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ON FAINTLY g-CONTINUOUS FUNCTIONS

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Abstract: A new class of functions called faintly g-continuous functions has been defined and studied in topological space. Also, the relationships between faintly g-continuous functions and graphs are investigated.

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized closed sets. In 1970, Levine [6] initiated the study of so-called g-closed sets, that is, a subset A of a topological space (X, τ) is g-closed if the closure of A included in every open superset of A and defined a $T_{1/2}$ space to be one in which the closed sets and the g-closed sets coincide. In this paper, a new class of functions called

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faintly g-continuous functions has been defined and studied in topological space. Relationships among this new class of functions and GO-connected spaces and GO-compact spaces are investigated. Also, the relationship between faintly g-continuous functions and graphs are investigated.

2. Preliminaries

In the present paper, (X, τ) and (Y, σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$ and A^c denote the closure of A, the interior of A and the complement of A in X, respectively. The complement of g-closed set is called g-open. The family of all g-open sets of (X, τ) is denoted by $GO(X)$. A point $x \in X$ is called a θ-cluster point of A if $\mathrm{Cl}(A) \cap A \neq \emptyset$ for every open set V of X containing x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $\mathrm{Cl}_\theta(A)$. If $A = \mathrm{Cl}_\theta(A)$, then A is said to be θ-closed. The complement of θ-closed set is said to be θ-open. The union of all θ-open sets contained in a subset A is called the θ-interior of A and is denoted by $\mathrm{Int}_\theta(A)$. It follows from [12] that the collection of θ-open sets in a topological space (X, τ) forms a topology τ_θ on X.

Definition 1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:

(i) faintly continuous [7] if $f^{-1}(V)$ is open in X for every θ-open set V of Y.

(ii) g-continuous [1] if $f^{-1}(V)$ is g-closed set in X for each closed set V of Y.

(iii) gc-irresolute [1] if $f^{-1}(V)$ is g-closed set in X for each g-closed set V of Y.

(iv) gc-homeomorphism [8] if it is bijective, gc-irresolute and its inverse f^{-1} is gc-irresolute.

(v) g-closed [9] if $f(F)$ is g-closed in Y for every closed set F of X.

3. Faintly g-Continuous Functions

Definition 2. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be faintly g-continuous if $f^{-1}(V)$ is g-open in X for every θ-open set V of Y.

Theorem 3. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(i) f *is faintly g-continuous;*

(ii) $f^{-1}(F)$ *is g-closed in X for every θ-closed subset F of Y;*

(iii) $f : (X, \tau) \to (Y, \sigma_\theta)$ *is g-continuous.*

Proof. Easy proof and hence omitted. □

Theorem 4. *Every g-continuous function is faintly g-continuous.*

Proof. Clear. □

Remark 5. The converse of Theorem 4 is not true in general as can be seen from the following example.

Example 6. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is faintly g-continuous but not g-continuous.

Definition 7. A topological space (X, τ) is said to be a $T_{1/2}$-space [6] if every g-closed subset of (X, τ) is in closed.

Theorem 8. *Let (Y, σ) be a $T_{1/2}$-space. Then a function $f : (X, \tau) \to (Y, \sigma)$ is faintly g-continuous if and only if it is faintly continuous.*

Proof. Follows from the Definition 7. □

Theorem 9. *If a function $f : (X, \tau) \to (Y, \sigma)$ is faintly g-continuous and (Y, σ) is a regular space, then f is g-continuous.*

Proof. Let V be any open set of Y. Since Y is regular, V is θ-open in Y. Since f is faintly g-continuous, by Theorem 3, we have $f^{-1}(V)$ is g-open and hence f is g-continuous. □

Theorem 10. *If a function $f : (X, \tau) \to (Y, \sigma)$ is faintly g-continuous, then for each point $x \in X$ and each θ-open set V containing $f(x)$, there exists a g-open set U containing x and $f(U) \subseteq V$.*

Proof. Clear. □

Remark 11. The following example shows that the composition of two faintly g-continuous functions need not be faintly g-continuous.

Example 12. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\eta = \{\emptyset, \{b\}, \{a, c\}, X\}$. Define a function $f : (X, \tau) \to (X, \sigma)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$ and $g : (X, \sigma) \to (X, \eta)$ an identity function. Then f and g are faintly g-continuous functions but their composition is not faintly g-continuous.

Now we investigate some basic properties of faintly g-continuous functions concerning composition and restriction. The proof of the first two results are straightforward and are omitted.

