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BERNOULLI'S n-FORMULA AND n_MULTI-SERIES BY THE GENERALIZED DIFFERENCE EQUATION

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Abstract: We derive the Bernoulli's n-Formula and n_multi-series by equating the numerical and complete solution of the n^{th} order generalized difference equation. We also illustrate 2_multi-series to product of arithmetic and geometric functions.

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1. Introduction

The Fractional Calculus is currently a very important research field in several different areas: physics (including classical and quantum mechanics and thermodynamics), chemistry, biology, economics and control theory ([5], [6], [7]). In 1989, K.S.Miller and Ross [8] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional difference operator. The main definition of fractional difference equation (as done in [8]) is the ν fractional sum of $f(t)$ by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma_{(\nu)}} \sum_{s=a}^{t-\nu} \frac{\Gamma_{(t-s)}}{\Gamma_{(t-s-(\nu-1))}} f(s), \tag{1}$$

where $\nu > 0$. On the other hand, when $\nu = m$ is a positive integer, if we replace

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the function $f(t)$ by $u(k)$ and Δ by Δ_ℓ defined as $\Delta_\ell u(k) = u(k+\ell) - u(k)$, (1) becomes

$$\underset{n(\ell)}{\mathrm{u}}(k) = \Delta_\ell^{-n} u(k) = \sum_{r=n}^{\left[\frac{k}{\ell}\right]} \frac{(r-1)^{(n-1)}}{(n-1)!} u(k - r\ell). \tag{2}$$

Let $\ell_i > 0$, $u(k)$ be real valued function on $[0, \infty)$, $u(k) = 0$ for all $k \in (-\infty, 0]$, $[k/\ell_i]$ be the integer part of k/ℓ_i, $\ell_i(k) = k - [k/\ell_i]\ell_i$ for $i = 1, 2, \cdots n$, $\ell_o(k) = k$ and $r^{(n)} = r(r-1)\cdots(r-(n-1))$. Then, for $n \geq 2$, (2) induces n_multi-series

$$\underset{\ell_{[1,n]}}{\mathrm{u}}(k) = \sum_{r_n=1}^{\left[\frac{k}{\ell_n}\right]} \underset{\ell_{[1,n-1]}}{\mathrm{u}}(k - r_n\ell_n), \tag{3}$$

where $\underset{\ell_{[1,1]}}{\mathrm{u}}(k) = \sum_{r_1=1}^{\left[\frac{k}{\ell_1}\right]} u(k - r_1\ell_1)$ (1_series with respect to ℓ_1),

$\underset{\ell_{[1,2]}}{\mathrm{u}}(k) = \sum_{r_2=1}^{\left[\frac{k}{\ell_2}\right]} \underset{\ell_{[1,1]}}{\mathrm{u}}(k - r_2\ell_2)$ (2_multi-series with respect to ℓ_1, ℓ_2)
and in general

$\underset{\ell_{[1,i]}}{\mathrm{u}}(k) = \sum_{r_i=1}^{[\frac{k}{\ell_i}]} \underset{\ell_{[1,i-1]}}{\mathrm{u}}(k - r_i\ell_i)$ (i_ multi-series w.r.to $\ell_1, \ell_2, \cdots, \ell_n$).
Substituting $\underset{\ell_{[1,1]}}{\mathrm{u}}, \underset{\ell_{[1,2]}}{\mathrm{u}}, \cdots, \underset{\ell_{[1,n-1]}}{\mathrm{u}}$ in (3), we get

$$\underset{\ell_{[1,n]}}{\mathrm{u}}(k) = \sum_{r_n=1}^{\left[\frac{k}{\ell_n}\right]} \sum_{r_{n-1}=1}^{\left[\frac{k-r_n\ell_n}{\ell_{n-1}}\right]} \sum_{r_{n-2}=1}^{\left[\frac{k-r_n\ell_n-r_{n-1}\ell_{n-1}}{\ell_{n-2}}\right]} \cdots \sum_{r_1=1}^{\left[\frac{k-\sum_{i=2}^{n} r_i\ell_i}{\ell_1}\right]} u(k - \sum_{i=1}^{n} r_i\ell_i), \tag{4}$$

which is a numerical solution of the generalized difference equation

$$\underset{\ell_{[1,n]}}{\Delta} v(k) = \Delta_{\ell_1}(\Delta_{\ell_2} \cdots \Delta_{\ell_n}(v(k)) \cdots) = u(k), \qquad k \geq 0. \tag{5}$$

By denoting R.H.S of (4) as $\sum_{\ell_{[1,n]}} u(\tilde{k})$, (3) and (4) become

$$\underset{\ell_{[1,n]}}{\mathrm{u}}(k) = \sum_{\ell_{[1,n]}} u(\tilde{k}). \tag{6}$$

$$\text{In particular, when } n = 1, \underset{\ell_{[1,1]}}{\mathrm{u(k)}} = \underset{\ell_{[1,1]}}{\Delta^{-1}} u(k)\Big|_{\ell_1(k)}^{k} = \sum_{\ell_{[1,1]}} u(\tilde{k}). \tag{7}$$

