrence formulae for Riemann Zeta Functions, Dirichlet series and Other Allied Series by selecting only different forms of complete set of orthonormal function (CSOF) in contrast to the expansion method (EM) where one needs to select not only different kinds of CSOF but also suitable arbitrary function. A new proof is also given by selecting only orthogonal Bessel functions for the well known identity corresponding to the sum of squares of the reciprocals of zeros for the m-th order Bessel function. In addition, we have shown here that, in comparison to the EM and other methods, our present method has far-reaching implications, viz. (i) All proofs are based on completeness relation and proper selection of orthogonal functions without selecting any arbitrary functions. (ii) Simpler proofs are possible for new identities corresponding to infinite number of infinite series, sum of each having a fixed value pi (π). (iii) New proofs emerge not only for the identities corresponding to the Associated Clausen functions but also the sum of new classes of infinite series which resemble the associated Clausen functions.

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1. Introduction

The Riemann zeta function is one of the most important and fascinating functions in mathematics. In the case of a real argument, this function was introduced and studied by Euler in the first half of the eighteenth century. For an even real positive argument it is defined as

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \tag{1}$$

Many years after Euler's discovery, other important investigations had been carried out for Riemann zeta function $\zeta(2k)$ and all these results can be found in the literature [1],[2],[3],[4],[5],[6],[7],[8]. Another important series is the Dirichlet series, defined for the positive even argument as

$$D(2k) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{2k}} \tag{2}$$

There are two other important allied series, defined respectively as

$$\delta(2k) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k}} \tag{3}$$

and

$$\sigma(2k+1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2k+1}} \tag{4}$$

There are research papers and also many standard text books where expansion methodology (EM) [6],[7],[8] has been employed to evaluate the sum of infinite series. Basic principle in EM methodology is to expand any arbitrary function $f(x)$ in terms of a complete set of orthonormal function (CSOF)$\{\psi_n(x)\}$

$$f(x) = \sum_{n=n_0}^{\infty} C_n \psi_n(x) \tag{5}$$

where is the starting index n_0 having a value of 0, 1 or $-\infty$ depending upon the indexing assigned to the eigen values. In order to evaluate the Riemann series one consider Fourier cosine series expansion viz.

$$f(x) = x^{2k} = \frac{C_{k,0}}{2} + \sum_{n=1}^{\infty} C_{k,n} \cos(nx) \tag{6}$$

where

$$C_{k,n} = \frac{1}{\pi} \int_{-\pi}^{\pi} x^{2k} \cos(nx) dx \tag{7}$$

whereas in $\delta(2k)$ and $\sigma(2k+1)$, one consider

$$f(x) = x^{2k+1} = \sum_{n=1}^{\infty} C_{k+1,n} \sin(nx) \tag{8}$$

where

$$C_{k+1,n} = \frac{1}{\pi} \int_{-\pi}^{\pi} x^{2k+1} \sin(nx) dx \tag{9}$$

In order to evaluate the sum of squares of the reciprocals of zeros for the m-th order Bessel function, one consider

$$f(x) = x^m = \sum_{n=1}^{\infty} C_n J_m(\alpha_{nm} x) \tag{10}$$

where $J_m(\alpha_{nm} x)$ corresponds to m-th order Bessel functions It is clear from the above discussion that, to evaluate the different kinds of infinite series one needs to select not only different kinds of CSOF but also a suitable form for the arbitrary function $f(x)$. The difficulty of selecting arbitrary function $f(x)$ to prove various identities can be eliminated if one can find an alternative relation which does not involve an arbitrary function $f(x)$ but depends only on CSOF. The completeness relation defined as

$$\sum_{n=n_0}^{\infty} \psi_n(x) \psi_n^*(x') = \delta(x - x') \tag{11}$$

belongs to such type of equation. where n_0 is the starting index having a value of 0, 1 or $-\infty$ depending upon the indexing assigned to the eigen values. Here $\delta(x)$ represents the Dirac Delta function. We will show that the completeness relation (Eq.(11)) provides a unified theoretical framework for deriving different kinds of new recurrence formulae and new identities corresponding to the sum of infinite series very easily by selecting only different forms of CSOF in contrast to EM where one needs to select not only different kinds of CSOF but also suitable arbitrary function $f(x)$.

