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ON SUBCLASSES OF BIUNIVALENT FUNCTIONS OF BAZILEVIC TYPE INVOLVING LINEAR AND SALAGEAN OPERATOR

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Abstract: In this present work, the authors introduce two subclasses of biunivalent functions using linear and Salagean operator and the coefficient estimates is obtained for $|a_2|$ and $|a_3|$ for these subclasses of functions.

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1. First Section of the Paper

Let Γ denote a class of function of the form

$$z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disc $U = \{|z| : z < 1\}$ and normalized with $f(0) = f'(0) - 1 = 0$. With this equation (1), we recall our well known subclasses of Γ which are univalent inside the unit disk $U = \{|z| : z < 1\}$ and normalized

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with $f(0) = f'(0) - 1 = 0$, and it is denoted by S.

By definition, the class $S^*(\beta)$ is known as starlike of order β and it is geometrically written as

$$S^*(\beta) = Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, 0 \leq \beta < 1 \quad z \in U. \tag{2}$$

The class $K(\beta)$ is known as convex of order β and it is geometrically written as

$$K(\beta) = Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta, 0 \leq \beta < 1 \quad z \in U. \tag{3}$$

Let A denote the class of function of the form

$$f(z)^\alpha = \left(z + \sum_{k=2}^{\infty} a_k z^k\right)^\alpha. \tag{4}$$

Using binomial expansion for (4) we have

$$f(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1}. \tag{5}$$

[12] define an operator as follows:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z)) = z + \sum_{k=2}^{\infty} k^n a_k z^k. \tag{6}$$

Applying (6) on (5) we obtain

$$D^n f(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} \left(\frac{\alpha + k - 1}{\alpha}\right)^n a_k(\alpha) z^{\alpha+k-1}, \tag{7}$$

where $\alpha > 0, n \in N$ and D^n is the Salagean derivative operator.

In 1994, [11] used equation (7) to define a class of $T_n^\alpha(\beta)$ as

$$Re\left(\frac{D^n f(z)^\alpha}{\alpha^n z^n}\right) > \beta, \tag{8}$$

and the first few coefficient bounds for the class $T_n^\alpha(\beta)$ were obtained. The characterization and the inclusion properties for this class were also derived.

The class defined by [11] arose from the discovery of a Russian Mathematician called [13] in 1995. He studied the functions

$$f(z) = \{\frac{\alpha}{1+\varepsilon^2}\int_0^z \frac{p(v)-i\varepsilon}{V\left(1+\frac{i\alpha\varepsilon}{1+\varepsilon^2}\right)} g(v)^{\frac{\alpha}{1+\varepsilon^2}} dv\}^{\frac{1+i\varepsilon}{\alpha}} \tag{9}$$

where $p \in P$ and $g \in S^{*}$. The number $\alpha > 0$ and ε are real and all powers are meant as principal determinant only. The family of functions in (9) became known as Bazilevic functions and is, in this work denoted by $B(\alpha, \varepsilon)$. Except that he, [13] showed that each function $f \in B(\alpha, \varepsilon)$ is univalent in U, very little is known regarding the family as a whole. However, with some simplifications, it may be possible to understand and investigate the family. Indeed it is easy to verify that, with special choices of the parameters α and ε, and the function $g(z)$, the family $B(\alpha, \varepsilon)$ cracks down to some well-known subclasses of univalent functions.

For instance, if we take $\epsilon = 0$, we have

$$f(z) = \{\alpha \int_0^z \frac{p(v)}{v} g(v)^\alpha dv\}^{\frac{1}{\alpha}}. \tag{10}$$

On differentiation, the expression (10) yields

$$\frac{zf'(z)f(z)^{\alpha-1}}{g(z)^\alpha} = p(z), \quad z \in U. \tag{11}$$

Or equivalently

$$Re\frac{zf'(z)f(z)^{\alpha-1}}{g(z)^\alpha} > 0, \quad z \in U. \tag{12}$$

The subclasses of Bazilevic functions satisfying (11) are called Bazilevic functions of type α and are denoted by $B(\alpha)$ (see [9].) In 1973, [8] gave a plausible description of functions of the class $B(\alpha)$ as those functions in S for which for each $r < 1$, the tangent to the curve $U_\alpha(r) = \{\varepsilon f(r \exp^{i\theta})^\alpha, 0 \le \theta < 2\pi\}$ never turns back on itself as much as π radian. If $\alpha = 1$, the class $B(\alpha)$ reduces to the family of close - to - convex functions, that is