Remark 1.1. $\sum\limits_{\ell_{[m,n]}} \underset{\ell_{[1,m-1]}}{\mathrm{u}}(\ell_{m-1}(\tilde{k}))$ denotes

$$\sum_{r_n=1}^{\left[\frac{k}{\ell_n}\right]} \sum_{r_{n-1}=1}^{\left[\frac{k-r_n\ell_n}{\ell_{n-1}}\right]} \cdots \sum_{r_m=1}^{\left[\frac{k-\sum_{i=m+1}^{n} r_i\ell_i}{\ell_m}\right]} \underset{\ell_{[1,m-1]}}{\mathrm{u}}(\ell_{m-1}(k-\sum_{i=m}^{n} r_i\ell_i)).$$

When $\ell_1 = \ell_2 = \cdots \ell_n = \ell$, the above n_multi-series $\sum\limits_{\ell_{[1,n]}} u(\tilde{k})$ becomes $\underset{n(\ell)}{\mathrm{u}}(k)$ given in (2). We find that, by expanding the terms, $\underset{\ell_{[1,n]}}{\mathrm{u}}(k)$ is independent of the order of the parameters $\ell_1,\ell_2,\cdots,\ell_n$. There are direct formulas to find the n_series when $u(k) = k^m$, $k_\ell^{(m)}$, a^k, $k^m a^k$ *etc* and $\ell_1 = \ell_2 = \cdots = \ell_n = \ell$ [3, 4].

There is no direct formula to find the sum of n_multi-series in the existing literature. We find that the n_multi-series $\sum\limits_{\ell_{[1,n]}} u(\tilde{k})$ is the numerical solution as well as the complete solution (closed form solution with lower limits) of equation (5), so we call $\underset{\ell_{[1,n]}}{\mathrm{u}}(k)$ as the complete solution and $\sum\limits_{\ell_{[1,n]}} u(\tilde{k})$ as the numerical solution of (5). Hence in this paper, we obtain the numerical-complete relation (6) and derive n_multi-series with Bernoulli's n-formula.

2. Preliminaries

In this section, we present some notations, basic definitions and preliminary results. Let $J_n = \{1, 2, \ldots, n\}$, $0(J_n) = \{\phi\}$, ϕ is the empty set, $1(J_n) = \{\{1\}, \{2\}, \cdots, \{n\}\}$, $2(J_n) = \{\{1,2\}, \{1,3\}, \cdots, \{1,n\}, \{2,3\}, \cdots, \{2,n\}, \cdots, \{n-2, n-1\}\}$. In general, $t(J_n)$ is the set of all subsets of size t in ascending order from the set J_n, $\wp(J_n) = \bigcup\limits_{t=0}^{n} t(J_n)$ is the power set of J_n, $\sum\limits_{t=1}^{n} f(t) = 0$ for $n < 1$, $\prod\limits_{i=2}^{t} f(i) = 1$ if $t \le 1$, $\underset{\ell_{[p,q]}}{\Delta^{-1}} u(k) = \Delta_{\ell_p}^{-1}(\Delta_{\ell_{p+1}}^{-1} \cdots \Delta_{\ell_q}^{-1}(u(k)) \cdots)$ for $1 \le p < q \le n$, $\underset{\ell_{[1,0]}}{\mathrm{u}} = u(k)$, $\underset{\ell_{[1,1]}}{\mathrm{u}}(k) = \Delta_{\ell_1}^{-1} u(k)$ and for $2 \le i \le n$, $\underset{\ell_{[1,i]}}{\mathrm{u}}(k) = \Delta_{\ell_i}^{-1}\big(\underset{\ell_{[1,i-1]}}{\mathrm{u}}(k)\big|_{\ell_{i-1}(k)}^{k}\big) = \Delta_{\ell_i}^{-1} \underset{\ell_{[1,i-1]}}{\mathrm{u}}(k) - \Delta_{\ell_i}^{-1} \underset{\ell_{[1,i-1]}}{\mathrm{u}}(\ell_{i-1}(k))$.

Lemma 2.1. *[2] If s_r^n and S_r^n are the Stirling numbers of the first and* *second kinds respectively,* $k_\ell^{(n)} = k(k-\ell)\cdots(k-(n-1)\ell)$*, then*

$$k_\ell^{(n)} = \sum_{r=1}^{n} s_r^n \ell^{n-r} k^r \quad and \quad k^n = \sum_{r=1}^{n} S_r^n \ell^{n-r} k_\ell^{(r)}. \tag{8}$$

$$\Delta_\ell k_\ell^{(n)} = (n\ell) k_\ell^{(n-1)} \quad and \quad \Delta_\ell^{-1} k_\ell^{(\nu)} = \frac{k_\ell^{(\nu+1)}}{(\nu+1)\ell}. \tag{9}$$

Theorem 2.2. *[1]* ***(Discrete Bernoulli's Formula)***
Let $u(k)$ *and* $v(k)$ *be two real valued functions,* $u^{(t)}(k) = \Delta_\ell^t u(k)$*,* $v_t(k) = \Delta_\ell^{-t} v(k)$ *for* $t = 1, 2, \cdots$ *and* $u^{(0)}(k) = u(k)$*. Then,*

$$\Delta_\ell^{-1}[u(k)v(k)] = \sum_{t=0}^{\infty} (-1)^t u^{(t)}(k) v_{t+1}(k+t\ell). \tag{10}$$

3. Main Results And Applications

Here, by introducing Striling numbers of third kind and expressing the polynomial factorial $k_{\ell_a}^{(n)}$ interms of $k_{\ell_b}^{(r)}$, $r = 1, 2, \cdots n$ and Stirling numbers of third kind, we derive the Bernoulli's n-formula and n_multi-series to product of arithmetic and geometric functions with examples.