2. Derivation of New Recursion Relation for Rieman Zeta Function

It is interesting to note that the natural numbers appear in the Riemann Zeta function defined in (1), as inverse power of positive numbers. One therefore needs to select only those class of orthonormal functions in the completeness relation (11) that are explicit function of natural numbers. We consider here the periodic function corresponding to a quantum particle in a one-dimensional ring that can be expressed as

$$\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} \exp[im\phi], 0 < \phi < 2\pi \tag{12}$$

where $m = \pm 1, \pm 2......... \pm \infty$.. The completeness relation corresponding to theses orthogonal functions can be written as

$$\delta(\phi - \phi') = \sum_{-\infty}^{\infty} \frac{\exp[im(\phi - \phi')]}{2\pi} \tag{13}$$

The completeness relation that is defined in (13) forms the basis for obtaining the recurrence relation for Riemann Zeta function. It is interesting to note that the natural numbers that are appearing in the exponent of (13) should be brought to the denominator by proper mathematical operation to evaluate the Riemann Zeta function (1) or to derive the recurrence relation for the same. In order to derive the a general recurrence relation formula for $\zeta(2n)$, we multiply both sides of (13) by, $(\phi)^n(\phi')^n$ and integrate over (ϕ) as well as ϕ' to obtain

$$\int_0^{2\pi} \int_0^{2\pi} (\phi)^n(\phi')^n d\phi d\phi' \delta(\phi - \phi') = \int_0^{2\pi} \int_0^{2\pi} (\phi)^n(\phi')^n d\phi d\phi'$$
$$+2Real \sum_{m=1}^{\infty} [\int_0^{2\pi} (\phi)^n \frac{\exp[im\phi]}{\sqrt{2\pi}} d\phi \int_0^{2\pi} (\phi')^n \frac{\exp[-im\phi')]}{\sqrt{2\pi}} d\phi'] \tag{14}$$

Now using the identities

$$\int_0^{2\pi} \exp[im\phi](\phi)^n d\phi = \sum_{j=0}^{n-1} \frac{(2\pi)^{n-j}(n)!}{(im)^{j+1}(n-j)!}(-1)^j \tag{15}$$

$$\int_0^{2\pi}\int_0^{2\pi}(\phi)^n(\phi^{'})^n d\phi d\phi^{'}\delta(\phi-\phi^{'}) = \frac{(2\pi)^{2n+1}}{2n+1} \tag{16}$$

(14) can be expressed as

$$\frac{(2\pi)^{2n+1}n^2}{(2n+1)(n+1)^2} = Real[\sum_{j=0}^{n-1}\sum_{j^{'}=0}^{n-1}\frac{\xi(j+j^{'}+2)(2\pi)^{2n-j-j^{'}}(n!)^2}{\pi(i)^{j+j^{'}}(n-j^{'})!(n-j)!}(-1)^j] \tag{17}$$

which, to the best of our knowledge, is a new recurrence relation. The double summation appearing in the recurrence relation (17) can be eliminated if we fix $\phi^{'} = 2\pi$ in (13), to have

$$\delta(\phi-2\pi) - \frac{1}{2\pi} = \sum_{m=1}^{\infty}\frac{2\cos(m\phi)}{2\pi} \tag{18}$$

and then multiply both sides by ϕ^{2k} followed by integration, to obtain the recurrence relation

$$\frac{(2\pi)^{2k}}{2} - \frac{(2\pi)^{2k}}{(2k+1)} = 2\sum_{j=0}^{k-1}(-1)^j\frac{\xi(2j+2)(2\pi)^{2k-(2j+2)}2k!}{(2k-(2j+1))!} \tag{19}$$

which contains only one sum. This is clearly another new recurrence relation for the Riemann zeta function. Now solving the recurrence relations defined in (17) and (19), we obtain the well known results viz.

$$\xi(2) = \frac{\pi^2}{6}, \xi(4) = \frac{\pi^4}{90}, \xi(6) = \frac{\pi^6}{945}, \xi(8) = \frac{\pi^8}{9450} \tag{20}$$

3. Derivation of New Recursion Relation for Dirichlet Series

We now consider the Dirichlet series

$$D(2k) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{2k}} \tag{21}$$

It is interesting to note that the difference between the Riemann series $\xi(2k)$ (1) and the Dirichlet series D(2k) (21) is in the multiplicative factor $(-1)^{n-1}$. However, the factor $(-1)^{n-1}$ can be incorporated in the completeness relation defined in (13) if we multiply both sides by (-1) and then consider $\phi^{'} = \pi$ in the same i.e.