$$Re\frac{zf(z)}{g(z)} > 0, \quad z \in U \tag{13}$$

If we decide to choose $g(z) = f(z)$ in (13) we have

$$Re\frac{zf'(z)}{f(z)} > 0, \quad z \in U \tag{14}$$

which implies that $f(z)$ is starlike. Furthermore, if we replace $f(z)$ by $zf'(z)$ in (14) we obtain

$$Re\{1 + \frac{zf''(z)}{f'(z)}\} > 0, \quad z \in U. \tag{15}$$

which shows that $f(z)$ is convex. Moreover, if $g(z) = z$ in (13) then we have the family $B_1(\alpha)$ [4] of functions satisfying

$$Re\frac{zf'(z)f(z)^{\alpha-1}}{z^{\alpha}} > 0, \quad z \in U \tag{16}$$

The various subfamilies of Bazilevic functions are being studied repeatedly by many authors, the literatures in this direction littered everywhere (see [4]).

In 1992, [1] introduced a generalization of functions satisfying (11) as

$$Re\frac{D^n f(z)^{\alpha}}{z^{\alpha}} > 0, \quad z \in U \tag{17}$$

where the parameter α and the operator D^n are as earlier defined. He denoted this class of functions by $B_n(\alpha)$. It is easily seen that his generalization has extraneously included analytic functions satisfying

$$Re\frac{f(z)^{\alpha}}{z^{\alpha}} > 0, \quad z \in U \tag{18}$$

which are largely non-univalent in the unit disk. By proving the inclusion

$$B_{n+1}(\alpha) \subset B_n(\alpha) \tag{19}$$

[1] was able to show that for all $n \in N$, each function of the class $B_n(\alpha)$ is univalent in U.

Notable contributors like [7], [8], [13], [15], [16], [17], [11], had earlier considered various special cases of the parameters n and α of (17) and established many interesting properties of function in those particular cases (see detail [9]).

In some general sense, it is possible to further improve work on function defined by the geometric condition (17).

Let us also define the function $\varphi_{\alpha}(a, c, z)$ by

$$\varphi_\alpha(a,c,z) = z^\alpha + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^{\alpha+k-1},$$

$$z \in U, \alpha \in R^+, a \in R, c \in R = 0,-1,,-2,... \quad (20)$$

where $(a)_k$ is the Pochammer symbol defined by

$$(a)_k = \frac{\gamma(a+k)}{\gamma(a)}.$$

Corresponding to the function $\varphi_\alpha(a,c,z)$, [10] defined a linear operator

$$J_n^\alpha(a,c)f(z)^\alpha = \varphi_\alpha(a,c,z) * D^n f(z)^\alpha, f(z)^\alpha \in A_\alpha.$$

Or equivalently

$$J_n^\alpha(a,c)f(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} \left(\frac{\alpha+k-1}{\alpha}\right)^n z^{\alpha+k-1}. \quad (21)$$

It is well known that every function $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad z \in U,$$

$$f\left(f^{-1}(\omega)\right) = \omega \quad \left(|\omega| < r_0(f); r_0(f) \geq \frac{1}{4}\right), \quad (22)$$

where $f^{-1}(\omega) = \omega - a_2\omega^2 + \left(2a_2^2 - a_3\right)\omega^3 - ....$

A function $f \in \Gamma$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Authors like [14], [5], [2], [6] and some others have worked in different directions of biunivalent functions and very interesting result were obtained.

Then, it is also clear that $J_n^\alpha(a,c)f(z)^\alpha$ has an inverse $J_n^\alpha(a,c)g(\omega)^\alpha$

$$J_n^\alpha(a,c)g\left(J_n^\alpha(a,c)f(z)^\alpha\right) = z^\alpha$$

$$J_n^\alpha(a,c)f\left(J_n^\alpha(a,c)g(\omega)^\alpha\right) = \omega^\alpha$$

where

$$J_n^\alpha(a,c)g(\omega)^\alpha = \omega^\alpha - \alpha\left(\frac{\alpha+1}{\alpha}\right)^n \frac{(a)_1}{(c)_1} a_2(\alpha)\omega^{\alpha+1} + \left(\frac{\alpha(\alpha+1)}{2}a_2^2 - \alpha a_3\right)$$

$$\left(\frac{\alpha+2}{\alpha}\right)^n \frac{(a)_2}{(c)_2}\omega^{\alpha+2}$$
$$+\left(\alpha(\alpha+1)a_2a_3-\alpha a_4-\frac{\alpha(\alpha+1)(\alpha+2)}{6}a_2^3\right)\left(\frac{\alpha+3}{\alpha}\right)^n \frac{(a)_3}{(c)_3}\omega^{\alpha+3}+...$$