Definition 3.1. Let $1 \le p \le n$, The Stirling number of third kind for the pair of positive reals ℓ_a and ℓ_b is defined by

$$S_{p\_\ell_a}^{n\_\ell_b} = \sum_{t=p}^{n} s_t^n S_p^t \ell_a^{n-t} \ell_b^{t-p}. \tag{11}$$

Lemma 3.2. *The expression of* $k_{\ell_a}^{(n)}$ *interms of* $k_{\ell_b}^{(p)}$ *is given by*

$$k_{\ell_a}^{(n)} = \sum_{p=1}^{n} S_{p-\ell_a}^{n-\ell_b} k_{\ell_b}^{(p)}. \tag{12}$$

Proof. The proof follows from (11) and first, second terms of (8). □

Example 3.3. Since $S_{1-2}^{5-4} = 0$, $S_{2-2}^{5-4} = 0$, $S_{3-2}^{5-4} = 60$, $S_{4-2}^{5-4} = 20$ and $S_{5-2}^{5-4} = 1$, (12) yields $k_2^{(5)} = (60 \times k_4^{(3)}) + (20 \times k_4^{(4)}) + (1 \times k_4^{(5)})$.

Lemma 3.4. *Let* $p_1 = 1$, $\ell_1, \ell_2, \cdots \ell_n$ *be a set of positive reals and* $\underset{\ell_{[1,n]}}{\Delta^{-1}} = \Delta^{-1}_{\ell_1}\Delta^{-1}_{\ell_2}\cdots\Delta^{-1}_{\ell_n}$. *Then,* $\Delta^{-1}_{\ell_1}k^{(0)} = \frac{k^{(1)}_{\ell_1}}{\ell_1}$ *and*

$$\underset{\ell_{[1,n]}}{\Delta^{-1}} k^{(0)} = \left[\prod_{r=2}^{n-1}\sum_{p_r=1}^{1+p_{r-1}} \frac{S^{1+p_{r-1}\_\ell_{r+1}}_{p_r\_\ell_r}}{(1+p_{r-1})\ell_r}\right] \frac{k^{(1+p_{n-1})}_{\ell_n}}{\ell_1(1+p_{n-1})\ell_n}. \tag{13}$$

Proof. Since $1 = k^{(0)}_{\ell_1} = k^{(0)}$ and $k^{(1)}_{\ell_1} = k^{(1)}_{\ell_2}$ from second relation of (9), we get $\Delta^{-1}_{\ell_1}k^{(0)}_{\ell_1} = \dfrac{k^{(1)}_{\ell_1}}{\ell_1}$. Taking $\Delta^{-1}_{\ell_2}$, we get $\underset{\ell_{[1,2]}}{\Delta^{-1}} k^{(0)} = \dfrac{k^{(2)}_{\ell_2}}{2\ell_1\ell_2}$. Again taking $\Delta^{-1}_{\ell_3}$ on both sides of the above and applying (12), we obtain $\underset{\ell_{[1,3]}}{\Delta^{-1}} k^{(0)} = \frac{1}{2\ell_1\ell_2}\Delta^{-1}_{\ell_3}k^{(2)}_{\ell_2} = \sum_{p_2=1}^{2} S^{2\_\ell_3}_{p_2\_\ell_2}\frac{k^{(1+p_2)}_{\ell_3}}{2\ell_1\ell_2\ell_3(1+p_2)}$.
Now the proof is completed by taking $\Delta^{-1}_{\ell_i}$ and applying second relation of (9) and (12) for $i = 4, 5, \cdots, n$ respectively. □

Lemma 3.5. *Let* $p_0 \in N(0)$ *and* $S^{n-\ell_b}_{p-\ell_a}$ *be the Stiriling number of third kind. Then,*

$$\underset{\ell_{[1,n]}}{\Delta^{-1}} k^{(p_o)}_{\ell_1} = \prod_{r=1}^{n-1}\sum_{p_r=1}^{1+p_{r-1}} \frac{S^{1+p_{r-1}\_\ell_{r+1}}_{p_r\_\ell_r}}{(1+p_{r-1})\ell_r}\frac{k^{(1+p_{n-1})}_{\ell_n}}{(1+p_{n-1})\ell_n}. \tag{14}$$

Proof. Taking $\nu = p_o$ and $\ell = \ell_1$ in second relation of (9), we get

$$\Delta^{-1}_{\ell_1}k^{(p_o)}_{\ell_1} = \frac{k^{(p_o+1)}_{\ell_1}}{(p_o+1)\ell_1}.$$

The proof follows by repeatedly applying (12), $\Delta^{-1}_{\ell_i}$ and second relation of (9) for $i = 2, 3, \cdots, n$. □

Theorem 3.6. *A closed form solution of (5) for* $u(k) = k^{p_o}$ *is*

$$\underset{\ell_{[1,n]}}{\Delta^{-1}} k^{p_o} = \prod_{r=2}^{n}\sum_{p_r=1}^{1+p_{r-1}}\sum_{p_1=1}^{p_o} S^{p_o}_{p_1}\ell_1^{p_o-p_1}\frac{S^{1+p_{r-1}\_\ell_r}_{p_r\_\ell_{r-1}}}{(1+p_{r-1})\ell_{r-1}}\frac{k^{(1+p_n)}_{\ell_n}}{(1+p_n)\ell_n}. \tag{15}$$