$$-\delta(\phi - \pi) + \frac{1}{2\pi} = \sum_{m=1}^{\infty} \frac{2\cos(m\phi)}{2\pi}(-1)^{m-1} \tag{22}$$

In order to derive the recurrence formula for Dirichlet $D(2k)$, we multiply both sides of (22) by ϕ^{2k} followed by integration, obtain the recurrence relation

$$-(\pi)^{2k} + \frac{(2\pi)^{2k}}{(2k+1)} = 2\sum_{j=0}^{k-1}(-1)^j \frac{D(2j+2)(2\pi)^{2k-(2j+2)}2k!}{(2k-(2j+1))!} \tag{23}$$

This is, to the best of our knowledge, a new recursion relation for the Dirichlet series. It is to be noted that right hand side of the recurrence relations for Dirichlet series (23) and Rieman zeta function (19) are identical. This arises due to the similarity of the right hand side of the respective completeness relations (22) and (18). Now substituting the value of k =1, 2 and 3 in (23) and solving the resultant recurrence relation we obtain the values of $D(2k)$ as

$$D(2) = \frac{\pi^2}{12}, D(4) = \frac{7\pi^4}{720}, D(6) = \frac{31\pi^6}{30240} \tag{24}$$

4. Derivation of New Recursion Relations for $\delta(2k)$ and $\sigma(2k+1)$ Series

It is to noted that each term of the infinite series representing $\delta(2k)$ (3) and $\sigma(2k+1)$ (4) contain even powers or odd powers of odd numbers only. Therefore, in order to derive the recurrence formula corresponding to these series, we consider the odd function corresponding to a quantum particle in a one dimensional box of finite length a.

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a}),\ 0 < x < a \tag{25}$$

where $n = 1, 2, 3....\infty$. Justification for selecting of the function (25) will be discussed while deriving the recursion relation corresponding to the infinite series viz. $\delta(2k)$ and $\sigma(2k+1)$. The completeness relation corresponding to these orthogonal functions can be written as

$$\delta(x - x^{'}) = \frac{2}{a}\sum_{n=1}^{\infty}\sin(\frac{n\pi x}{a})\sin(\frac{n\pi x^{'}}{a}),\ 0 < x < a \tag{26}$$

The completeness relation defined in (26) form the basis for obtaining the recurrence relation for the series $\delta(2k)$ and $\sigma(2k+1)$. We first consider the infinite series $\delta(2k)$. From (26), one can obtain

$$\int_0^a x^{2p}dx \int_0^a dx^{'}\delta(x-x^{'}) = \frac{2}{a}\sum_{n=1}^{\infty}\int_0^a x^{2p}dx\sin(\frac{n\pi x}{a})\int_0^a dx^{'}\sin(\frac{n\pi x^{'}}{a}), \quad (27)$$

It is interesting to note that the second integral in the right hand side of (27) does eliminate the even terms from the series justifying the selection of the function (25) in the completeness relation (26). Now, on evaluation of the relevant integrals, we obtain the recursion formula $\delta(2k)$ defined as

$$4\sum_{j=0}^{p}(-1)^j(\frac{1}{\pi})^{2j+2}(1+\delta_{jp})\frac{2p!}{(2p-2j)!}\delta(2j+2) = \frac{1}{2p+1} \quad (28)$$

which is another new recurrence relation for the series $\delta(2k)$. Here δ_{ij} denotes the Kronecker delta. Now substituting the value of p =0, 1 and 2 in (28) and after solving the resultant recurrence relation we obtain the values of $\delta(2k)$ as

$$\delta(2) = \frac{\pi^2}{8}, \delta(4) = \frac{\pi^4}{96}, \delta(6) = \frac{\pi^6}{960}. \quad (29)$$

We now consider the series $\sigma(2k+1)$. It is to be noted that one of the differences between the series $\delta(2k)$ and $\sigma(2k+1)$ is only the multiplicative factor $(-1)^n$. However, the factor $(-1)^n$ can be incorporated in the completeness relation defined in (26) if we consider $x^{'} = a/2$ in the same. In order to derive the recurrence relation, we first substitute $x^{'} = a/2$ into (26), then multiply both sides by x^{2p+1} , and carry out the integration, which leads to the recurrence relation for $\sigma(2k+1)$ as given by

$$2\sum_{j=0}^{p}(-1)^j(\frac{1}{\pi})^{2j+1}\frac{(2p+1)!}{(2p+1-2j)!}\sigma(2j+1) = \frac{1}{2^{2p+1}} \quad (30)$$