Theorem 13. *If $f : (X, \tau) \to (Y, \sigma)$ is gc-irresolute and $g : (Y, \sigma) \to (Z, \eta)$ is faintly g-continuous, then $g \circ f : (X, \tau) \to (Z, \eta)$ is faintly g-continuous.*

It is well known that a function $f : (X, \tau) \to (Y, \sigma)$ is said to be quasi-θ-continuous [10] if $f^{-1}(V)$ is θ-open in (X, τ) for every θ-open set V of (Y, σ).

Theorem 14. *If $f : (X, \tau) \to (Y, \sigma)$ is faintly g-continuous and $g : (Y, \sigma) \to (Z, \gamma)$ is quasi-θ-continuous, then $g \circ f : (X, \tau) \to (Z, \gamma)$ is faintly g-continuous.*

Let $\{X_\alpha : \alpha \in \Lambda\}$ and $\{Y_\alpha : \alpha \in \Lambda\}$ be two families of topological spaces with the same index set Λ. The product space of $\{X_\alpha : \alpha \in \Lambda\}$ is denoted by Π $\{X_\alpha{:}\alpha \in \Lambda\}$ (or simply ΠX_α). Let f_α: $X_\alpha \to Y_\alpha$ be a function for each $\alpha \in \Lambda$. The product function f: $\Pi X_\alpha \to \Pi Y_\alpha$ is defined by $f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\}$ for each $\{x_\alpha\} \in \Pi X_\alpha$.

Theorem 15. *If a function $f : X \to \Pi Y_\alpha$ is faintly g-continuous, then $P_\alpha \circ f$: $X \to Y_\alpha$ is faintly g-continuous for each $\alpha \in \Lambda$, where P_α is the projection of ΠY_α onto Y_α.*

Proof. Let V_α be any θ-open set of Y_α. Then, $P_\alpha^{-1}(V_\alpha)$ is θ-open in ΠY_α and hence $(P_\alpha \circ f)^{-1}$ $(V_\alpha) = f^{-1}(P_\alpha^{-1}(V_\alpha))$ is g-open in X. Therefore, $P_\alpha \circ f$ is faintly g-continuous. □

Theorem 16. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \gamma)$ be functions. If f is bijective, continuous and g-closed, and if $g \circ f : (X, \tau) \to (Z, \gamma)$ is faintly g-continuous, then $g : (Y, \sigma) \to (Z, \gamma)$ is faintly g-continuous.*

Proof. Let V be a θ-open set of Z. Then $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is g-open in X. Since f is bijective, continuous and g-closed, f maps g-open sets to g-open sets [[3], Theorem 3] hence it follows that $g^{-1}(V) = f(f^{-1}(g^{-1}(V)))$ is g-open in Y. □

Corollary 17. *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective gc-homeomorphism and let $g : (Y, \sigma) \to (Z, \gamma)$ be a function. Then $g \circ f : (X, \tau) \to (Z, \gamma)$ is faintly g-continuous if and only if g is faintly g-continuous.*

Theorem 18. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is faintly g-continuous and A is a closed subset of X, then $f_{|A}$: $(A, \tau_{|A}) \to (Y, \sigma)$ is faintly g-continuous.*

Proof. The proof follows from Corollary 2.7 of [6]. □

Theorem 19. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ the graph function of f, defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is faintly g-continuous, then f is faintly g-continuous.*

Proof. Let U be an θ-open set in (Y, σ), then $X \times U$ is a θ-open set in $X \times Y$. It follows that $f^{-1}(U) = g^{-1}(X \times U) \in GO(X)$. This shows that f is faintly g-continuous. □

Definition 20. A topological space (X, τ) is said to be *GO*-connected [1] if X cannot be written as a disjoint union of two nonempty g-open sets.

Theorem 21. *If $f : (X, \tau) \to (Y, \sigma)$ is a faintly g-continuous function and (X, τ) is a GO-connected space, then Y is a connected space.*

Proof. Assume that (Y, σ) is not connected. Then there exist nonempty open sets V_1 and V_2 such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = Y$. Hence we have $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and $f^{-1}(V_1) \cup f^{-1}(V_2) = X$. Since f is surjective, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty subsets of X. Since V_i is open and closed, V_i is θ-open for each $i = 1, 2$. Since f is faintly g-continuous, $f^{-1}(V_i) \in GO(X)$. Therefore, (X, τ) is not *GO*-connected. This is a contradiction and hence (Y, σ) is connected. □

Definition 22. A space (X, τ) is said to be *GO*-compact [1] (resp. θ-compact [11]) if each cover of X by g-open (resp. θ-open) has a finite subcover.