Proof. Replacing n by p_o, r by p_1 and ℓ by ℓ_1 in second relation of (8), we obtain $k^{p_o} = \sum\limits_{p_1=1}^{p_o} S_{p_1}^{p_o} \ell_1^{p_o-p_1} k_{\ell_1}^{(p_1)}$.
Applying $\Delta_{\ell_1}^{-1}$ and using second relation of (9), we get

$$\Delta_{\ell_1}^{-1} k^{p_o} = \sum_{p_1=1}^{p_o} S_{p_1}^{p_0} \ell_1^{p_o-p_1} \frac{k_{\ell_1}^{(1+p_1)}}{(1+p_1)\ell_1}. \tag{16}$$

Using (12) in (16), we obtain

$$\Delta_{\ell_1}^{-1} k^{p_o} = \sum_{p_1=1}^{p_o} S_{p_1}^{p_o} \ell_1^{p_o-p_1} \sum_{p_2=1}^{1+p_1} S_{p_{2\_\ell_1}}^{\ell_{1+p_1\_}\ell_2} \frac{k_{\ell_2}^{(p_2)}}{(1+p_1)\ell_1}.$$

The proof of (15) follows by contuining this process n times, □

The following Theorem is the generalization of Theorem 2.2.

Theorem 3.7. *(DISCRETE BERNOULLI'S n-FORMULA)*

$$\Delta_{\ell_{[1,n]}}^{-1}[u(k)v(k)] = \prod_{p=1}^{n} \sum_{t_p=0}^{\infty} (-1)^{\sum\limits_{p=1}^{n} t_p} (u(k))^{(t_p)_1^n} (v(k + \sum_{p=1}^{n} t_p \ell_p))_{(1+t_p)_1^n}, \tag{17}$$

$$\textit{where } (u(k))^{(t_p)_1^n} = \Delta_{\ell_1}^{t_1}(\cdots(\Delta_{\ell_n}^{t_n} u(k))\cdots), (v(k + \sum_{p=1}^{n} t_p \ell_p))_{(1+t_n)_1^n}$$

$=\Delta_{\ell_1}^{-(1+t_1)}(\Delta_{\ell_2}^{-(1+t_2)} \cdots (\Delta_{\ell_n}^{-(1+t_n)} v(k + \sum\limits_{p=1}^{n} t_p \ell_p) \cdots))$ *and*
$\prod\limits_{p=1}^{n} \sum\limits_{t_p=0}^{\infty} = \sum\limits_{t_1=0}^{\infty} \sum\limits_{t_2=0}^{\infty} \cdots \sum\limits_{t_n=0}^{\infty}$.

Proof. From Theorem 2.2, we have

$$\Delta_{\ell_1}^{-1}[u(k)v(k)] = \sum_{t_1=0}^{\infty} (-1)^{t_1} (u(k))^{(t_1)} (v(k + t_1\ell_1))_{(1+t_1)}.$$

Taking $\Delta_{\ell_2}^{-1}$ on both sides, we get

$$\Delta_{\ell_{[1,2]}}^{-1}[u(k)v(k)] = \prod_{p=1}^{2} \sum_{t_p=0}^{\infty} (-1)^{t_1+t_2} (u(k))^{(t_p)_1^2} (v(k + t_1\ell_1 + t_2\ell_2))_{(1+t_p)_1^p}. \tag{18}$$

The proof follows by taking $\Delta_{\ell_i}^{-1}$, for $i = 3, 4, \cdots, n$ successively on both sides of (18). □

Lemma 3.8. *Let t_i's and p_o are positive integers. Then,*

$$\prod_{i=1}^{n} \Delta_{\ell_i}^{t_i}(k_{\ell_1}^{(p_o)}) = p_o^{(t_1)}\ell_1^{t_1} \prod_{r=1}^{n-1} \sum_{p_r=1}^{p_{r-1}-t_r} p_r^{t_{r+1}} \ell_{r+1}^{t_{r+1}} S_{p_r \_ \ell_r}^{p_{r-1}-t_r \_ \ell_{r+1}} k_{\ell_n}^{(p_{n-1}-t_n)}. \tag{19}$$

Proof. From first relation of (9), we have

$$\Delta_{\ell_1}^{t_1}(k_{\ell_1}^{(p_o)}) = p_o^{(t_1)}\ell_1^{t_1} k_{\ell_1}^{(p_o-t_1)}.$$

Taking $\Delta_{\ell_2}^{t_2}$ on both sides of the above equation, we obtain

$$\Delta_{\ell_2}^{t_2}(\Delta_{\ell_1}^{t_1}(k_{\ell_1}^{(p_o)})) = p_o^{(t_1)}\ell_1^{t_1} \Delta_{\ell_2}^{t_2}(k_{\ell_1}^{(p_o-t_1)}).$$

Applying (12), we get

$$\Delta_{\ell_2}^{t_2}(\Delta_{\ell_1}^{t_1}(k_{\ell_1}^{(p_o)})) = p_o^{(t_1)}\ell_1^{t_1} \Delta_{\ell_2}^{t_2}\left(\sum_{p_1=1}^{p_o-t_1} S_{p_1 \_ \ell_1}^{p_o-t_1 \_ \ell_2} k_{\ell_2}^{(p_1)}\right).$$