This is a new recurrence relation for the series $\sigma(2k+1)$. Now substitute the value of p =0, 1 and 2 in (28) and after solving the resultant recurrence relations, we obtain the values of $\sigma(2k+1)$ as

$$\sigma(2) = \frac{\pi}{4}, \sigma(3) = \frac{\pi^3}{32}, \sigma(5) = \frac{5\pi^5}{1536}. \quad (31)$$

5. A New Proof for the Derivation of an Identity Corresponds to the Sum of Squares of the Reciprocals of Zeros for the m-th Order Bessel Function

We now consider the CSOF corresponding to the eigen functions of the Schrodinger equation for a quantum particle confined inside a sphere of radius a. The completeness relation in three dimensions for these CSOF is defined as

$$\frac{\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')}{r^2\sin\theta} = \sum_{i=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-\infty}^{\infty} \frac{\frac{J_{l+\frac{1}{2}}(k_{il}r)}{\sqrt{r}} Y_{lm}(\theta,\phi) \frac{J_{l+\frac{1}{2}}(k_{il}r')}{\sqrt{r'}} Y^{*}_{lm}(\theta',\phi')}{\frac{a^2}{2}(J_{l+\frac{1}{2}}(k_{il}a))^2} \tag{32}$$

where $J_{l+\frac{1}{2}}(r)$ and $Y_{lm}(\theta,\phi)$ respectively represent the Bessel function and spherical harmonics and k_{1l}, k_{2l}, k_{3l} .. are the positive roots of the equation $J_{l+\frac{1}{2}}(k_{il}a) = 0$, numbered in increasing order. We first integrate both sides of (32) over the domain of (r,r'),(θ,θ') and (ϕ,ϕ') to obtain

$$\begin{aligned}
&\int_0^a r^{l+2}dr \int_0^a (r')^{l+2}dr' \int_0^{\pi}\int_0^{2\pi} Y^{*}_{l'm'}(\theta,\phi)\sin\theta d\theta d\phi \times \\
&\int_0^{\pi}\int_0^{2\pi} Y_{l'm'}(\theta',\phi')\sin\theta' d\theta' d\phi' \frac{\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')}{r^2\sin\theta} \\
&= \sum_{i=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{\frac{a^2}{2}(J_{l+\frac{1}{2}}(k_{il}a))^2} \int_0^a r^{l+2}dr \frac{J_{l+\frac{1}{2}}(k_{il}r)}{\sqrt{r}} \times \\
&\int_0^{\pi}\int_0^{2\pi} Y^{*}_{l'm'}(\theta,\phi) Y_{lm}(\theta,\phi)\sin\theta d\theta d\phi \times \\
&\int_0^a (r')^{l+2}dr' \frac{J_{l+\frac{1}{2}}(k_{il}r')}{\sqrt{r'}} \int_0^{\pi}\int_0^{2\pi} Y_{l'm'}(\theta',\phi') Y^{*}_{lm}(\theta',\phi')\sin\theta' d\theta' d\phi'
\end{aligned} \tag{33}$$

using the identities

$$\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi Y_{im}(\theta,\phi) Y^{*}_{i'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'} \tag{34}$$

$$\int_0^a r^{v+1} dr J_v(k_0 r) = \frac{a^{v+1}}{k_0} J_{v+1}(k_0 a), \tag{35}$$

we obtain the well known relation

$$\frac{1}{4[(l+\frac{1}{2})+1]}=\sum_{i=1}^{\infty}\frac{1}{(\alpha^{i}_{l+\frac{1}{2}})^2},\alpha^{i}_{l+\frac{l}{2}}=k_{il}a \tag{36}$$

6. Derivation of Infinite Number of New Expressions for π

The vanishing property of Dirac delta function corresponding to two different points ($x \neq x^{'}$) provides a scheme to obtain infinite expressions for pi (π). In order to derive infinite expressions for pi (π), we multiply both sides of the (26) by $\exp[im\pi x/a]$ and $\exp[-im\pi x^{'}/a]$ and then integrate over x and $x^{'}$ to obtain the result

$$\frac{\pi^2}{4}=\sum_{n=1}^{\infty}\frac{n^2}{[n^2-m^2]^2}[1-(-1)^n\cos(m\pi)], n\neq m \tag{37}$$

