STARLIKE AND UNIFORMLY CONVEX FUNCTIONS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION

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Abstract: In this paper, we obtained some conditions on the parameters of generalized hypergeometric function and also deals with mapping properties of various subclasses of starlike and uniformly convex functions defined through a generalized hypergeometric function.

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1. Introduction

Let $\mathcal{A}$ denotes the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $\mathbb{U} = \{z : |z| < 1\}$. As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in $\mathbb{U}$. Denote by $\mathcal{T}$ \textsuperscript{[13]} the subclass of $\mathcal{A}$ consisting

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of functions of the form
\[ f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad n = 2, 3, \ldots \]  \hspace{1cm} (2)

Also, for functions \( f \in A \) given by (1) and \( g \in A \) given by \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \),
we define the Hadamard product (or convolution) of \( f \) and \( g \) by
\[ (f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \]  \hspace{1cm} (3)

A function \( f \in A \) is said to be starlike of order \( \alpha \) (\( 0 \leq \alpha < 1 \)), if and only if \( \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \) (\( z \in \mathbb{U} \)). This function class is denoted by \( S^*(\alpha) \). We also write \( S^*(0) \equiv S^* \), where \( S^* \) denotes the class of functions \( f \in A \) that \( f(\mathbb{U}) \) is starlike with respect to the origin. A function \( f \in A \) is said to be convex of order \( \alpha \) (\( 0 \leq \alpha < 1 \)) if and only if \( \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \) (\( z \in \mathbb{U} \)). This class is denoted by \( K(\alpha) \). Further, \( K(0) = K \), the well-known standard class of convex functions. It is an established fact that \( f \in K(\alpha) \iff zf \in S^*(\alpha) \).

We recall the definitions of the class of uniformly convex functions denoted by \( UCV \), introduced by Goodman [5, 6], studied extensively by Rønning [11, 12], and independently by Ma and Minda [7].

**Definition 1.1.** A function \( f \) of the form (1) is said to be uniformly convex in \( \mathbb{U} \), if and only if
\[ \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq \left| \frac{zf''(z)}{f'(z)} - 1 \right|, \quad (z \in \mathbb{U}). \]

Further the class \( \alpha - S_P \) related to the class \( \alpha - UCV \) by means of the well-known Alexander equivalence between the usual classes of convex \( K \) and starlike \( S^* \) functions are defined and the analytic criterion for functions in these classes are given as below:

**Definition 1.2.** A function \( f \) of the form (1) is said to be in the class \( S_p(\alpha) \) if
\[ \Re \left( \frac{zf'(z)}{f(z)} \right) \geq \alpha \left| \frac{zf''(z)}{f'(z)} - 1 \right|, \quad (z \in \mathbb{U}) \]
and \( f \in UCV(\alpha), \alpha \geq 0 \) if
\[ \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq \alpha \left| \frac{zf''(z)}{f'(z)} \right|, \quad (z \in \mathbb{U}) \]
Also, $\mathcal{UCT}(\alpha) = \mathcal{UCV}(\alpha) \cap \mathcal{T}$ and $\mathcal{T}S_p(\alpha) = S_p(\alpha) \cap \mathcal{T}$. It is of interest to note that $\mathcal{UCV}(1) \equiv \mathcal{UCV}$ and $\equiv \mathcal{UCV}(0) \equiv \mathcal{K}$; $S_p(0) \equiv S^*$ and $S_p(1) \equiv S_p$.

(see [16]) we recall the following results due to Subramanian et al., [16].