The proof follows by repeating this process n times. $\square$

Lemma 3.9. *If $a^{\ell_i} - 1 \neq 0$, for $i = 1, 2, \cdots n$, then*

$$\Delta_{\ell_1}^{-(1+t_1)}\left(\Delta_{\ell_2}^{-(1+t_2)}\left(\cdots \Delta_{\ell_n}^{-(1+t_n)} a^k\right)\right) = \frac{a^k}{\prod_{i=1}^{n}(a^{\ell_i}-1)^{(1+t_i)}}. \tag{20}$$

Proof. Since $\Delta_\ell a^k = a^{k+\ell} - a^k = a^k(a^\ell - 1)$, we have

$$\Delta_{\ell_1}^{-(1+t_1)} a^k = \frac{a^k}{(a^{\ell_1}-1)^{(1+t_1)}}.$$

Taking $\Delta_{\ell_2}^{-(1+t_2)}$ on both sides of the above equation, we get

$$\Delta_{\ell_1}^{-(1+t_1)}\left(\Delta_{\ell_2}^{-(1+t_2)} a^k\right) = \frac{a^k}{(a^{\ell_1}-1)^{(1+t_1)}(a^{\ell_2}-1)^{(1+t_2)}}.$$

The proof follows by repeatedly taking $\Delta_{\ell_i}^{-(1+t_i)}$ on both sides for $i = 3, 4, \cdots, n$. $\square$

Theorem 3.10. *Bernoulli's n_Formula to the product $k_\ell^{(p_o)} a^k$ is*

$$\Delta_{\ell_{[1,n]}}^{-1}(k_{\ell_1}^{(p_o)} a^k) = \prod_{p=1}^{n} \sum_{t_p=0}^{p_o} (-1)^{\sum_{p=1}^{n} t_p} (k_{\ell_1}^{(p_o)})^{(t_p)_1^n} (a^{(k+\sum_{p=1}^{n} t_p \ell_p)})_{(1+t_p)_1^n}. \tag{21}$$

Proof. Since $(k_{\ell_1}^{(p_o)})^{(t_1)(t_2)\cdots(t_n)} = 0$ when $t_1+t_2+\cdots+t_n > p_o$ and $t_1, t_2, \cdots, t_n$ take the values from 0 to p_0, we have

$$\underset{\ell_{[1,n]}}{\Delta^{-1}}(k_{\ell_1}^{(p_o)} a^k) = \prod_{p=1}^{n} \sum_{t_p=0}^{p_o} (-1)^{\sum_{p=1}^{n} t_p} (k_{\ell_1}^{(p_o)})^{(t_p)_1^n} (a^{(k+\sum_{p=1}^{n} t_p\ell_p)})_{(1+t_p)_1^n},$$

$$where\ (k_{\ell_1}^{(p_o)})^{(t_p)_1^n} = (k_{\ell_1}^{(p_o)})^{(t_1)(t_2)\cdots(t_n)} = \begin{cases} \Delta_{\ell_1}^{t_1}(\Delta_{\ell_2}^{t_2}(\cdots \Delta_{\ell_n}^{t_n} k_{\ell_1}^{(p_o)})) \\ 0 \text{ if } t_1, t_2, \cdots, t_n > p_o, \end{cases} \tag{22}$$

$$(a^{k+\sum_{p=1}^{n} t_p\ell_p})_{(1+t_p)_1^n} = \Delta_{\ell_1}^{-(1+t_1)}(\Delta_{\ell_2}^{-(1+t_2)} \cdots (\Delta_{\ell_n}^{-(1+t_n)} a^{k+\sum_{p=1}^{n} t_p\ell_P}))). \tag{23}$$

(21) follows by taking $u(k) = k_\ell^{(p_o)}$ and $v(k) = a^k$ in (17). □

The following theorem gives complete solution of equation (5).

Theorem 3.11. *Consider the functions* $\underset{\ell_{[1,i]}}{\mathrm{u}}(k)$, $\ell_i(k)$ *for* $i = 1, 2, \cdots, n$, *given in the notations and above. Assume that for each* i, $1 \le i \le n$, $\underset{\ell_{[1,i]}}{\Delta^{-1}} u(k)$ *be any closed form solution of the difference equation* $\underset{\ell_{[1,i]}}{\Delta} v(k) = u(k)$. *Then, for* $k \ge \max_{1\le i\le n} \ell_i$,

$$\underset{\ell_{[1,n]}}{\mathrm{u}}(k)\Big|_{\ell_n(k)}^{k} = \underset{\ell_{[1,n]}}{\Delta^{-1}} u(k) + \sum_{t=1}^{n}(-1)^t \sum_{\{m_s\}_{s=1}^{t}\in t(J_n)} \underset{\ell_{[1,m_1]}}{\Delta^{-1}} u(\ell_{m_1}(k))\times$$

$$\times \prod_{i=1}^{t} \underset{\ell_{[1+m_i,m_{i+1}]}}{\Delta^{-1}} (\ell_{m_{i+1}}(k))^{(0)} \underset{\ell_{[1+m_t,n]}}{\Delta^{-1}} k^{(0)} \tag{24}$$

is the complete solution of equation (5).