It is to be noted that the left hand side of (37) is independent of m which has arisen due to the presence of delta function in the completeness relation defined in (26). Now selecting different m , one can obtain infinite number of different kinds of infinite series, with the sum of each equal to $\frac{\pi^2}{4}$. Specifically, for the choice, $m = 2k + (1/2)$, with integer k, the cosine term disappears resulting into a simpler series.

$$\frac{\pi^2}{4}=\sum_{n=1}^{\infty}\frac{n^2}{[n^2-(2k+\frac{1}{2})^2]^2}, k=0,1,2,.... \tag{38}$$

Now differentiating the (37) with respect to m then considering the value $m = 2k + (1/2)$ one can obtain infinite number of infinite series sum of each equal to π

$$\pi=\frac{\sum_{n=1}^{\infty}\frac{4n^2}{[\frac{(4n^2-(4k+1)^2)}{4}]^3}+6(4k+1)^2\sum_{n=1}^{\infty}\frac{n^2}{[\frac{(4n^2-(4k+1)^2)}{4}]^4}}{\sum_{n=1}^{\infty}\frac{4(4k+1)n^2}{[\frac{(4n^2-(4k+1)^2)}{4}]^3}[(-1)^{n+1}]} \tag{39}$$

7. New Proofs for the Identities Corresponding to Associated Clausen Functions

In this section, our objective is to evaluate the sum of infinite series corresponding to a few Associated Clausen functions [9],[10]. In order to arrive at the results, we multiply both side of (13) by $\phi^{'}$ and integrate over the same to obtain

$$\int_0^{2\pi} \phi^{'} d\phi^{'} \delta(\phi - \phi^{'}) = \sum_{m=-\infty}^{\infty} [\frac{\exp[im\phi]}{\sqrt{2\pi}} \int_0^{2\pi} \phi^{'} \frac{\exp[-im\phi^{'})]}{\sqrt{2\pi}} d\phi^{'}] \tag{40}$$

which on substitution of the identity

$$\int_0^{2\pi} \phi^{'} \exp[im\phi^{'}] d\phi^{'} = \frac{2\pi}{im} \tag{41}$$

leads to one of the Clausen Functions defined as

$$\sum_{m=1}^{+\infty} \frac{\sin m\phi}{m} = \frac{\pi - \phi}{2}, 0 \leq \phi \leq 2\pi \tag{42}$$

Similarly, another important identity involving the Clausen functions can be derived by multiplying both sides of (13) by $(\phi^{'})^2$, followed by integration, i.e.

$$\int_0^{2\pi} d\phi^{'} (\phi^{'})^2 \delta(\phi - \phi^{'}) = \sum_{m=-\infty}^{+\infty} \frac{\exp[im\phi]}{\sqrt{2\pi}} \int_0^{2\pi} (\phi^{'})^2 d\phi^{'} \frac{\exp[-im\phi^{'}]}{\sqrt{2\pi}}. \tag{43}$$

On evaluation of relevant integrals, we obtain

$$\sum_{m=1}^{\infty} \frac{\cos m\phi}{m^2} = \frac{(\pi - \phi)^2}{4} - \frac{\xi(2)}{2} \tag{44}$$

Similarly, the sum of other infinite series corresponding to the Clausen functions can be obtained by multiplying $(\phi^{'})^n, (n > 2)$ to both sides of (13), followed by integration and after some rearrangement

8. New Proofs for Identities Corresponding to New Classes of Infinite Series

In order to evaluate the sum of new classes of infinite series similar to the Associated Clausen functions [9], [10] we first rewrite the (26) as

$$\int_0^a dx^{'} (x^{'}) \delta(x - x^{'}) = \frac{2}{a} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \int_0^a (x^{'}) dx^{'} \sin \frac{n\pi x^{'}}{a}. \tag{45}$$

On evaluation of relevant integrals, we obtain a new mathematical identity which resembles that for Associated Clausen type of functions, viz.

$$\sum_{n=1}^{m}(-1)^{n+1}\frac{\sin(n\pi y)}{n} = \frac{\pi}{2}y.\ 0 < y < 1 \tag{46}$$