Theorem 23. *The surjective faintly g-continuous image of a GO-compact space is θ-compact.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a faintly g-continuous function from a *GO*-compact space X onto a space Y. Let $\{G_\alpha$: $\alpha \in I\}$ be any θ-open cover of Y. Since f is faintly g-continuous, $\{f^{-1}(G_\alpha)$: $\alpha \in I\}$ is a g-open cover of X. Since X is g-compact, there exists a finite subcover $\{f^{-1}(G_i) : i = 1, 2, ...\ n\}$ of X. Then it follows that $\{G_i$: $i = 1, 2, ...\ n\}$ is a finite subfamily which cover Y. Hence Y is θ-compact. □

Definition 24. A topological space (X, τ) is said to be:

(i) countably g-compact [5] (resp. countably θ-compact) if every countable cover of X by g-open (resp. θ-open) sets has a finite subcover;

(ii) g-Lindelof [5] (resp. θ-Lindelof) if every cover of X by g-open (resp. θ-open) sets has a countable subocver.

Theorem 25. *Let $f : (X, \tau) \to (Y, \sigma)$ be a faintly g-continuous surjective function. Then the following hold:*

(i) If X is g-Lindelof, then Y is θ-Lindelof;

(ii) If X is countably g-compact, then Y is countably θ-compact.

Proof. The proof is similar to Theorem 23. □

Definition 26. A topological space (X, τ) is said to be:

(i) g-T_1 [4] (resp. θ-T_1) if for each pair of distinct points x and y of X, there exists g-open (resp. θ-open) sets U and V containing x and y, respectively such that $y \notin U$ and $x \notin V$.

(ii) g-T_2 [2] (resp. θ-T_2 [11]) if for each pair of distinct points x and y in X, there exists disjoint g-open (resp. θ-open) sets U and V in X such that $x \in U$ and $y \in V$.

Theorem 27. *If $f : (X, \tau) \to (Y, \sigma)$ is faintly g-continuous injection and Y is θ-T_1, then X is a g-T_1.*

Proof. Suppose that Y is θ-T_1. For any distinct points x and y in X, there exist $V, W \in \sigma_\theta$ such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$ and $f(y) \in W$. Since f is faintly g-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are g-open subsets of (X, τ) such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that X is g-T_1. □

Theorem 28. *If $f : (X, \tau) \to (Y, \sigma)$ is faintly g-continuous injection and Y is a θ-T_2 space, then X is a g-T_2 space.*

Proof. Suppose that Y is θ-T_2. For any pair of distinct points x and y in X, there exist disjoint θ-open sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is faintly g-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are g-open in X containing x and y, respectively. Therefore, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ because $U \cap V = \emptyset$. This shows that X is g-T_2. □

Recall that for a function $f : (X, \tau) \to (Y, \sigma)$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Theorem 29. *If $f : (X, \tau) \to (Y, \sigma)$ is faintly g-continuous injection and (Y, σ) is θ-T_2, then $G(f)$ is g-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y)\backslash G(f)$, then $f(x) \neq y$. Since Y is θ-T_2, there exist θ-open sets V and W in Y such that $f(x) \in V$, $y \in W$ and $V \cap W = \emptyset$. Since f is faintly g-continuous, $f^{-1}(V) \in GO(X, x)$. Take $U = f^{-1}(V)$. We have $f(U) \subset V$. Therefore, we obtain $f(U) \cap V = \emptyset$. This shows that $G(f)$ is g-closed. □

Theorem 30. *If $f : (X, \tau) \to (Y, \sigma)$ is a surjective function with a g-closed graph, then (Y, σ) is Hausdorff.*

Proof. Let y_1 and y_2 be any distinct points of Y. Then since f is surjective, there exists $x_1 \in X$ such that $f(x_1) = y_1$; hence $(x_1, y_2) \in (X \times Y) \setminus G(f)$. Since $G(f)$ is g-closed, there exist $U \in GO(X, x_1)$ and an open set V of Y containing y_2 such that $f(U) \cap \mathrm{Cl}(V) = \emptyset$. Therefore, we have $y_1 = f(x_1) \in f(U) \subset Y \setminus \mathrm{Cl}(V)$. Then there exists an open set H of Y such that $y_1 \in H$ and $H \cap V = \emptyset$. Moreover, we have $y_2 \in V$ and V is open in Y. This shows that Y is Hausdorff. □

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