Proof. Since $1 = k^{(0)}$, applying the limit from $\ell_1(k)$ to k for $\Delta_{\ell_1}^{-1} u(k)$, we have $\Delta_{\ell_1}^{-1} u(k)\big|_{\ell_1(k)}^{k} = \Delta_{\ell_1}^{-1} u(k) - \Delta_{\ell_1}^{-1} u(\ell_1(k)) k^{(0)}$,
which is the complete solution of equation (5) for $n = 1$.
Taking $\Delta_{\ell_2}^{-1}$ on both sides and applying the limits from $\ell_2(k)$ to k and keeping $\Delta_{\ell_1}^{-1} u(\ell_1(k))$ as a constant, we obtain

$$\Delta_{\ell_2}^{-1}(\Delta_{\ell_1}^{-1} u(k)\big|_{\ell_1(k)}^{k})\big|_{\ell_2(k)}^{k} = \underset{\ell_{[1,2]}}{\Delta^{-1}} u(k)\big|_{\ell_2(k)}^{k} - \Delta_{\ell_1}^{-1} u(\ell_1(k)) \Delta_{\ell_2}^{-1} k^{(0)}\big|_{\ell_2(k)}^{k}$$

which is the complete solution of the equation (5), and it can be expressed as

$$\underset{\ell_{[1,2]}}{\mathrm{u}}(k)\big|_{\ell_2(k)}^{k} = \underset{\ell_{[1,2]}}{\Delta^{-1}} u(k) - \Delta_{\ell_1}^{-1} u(\ell_1(k)) \Delta_{\ell_2}^{-1} k^{(0)}$$

$$-\underset{\ell_{[1,2]}}{\Delta^{-1}} u(\ell_2(k)) + \Delta_{\ell_1}^{-1} u(\ell_1(k)) \Delta_{\ell_2}^{-1}(\ell_2(k))^{(0)}.$$

In the R.H.S of the above expression, second term is associated to $\{m_1\} = \{1\} \in 1(J_2)$, third term to $\{m_1\} = \{2\} \in 1(J_2)$ and the fourth term to $\{m_1, m_2\} = \{1, 2\} \in 2(J_2)$. Taking $\Delta_{\ell_3}^{-1}$ on $u_2(k)$, applying the limits $\ell_3(k)$ and k, and as $\Delta_{\ell_1}^{-1} u(\ell_1(k))$, $\Delta_{\ell_2}^{-1}(\ell_2(k))^{(0)}$ and $\underset{\ell_{[1,2]}}{\Delta^{-1}} u(\ell_2(k))$ are constants, we get a relation of the form $\underset{\ell_{[1,3]}}{\mathrm{u}}(k)\Big|_{\ell_3(k)}^{k} = \Delta_{\ell_3}^{-1} \underset{\ell_{[1,2]}}{\mathrm{u}}(k) - \Delta_{\ell_3}^{-1} \underset{\ell_{[1,2]}}{\mathrm{u}}(\ell_3(k))$, and this relation can be expressed as

$$\underset{\ell_{[1,3]}}{\mathrm{u}}(k)\Big|_{\ell_3(k)}^{k} = \underset{\ell_{[1,3]}}{\Delta^{-1}} u(k) + \sum_{t=1}^{3} \sum_{\{m_s\}_1^t \in t(J_n)} (-1)^t \underset{\ell_{[1,m_1]}}{\Delta^{-1}} u(\ell_{m_1}(k))$$

$$\times \prod_{i=1}^{t} \underset{\ell_{[1+m_i, m_{i+1}]}}{\Delta^{-1}} (\ell_{m_{i+1}}(k))^{(0)} \underset{\ell_{[1+m_t,3]}}{\Delta^{-1}} k^{(0)},$$

which is a complete solution of the equation (5) for $n = 3$.
As all the lower limit values are constants, the proof is completed by taking $\Delta_{\ell_i}^{-1}$ and applying the limit from $\ell_i(k)$ to k on $\underset{[1,3]}{\mathrm{u}}(k)$ successively for $i = 4, 5, \cdots, n$. □

The following theorem gives the numerical solution of the equation (5).

Theorem 3.12. *Consider the terms of Theorem 3.11. Then,*

$$v(k) = \sum_{m=1}^{n} \sum_{\ell_{[m,n]}} \underset{\ell_{[1,m-1]}}{\mathrm{u}}(\ell_{m-1}(\tilde{k})), \quad k \geq \sum_{i=1}^{n} \ell_i, \tag{25}$$

is the numerical solution of the difference equation (5).

Proof. From equation (7), we have

$$\Delta_{\ell_1}^{-1} u(k)\Big|_{\ell_1(k)}^{k} = \sum_{\ell_{[1,1]}} u(\tilde{k}) = \underset{\ell_{[1,1]}}{\mathrm{u}}(k) - \underset{\ell_{[1,1]}}{\mathrm{u}}(\ell_1(k)) = z_1(k), \text{ (say)} \tag{26}$$

is a numerical solution of the equation (5) for $n = 1$. Again taking $\Delta_{\ell_2}^{-1}$ on $z_1(k)$ and applying equation (7), we get

$$\Delta_{\ell_2}^{-1} z_1(k)\Big|_{\ell_2(k)}^{k} = \sum_{\ell_{[2,2]}} z_1(\tilde{k}) = z_2(k), \text{ (say)} \tag{27}$$

which is a numerical solution of the equation (5) for $n = 2$.
Replacing k by $k - r_2\ell_2$ in (26), we obtain

$$z_1(k - r_2\ell_2) = \underset{\ell_{[1,1]}}{\mathrm{u}}(k - r_2\ell_2) - \underset{\ell_{[1,1]}}{\mathrm{u}}(\ell_1(k - r_2\ell_2)). \tag{28}$$