Similarly, to obtain another new identity corresponding to the sum of new infinite series, we rewrite (26) as

$$\int_0^a dx^{'}(x^{'})^2\delta(x-x^{'}) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\int_0^a (x^{'})^2 dx^{'}\sin\frac{n\pi x^{'}}{a}. \tag{47}$$

on evaluation of the relevant integrals, leads to the result

$$\sum_{n=0}^{\infty}\frac{sin(\frac{(2n+1)\pi x^{'}}{a})}{(2n+1)^3} = \frac{\pi^3}{8a^2}[\frac{2a^2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(\frac{n\pi x^{'}}{a})}{n} - (x^{'})^2] \tag{48}$$

Now, substituting (46) into (48), we obtain a new mathematical identity similar to the one involving Associated Clausen type of functions, viz..

$$\sum_{n=0}^{\infty}\frac{\sin((2n+1)\pi y)}{(2n+1)^3} = \frac{\pi^3}{8}[y-y^2]. \tag{49}$$

To derive another new identity similar to the one defined in (49), we first write the integral form of (26) as

$$\int_0^a dx^{'}(x^{'})^3\delta(x-x^{'}) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\int_0^a (x^{'})^3 dx^{'}\sin\frac{n\pi x^{'}}{a}. \tag{50}$$

On evaluation of the relevant integrals mentioned above, we obtain

$$\sum_{n=1}^{\infty}\frac{\sin(n\pi y)}{n^3}(-1)^{n+1} = \frac{\pi^3}{12}[y-y^3].\ 0 < y < 1 \tag{51}$$

To obtain new proofs for a few more new identities, we first write the integral form of (26) as

$$\int_0^a dx^{'}(x^{'})^4\delta(x-x^{'}) = \frac{2}{a}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\int_0^a (x^{'})^4 dx^{'}\sin\frac{n\pi x^{'}}{a}. \tag{52}$$

$$\int_0^a dx^{'}(x^{'})^5\delta(x-x^{'}) = \frac{2}{a}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\int_0^a (x^{'})^5 dx^{'}\sin\frac{n\pi x^{'}}{a}. \tag{53}$$

and then evaluate the relevant integrals, to obtain the new identities

$$\sum_{n=0}^{\infty}\frac{\sin((2n+1)\pi y)}{(2n+1)^5} = \frac{\pi^5}{96}[y^4 - 2y^3 + y]. \tag{54}$$

$$\sum_{n=1}^{\infty}\frac{\sin(n\pi y)}{n^5}(-1)^{n+1} = \frac{\pi^5}{240}[y^5 - \frac{10}{3}y^3 + \frac{7}{3}y]. \tag{55}$$

Similarly, the sum of other new infinite series which do resemble to the Associated Clausen functions can be obtained by multiplying $(x^{'})^n (n \geq 5)$ to both sides of (26) and then carrying out the relevant integrals.

9. Conclusion

The completeness relation (CR) (11) was unexplored for long time in the field of Number theory. We have shown here, for the first time, that the completeness relation (11) provides a simple unified theoretical framework for deriving different kinds of new recurrence formulae for Riemann Zeta Functions (17), (19) Dirichlet series (23) and Other Allied Series (28), (30) by selecting only different forms of complete set of orthonormal function (CSOF) in contrast to the expansion method (EM) where one needs to select not only different kinds of CSOF but also suitable arbitrary function $f(x)$. The recurrence formulae derived here provide a scheme to obtain the well known results for Riemann Zeta Functions, Dirichlet series and Other Allied Series. However, in comparison to the EM and other methods, the present method has many extra advantages namely (i) All proofs are based on completeness relation and proper selection of orthogonal functions without selecting any arbitrary function $f(x)$. (ii) It provides a simpler new proof for the well known identity (36) corresponding to the sum of squares of the reciprocals of zeros for the m-th order Bessel function. (iii) Simpler proofs are possible for new identities (37), (38), (39) corresponding to infinite number of infinite series, sum of each having a fixed value pi (π) (iv) New proofs emerge not only for the identities (42), (44) corresponding to the Associated Clausen functions but also the sum of new classes (46), (49), (51), (54), (55) of infinite series which resemble the associated Clausen functions. The present theories has opened a new window to evaluate the sum of infinite series by proper selection of orthonormal functions only through the completeness relation.

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