The class of function $Q_\gamma(\beta)$ was introduced in 1995 precisely, and it is defined as follows:

$$Q_\lambda(\beta)=\left\{f\in\Gamma : Re\left((1-\gamma)\frac{f(z)}{z}+\gamma f'(z)\right)>\beta\right\}, \tag{23}$$

where $0\le\beta<1, \lambda\ge 0$. It is easily seen that $Q_{\lambda_1}(\beta)\subset Q_{\lambda_2}(\beta)$ for $\lambda_1>\lambda_2\ge 0$. Thus, for $\lambda\ge 1, 0\le\beta<1, Q_\lambda(\beta)\subset Q_1(\beta)=\{f\in\Gamma : Ref'(z)>\beta, 0\le\beta<1\}$ and hence $Q_{\lambda_1}(\beta)$ is univalent class [see [2]].

Also, a function $f\in\Gamma$ is in the class $S_\sigma^{'}(\alpha)$ of strongly bi-starlike function of order $\alpha(0<\alpha\le 1)$ if each of the following conditions is satisfied

$$f\in\sigma, \left|arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2}\,(0<\alpha\le 1, z\in U),$$

$$f\in\sigma, \left|arg\left(\frac{zg'(\omega)}{g(\omega)}\right)\right|<\frac{\alpha\pi}{2}\,(0<\alpha\le 1, z\in U), \tag{24}$$

where g is the extension of f^{-1} of U. The classes of bi-starlike functions and bi-convex functions were introduced and the first few coefficient bounds were obtained using taylor Maclaurin (see details [2]).

In this present work, the author introduce two subclasses of the bi-univalent functions defined by linear and Salagean operator and the first few coefficient bounds were obtained by employing the technique used by [2].

For the purpose of our results the follwing lemma and definitions shall be necessary.

Lemma. If $h\in P$, then $|c_k|\le 2$ for each K, where P is the family of all functions h analytic in U for which $Reh(z)>0, h(z)=1+c_1z+c^2z^2+...$ for $z\in U$.

Definition 1. Coefficient Bounds of the function Class $T_n^{\Sigma,\alpha}(a,c,\gamma,\beta)$ if it satisfies the following condition:

$$\left|arg\left(\frac{J_n^\alpha(a,c)f(z)^\alpha}{z^\alpha}\right)\right|<\frac{\beta\pi}{2},$$

$$\left|arg\left(\frac{J_n^\alpha(a,c)g(\omega)^\alpha}{\omega^\alpha}\right)\right|<\frac{\beta\pi}{2}, \tag{25}$$

where $\gamma \geq 1, \alpha > 0, 0 < \beta \leq 1$ and $J_n^{\alpha}(a,c)g(\omega)^{\alpha}$ as defined in (21).

Definition 2. Coefficient Bounds of the function Class $T_n^{\Sigma,\alpha}(a,c,\gamma,\rho)$, if it satisfies the following condition:

$$Re\left\{\left(\frac{J_n^{\alpha}(a,c)f(z)^{\alpha}}{z^{\alpha}}\right)\right\} > \rho,$$

$$Re\left\{\left(\frac{J_n^{\alpha}(a,c)g(\omega)^{\alpha}}{\omega^{\alpha}}\right)\right\} > \rho, \tag{26}$$

where $\gamma \geq 1, \alpha > 0, 0 \leq \rho < 1$ and $J_n^{\alpha}(a,c)g(\omega)^{\alpha}$ as defined in (21).