Substituting (28) in (27), we find that

$$z_2(k) = \sum_{\ell_{[2,2]}} \underset{\ell_{[1,1]}}{\mathrm{u}}(\tilde{k}) - \sum_{\ell_{[2,2]}} \underset{\ell_{[1,1]}}{\mathrm{u}}(\ell_1(\tilde{k})) \tag{29}$$

which is the same as

$$z_2(k) = \underset{\ell_{[1,2]}}{\mathrm{u}}(k) - \underset{\ell_{[1,2]}}{\mathrm{u}}(\ell_2(k)) - \sum_{\ell_{[2,2]}} \underset{\ell_{[1,1]}}{\mathrm{u}}(\ell_1(\tilde{k})). \tag{30}$$

Applying the numerical solution $z_2(k) = \sum\limits_{\ell_{[1,2]}} u(\tilde{k})$ on (30), we get

$$\underset{\ell_{[1,2]}}{\mathrm{u}}(k)\Big|_{\ell_2(k)}^{k} = \sum_{\ell_{[1,2]}} u(\tilde{k}) + \sum_{\ell_{[2,2]}} \underset{\ell_{[1,1]}}{\mathrm{u}}(\ell_1(\tilde{k})), \tag{31}$$

where the values $\underset{\ell_{[1,1]}}{\mathrm{u}}(\ell_1(k - r_2\ell_2))$ can be evaluated by replacing k by $\ell_1(k - r_2\ell_2)$ in the complete solution $\underset{\ell_{[1,1]}}{\mathrm{u}}(k)\Big|_{\ell_1(k)}^{k}$ given in Theorem 3.11 for $n = 1$.
The proof is completed by taking $\Delta_{\ell_i}^{-1}$ on $z_2(k)$ and applying the numerical solution of equation (5) successively for $i = 3, 4, 5, \cdots, n$. □

Theorem 3.13. *The numerical-complete relation of the equation (5) is given by*

$$\sum_{m=1}^{n} \sum_{\ell_{[m,n]}} \underset{\ell_{[1,m-1]}}{\mathrm{u}}(\ell_{m-1}(\tilde{k})) = \underset{\ell_{[1,n]}}{\Delta^{-1}} u(k) + \sum_{t=1}^{n} \sum_{\{m_s\}_{s=1}^{t} \in t(J_n)} (-1)^t \times$$

$$\times \underset{\ell_{[1,m_1]}}{\Delta^{-1}} u(\ell_{m_1}(k)) \prod_{i=1}^{t} \underset{\ell_{[1+m_i,m_{i+1}]}}{\Delta^{-1}} (\ell_{m_{i+1}}(k))^{(0)} \underset{\ell_{[1+m_t,n]}}{\Delta^{-1}} k^{(0)}. \tag{32}$$

Proof. The proof follows by equating the numerical solution given in Theorem 3.12 and the complete solution given in Theorem 3.11. □

Theorem 3.14. *n_multi-series to the polynomial factorial $k_{\ell_1}^{p_o}$ is*

$$\sum_{m=1}^{n} \sum_{\ell_{[m,n]}} \underset{\ell_{[1,m-1]}}{\mathrm{u}}(\ell_{m-1}(\tilde{k})) = \underset{\ell_{[1,n]}}{\Delta^{-1}} k_{\ell_1}^{(p_o)} + \sum_{t=1}^{n} \sum_{\{m_s\}_{s=1}^{t} \in t(J_n)} (-1)^t \times$$

$$\times \underset{\ell_{[1,m_1]}}{\Delta^{-1}} (\ell_{m_1}(k))^{(p_o)}_{\ell_1} \prod_{i=1}^{t} \underset{\ell_{[1+m_i,m_{i+1}]}}{\Delta^{-1}} (\ell_{m_{i+1}}(k))^{(0)} \underset{\ell_{[1+m_t,n]}}{\Delta^{-1}} k^{(0)}. \tag{33}$$

Proof. The proof follows by taking $u(k) = k^{(p_o)}_{\ell_1}$ in (32) and the terms in (33) are evaluated by (13) and (14). □

Theorem 3.15. *n_multi-series formula to the polynomial k^{p_o} is*

$$\sum_{m=1}^{n} \sum_{\ell_{[m,n]}} \underset{\ell_{[1,m-1]}}{\mathrm{u}} (\ell_{m-1}(\tilde{k})) = \underset{\ell_{[1,n]}}{\Delta^{-1}} k^{p_o} + \sum_{t=1}^{n} \sum_{\{m_s\}_{s=1}^{t} \in t(J_n)} (-1)^t \times$$

$$\times \underset{\ell_{[1,m_1]}}{\Delta^{-1}} (\ell_{m_1}(k))^{p_o} \prod_{i=1}^{t} \underset{\ell_{[1+m_i,m_{i+1}]}}{\Delta^{-1}} (\ell_{m_{i+1}}(k))^{(0)} \underset{\ell_{[1+m_t,n]}}{\Delta^{-1}} k^{(0)}. \tag{34}$$

Proof. The proof follows by taking $u(k) = k^{p_o}$ in (32) and the terms in (34) are evaluated by (15). □

Theorem 3.16. *n_multi-series formula to $U(k) = u(k)v(k)$ is*

$$\sum_{m=1}^{n} \sum_{\ell_{[m,n]}} \underset{\ell_{[1,m-1]}}{\mathrm{U}} (\ell_{m-1}(\tilde{k})) = \underset{\ell_{[1,n]}}{\Delta^{-1}} (u(k)v(k)) + \sum_{t=1}^{n} \sum_{\{m_s\}_{s=1}^{t} \in t(J_n)} (-1)^t$$

$$\times \underset{\ell_{[1,m_1]}}{\Delta^{-1}} u(\ell_{m_1}(k))v(\ell_{m_1}(k)) \prod_{i=1}^{t} \underset{\ell_{[1+m_i,m_{i+1}]}}{\Delta^{-1}} (\ell_{m_{i+1}}(k))^{(0)} \underset{\ell_{[1+m_t,n]}}{\Delta^{-1}} k^{(0)}. \tag{35}$$

Proof. The proof follows by taking $U(k) = u(k)v(k)$ in Theorem 3.13 and $\underset{\ell_{[1,n]}}{\Delta^{-1}} [u(k)v(k)]$ can be obtain by Bernoulli's n-Formula . □

The following example illustrates Therom 3.16.

Example 3.17. Taking $n = 2$, $u(k) = k^{(2)}_{\ell_1} = (u(k))^{(0)(0)}$, $v(k) = a^k$ in (35) and using (17), we get

$$(u(k))^{(t_p)_1^2} = \Delta^{t_1}_{\ell_1}(\Delta^{t_2}_{\ell_2} u(k)) = \Delta^{t_1}_{\ell_1}(\Delta^{t_2}_{\ell_2} k^{(2)}_{\ell_1}). \tag{36}$$

From (36), $(u(k))^{(0)(1)} = 2\ell_2 k^{(1)}_{\ell_2} + (\ell_2 - \ell_1)\ell_2$, $(u(k))^{(0)(2)} = 2\ell_2^2$, $(u(k))^{(1)(0)} = 2\ell_1 k^{(1)}_{\ell_2}$, $(u(k))^{(1)(1)} = 2\ell_1\ell_2$, $(u(k))^{(2)(0)} = 2\ell_1^2$ and

$$(v(k))_{(1+t_p)_1^2} = \Delta^{-(1+t_1)}_{\ell_1}(\Delta^{-(1+t_2)}_{\ell_2} a^k) = \frac{a^k}{\prod_{i=1}^{2} (a^{\ell_i} - 1)^{1+t_i}}. \tag{37}$$

(37) yields $(v(k))_{(1)(1)}, (v(k))_{(1)(2)}, (v(k))_{(1)(3)}, (v(k))_{(2)(1)}, (v(k))_{(2)(2)}, (v(k))_{(2)(3)}$, $(v(k))_{(3)(1)}$ and $(v(k))_{(3)(2)}$.
To find the corresponding (35), consider:

(i) $$\underset{\ell_{[1,2]}}{\Delta^{-1}}\left(k^{(2)}_{\ell_1}a^k\right)= \frac{k^{(2)}_{\ell_2}a^k}{(a^{\ell_1}-1)(a^{\ell_2}-1)} - \frac{2\ell_2 k^{(1)}_{\ell_2}a^{k+\ell_2}}{(a^{\ell_1}-1)(a^{\ell_2}-1)^2} + \frac{2\ell_2^2 a^{k+2\ell_2}}{(a^{\ell_1}-1)(a^{\ell_2}-1)^3} + \left\{\frac{(\ell_2-\ell_1)}{(a^{\ell_1}-1)} - \frac{2\ell_1 a^{\ell_1}}{(a^{\ell_1}-1)^2}\right\} \times \left\{\frac{k^{(1)}_{\ell_2}a^k}{(a^{\ell_2}-1)} - \frac{\ell_2 a^{k+\ell_2}}{(a^{\ell_2}-1)^2}\right\} + \frac{2\ell_1^2 a^{k+2\ell_1}}{(a^{\ell_1}-1)^3(a^{\ell_2}-1)}.$$

(ii) the terms for $1(J_2) = \{\{1\},\{2\}\}$:
$(-1)\underset{\ell_{[1,1]}}{\Delta^{-1}}(\ell_1(k))^{(2)}_{\ell_1}a^{\ell_1(k)}\underset{\ell_{[2,2]}}{\Delta^{-1}}k^{(0)}$ *and* $(-1)\underset{\ell_{[1,2]}}{\Delta^{-1}}(\ell_2(k))^{(2)}_{\ell_1}a^{\ell_2(k)}$.

(iii) the terms for $2(J_2) = \{1,2\}$: $\underset{\ell_{[1,1]}}{\Delta^{-1}}(\ell_1(k))^{(2)}_{\ell_1}a^{\ell_1(k)}\underset{\ell_{[2,2]}}{\Delta^{-1}}(\ell_2(k))^{(0)}_{\ell_2}$.

In particular taking $k = 5$, $\ell_1 = 1$, $\ell_2 = 2$, $\ell_1(k) = 0$, $\ell_2(k) = 1$ and $a = 2$ in (35), we get (i) 37.92592 (ii) 20 and 13.92592 (iii) 4.
R.H.S of (35) is the sum of the terms $(i) + (ii) + (iii) = 8$.
L.H.S of (35) is 8, which is obtained by adding the terms:
$m = 1;\quad \sum_{\ell_{[1,2]}} u(\tilde{k}) = 0$ and $m = 2;\quad \sum_{\ell_{[2,2]}} \underset{\ell[1,1]}{\mathrm{u}}(\ell_1(\tilde{k})) = 8$.

Conclusion: Finding the complete solution of higher order Generalized difference equations coincided with the numerical solution of that equation is the significant of the research work. This relation generates certain formulas and some applications for finding the growth of animal population of first n generations.

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