2. Main Result

Theorem 3. *Letting $J_n^{\alpha}(a,c)f(z)^{\alpha}$ given by (21) in the class $T_n^{\Sigma,\alpha}(a,c,\gamma,\beta)$, where $\gamma \geq 1, \alpha > 0, 0 < \beta \leq 1, n \in N$ then*

$$|a_2| \leq \frac{2\beta}{\left|\alpha\sqrt{\beta\left(\frac{\alpha+2}{\alpha}\right)^n \frac{(a)_2}{(c)_2} - (\beta-1)\left(\frac{\alpha+1}{\alpha}\right)^{2n}\frac{(a)_1^2}{(c)_1^2}}\right|}, \tag{27}$$

$$|a_3| \leq \frac{2\beta}{\left|\alpha\left(\frac{\alpha+2}{\alpha}\right)^n \frac{(a)_2}{(c)_2}\right|} + \frac{2\beta^2}{\left|\alpha^2\left(\frac{\alpha+1}{\alpha}\right)^{2n}\frac{(a)_1^2}{(c)_1^2}\right|}. \tag{28}$$

It follows from (25) that

$$\left(\frac{J_n^{\alpha}(a,c)f(z)^{\alpha}}{z^{\alpha}}\right) = [p(z)]^2,$$

$$\left(\frac{J_n^{\alpha}(a,c)g(\omega)^{\alpha}}{\omega^{\alpha}}\right) = [q(z)]^2,$$

where $p(z)$ and $q(z)$ in P have the form

$$1 + p_1 z + p^2 z^2 + p^3 z^3 + ...$$

$$1 + q_1 z + q^2 z^2 + q^3 z^3 + ... \tag{29}$$

Now, comparing the coefficient

$$\alpha\frac{(a)_1}{(c)_1}\left(\frac{\alpha+1}{\alpha}\right)^n a_2=\beta p_1 \tag{30}$$

$$\left(\alpha a_3+\frac{\alpha(\alpha-1)}{2}a_2^2\right)\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n=\beta p_2+\frac{\beta(\beta-1)}{2}p_1^2 \tag{31}$$

$$-\alpha\frac{(a)_1}{(c)_1}\left(\frac{\alpha+1}{\alpha}\right)^n a_2=\beta q_1 \tag{32}$$

$$\left(\frac{\alpha(\alpha+1)}{2}a_2^2-\alpha a_3\right)\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n=\beta q_2+\frac{\beta(\beta-1)}{2}q_1^2. \tag{33}$$

From (30) and (32), we get

$$p_1=-q_1 \tag{34}$$

$$2\alpha^2\frac{(a)_1^2}{(c)_1^2}\left(\frac{\alpha+1}{\alpha}\right)^{2n}a_2^2=\beta^2\left(p_1^2+q_1^2\right). \tag{35}$$

Now from (31),(33) and (35), we obtain

$$\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n\alpha^2a_2^2=\beta\left(p_2+q_2\right)+\frac{(\beta-1)}{\beta}\frac{(a)_1^2}{(c)_1^2}\left(\frac{\alpha+1}{\alpha}\right)^{2n}\alpha^2a_2^2. \tag{36}$$

Therefore, we have

$$a_2^2=\frac{\beta^2\left(p_2+q_2\right)}{\alpha^2\left[\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n\beta-(\beta-1)\frac{(a)_1^2}{(c)_1^2}\left(\frac{\alpha+1}{\alpha}\right)^{2n}\right]}. \tag{37}$$

Applying Lemma for the coefficients p_2 and q_2, we have

$$|a_2|\le\frac{2\beta}{\left|\alpha\sqrt{\beta\left(\frac{\alpha+2}{\alpha}\right)^n\frac{(a)_2}{(c)_2}-(\beta-1)\left(\frac{\alpha+1}{\alpha}\right)^{2n}\frac{(a)_1^2}{(c)_1^2}}\right|}. \tag{38}$$

Next, in order to find the bound on $|a_3|$, by subtracting (33) from (31) and using (35), we get

$$2\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n\alpha a_3-\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n\alpha a_2^2=\beta\left(p_2-q_2\right). \tag{39}$$

It follows from (35) and (39) that

$$2\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n \alpha a_3 = \frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n \alpha a_2^2 + \beta\left(p_2 - q_2\right). \tag{40}$$

And then

$$a_3 = \frac{\beta^2(p_1^2+q_1^2)}{4\alpha^2 \frac{(a)_1^2}{(c)_1^2}\left(\frac{\alpha+1}{\alpha}\right)^{2n}} + \frac{\beta\left(p_2-q_2\right)}{2\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n \alpha}. \tag{41}$$

Applying Lemma 1 once again for the coefficients p_1, p_2, q_1 and q_2, we readily have

$$|a_3| \leq \frac{2\beta}{\left|\alpha\left(\frac{\alpha+2}{\alpha}\right)^n \frac{(a)_2}{(c)_2}\right|} + \frac{2\beta^2}{\left|\alpha^2\left(\frac{\alpha+1}{\alpha}\right)^{2n} \frac{(a)_1^2}{(c)_1^2}\right|}. \tag{42}$$

Theorem 4. *: Letting $J_n^\alpha(a,c)f(z)^\alpha$ given by (21) in the class $T_n^{\Sigma,\alpha}(a,c,\gamma,\rho)$, where $\gamma \geq 1, \alpha > 0, 0 < \rho \leq 1, n \in N$ then,*

$$|a_2| \leq \frac{2\sqrt{1-\rho}}{\left|\alpha\sqrt{\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n}\right|}, \tag{43}$$

$$|a_3| \leq \frac{2(1-\rho)^2}{\left|\alpha^2 \frac{(a)_1^2}{(c)_1^2}\left(\frac{\alpha+1}{\alpha}\right)^{2n}\right|} + \frac{2(1-\rho)}{\left|\alpha\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n\right|}. \tag{44}$$

It follows from (26) that

$$\left(\frac{J_n^\alpha(a,c)f(z)^\alpha}{z^\alpha}\right) = \rho + (1-\rho)p(z),$$

$$\left(\frac{J_n^\alpha(a,c)g(\omega)^\alpha}{\omega^\alpha}\right) = \rho + (1-\rho)q(z), \tag{45}$$

where $p(z)$ and $q(z)$ in P have the form (29) respectively.

As in the proof Theorem 1, by suitably comparing coefficients in (45), we get

$$\alpha\frac{(a)_1}{(c)_1}\left(\frac{\alpha+1}{\alpha}\right)^n a_2 = (1-\rho)p_1 \tag{46}$$

$$\left(\alpha a_3 + \frac{\alpha(\alpha-1)}{2}a_2^2\right)\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n = (1-\rho)p_2 \tag{47}$$

$$-\alpha\frac{(a)_1}{(c)_1}\left(\frac{\alpha+1}{\alpha}\right)^n a_2 = (1-\rho)q_1 \tag{48}$$

$$\left(\frac{\alpha(\alpha+1)}{2}a_2^2 - \alpha a_3\right)\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n = (1-\rho)q_2. \tag{49}$$

From (46) and (48), we get

$$p_1 = -q_1 \tag{50}$$

$$2\alpha^2\frac{(a)_1^2}{(c)_1^2}\left(\frac{\alpha+1}{\alpha}\right)^{2n} a_2^2 = (1-\rho)^2\left(p_1^2+q_1^2\right). \tag{51}$$

Also, from (47) and (49) we find that

$$\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n \alpha^2 a_2^2 = (1-\rho)\left(p_2+q_2\right). \tag{52}$$

Therefore, we have

$$|a_2^2| = \frac{(1-\rho)}{\left|\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n\alpha^2\right|}\left(|p_2|+|q_2|\right). \tag{53}$$

Applying Lemma 1 for the coefficient p_2 and q_2, we immediately have

$$|a_2| \leq \frac{2\sqrt{1-\rho}}{\left|\alpha\sqrt{\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n}\right|}. \tag{54}$$

Next, in order to find the bound on $|a_3|$, by subtracting (49) from (47) and using (51), we get

$$2\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n \alpha a_3 - \frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n \alpha a_2^2 = (1-\rho)\left(p_2-q_2\right). \tag{55}$$

Or equivalently,

$$a_3 = \frac{a_2^2}{2} + \frac{(1-\rho)\left(p_2-q_2\right)}{2\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^n\alpha}, \tag{56}$$

and, then from (51), we find that

$$a_3 = \frac{(1-\rho)^2\left(p_1^2+q_1^2\right)}{4\alpha^2\frac{(a)_1^2}{(c)_1^2}\left(\frac{\alpha+1}{\alpha}\right)^{2n}} + \frac{(1-\rho)\left(p_2-q_2\right)}{2\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^{n}\alpha}. \tag{57}$$

Applying Lemma 1 once again for the coefficients p_1, p_2, q_1 and q_2, we readily have

$$|a_3| \leq \frac{2(1-\rho)^2}{\left|\alpha^2\frac{(a)_1^2}{(c)_1^2}\left(\frac{\alpha+1}{\alpha}\right)^{2n}\right|} + \frac{2(1-\rho)}{\left|\alpha\frac{(a)_2}{(c)_2}\left(\frac{\alpha+2}{\alpha}\right)^{n}\right|}. \tag{58}$$

This commplete the proof.

By varying various choices of parameter involved, many corollaries will be obtained.

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