**Lemma 1.3.** A function $f(z)$ of the form (2) is in the class $\mathcal{T}S_p(\alpha)$ if and only if

$$
\sum_{n=2}^{n=\infty} [(\alpha + 1)n - \alpha] a_n \leq 1
$$

and

(ii) $\mathcal{UCT}(\alpha)$ if and only if

$$
\sum_{n=2}^{n=\infty} n [(\alpha + 1)n - \alpha] a_n \leq 1.
$$

For, $a_1, a_2, ..., a_p$ and $b_1, b_2, ..., b_q$ be complex numbers with $b_j \neq 0, -1, ..., j = 1, 2, ..., q$, then the generalized hypergeometric functions $\mathcal{pFq}(z)$ is defined by

$$
\mathcal{pFq}(z) = \mathcal{pFq}(a_1, a_2, ..., a_p; b_1, b_2, ..., b_q, z) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n...(a_p)_n z^n}{(b_1)_n(b_2)_n...(b_p)_n(1)_n},
$$

(4) $p \leq q + 1$ where $(\lambda)_n$ is the Pochhammer symbol defined by

$$
(\lambda)_n = \begin{cases} 
1 & \text{if } n = 0 \\
\lambda(\lambda + 1) \ldots (\lambda + n - 1) & \text{if } n = 1, 2, \ldots.
\end{cases}
$$

(5)

We note that the function in the series in (4) converges absolutely in the entire complex plane for $p < q + 1$, and for $p = q + 1$ in the unit disc, the condition $p \leq q + 1$ stated with the definition (3) will be hold true throughout this paper. We note that the function $\mathcal{pFq}(1)$ converges whenever $\Re \left( \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_j \right) > 0$.

Merkes and Scott [8] used continued fractions to find sufficient conditions for the function $z_2F_1(z)$ to be in the class $S^*(\alpha), 0 \leq \alpha < 1$. Silverman [14] and Dixit and Pathak [2] determined sufficient conditions for the function $z_2F_1(z)$ to be in the class $S^*(\alpha)$ and $\mathcal{UCV}(\alpha)$ respectively and various geometric properties of this function was discussed in [3, 4, 9, 14, 15]. In this paper, we obtain sufficient condition for function $h(z)$, given by

$$
h(z) = (1 - \mu) \tau(z) + \mu z \tau'(z)
$$

(6) where $\mu \geq 0$ and $\tau(z) = \mathcal{pFq}(z)$ belonging to the classes $S_p(\alpha)$ and $\mathcal{UCT}(\alpha)$. 
2. Main Results

**Theorem 2.1.** If \(a_i > 0, (i = 1, 2, \ldots, p)\) and \(\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 2\), then a sufficient condition for the function \(h(z)\) to be in the class \(T S_p(\alpha), 0 \leq \alpha < 1\), is that

\[
\mu(\alpha + 1) \left(\frac{(a_1)^2(a_2)\ldots(a_p)^2}{(b_1)^2(b_2)\ldots(b_p)^2}\right) \, _pF_q(a_1 + 2; b_1 + 2, 1) \\
+ (\mu(\alpha + 2) + \alpha + 1) \left(\frac{(a_1)(a_2)\ldots(a_p)}{(b_1)(b_2)\ldots(b_p)}\right) \, _pF_q(a_1 + 1; b_1 + 1, 1) + \, _pF_q(1) \leq 2.
\]

(7)

The condition (7) is necessary and sufficient for the function \(h_1(z)\), defined by \(h_1(z) = z \left(2 - \frac{h(z)}{z}\right)\) to be in the class \(T S_p(\alpha)\).

**Proof.** Since

\[
h(z) = z + \sum_{n=2}^{\infty} (1 - \mu + \mu n) \frac{(a_1)_{n-1}(a_2)_{n-1}\ldots(a_p)_{n-1} z^n}{(b_1)_{n-1}(b_2)_{n-1}\ldots(b_p)_{n-1}(1)_{n-1}}.
\]

According to Lemma 1.3 we need only to show that

\[
\sum_{n=2}^{\infty} [n(\alpha + 1) - \alpha] (1 - \mu + \mu n) \frac{(a_1)_{n-1}(a_2)_{n-1}\ldots(a_p)_{n-1}}{(b_1)_{n-1}(b_2)_{n-1}\ldots(b_p)_{n-1}(1)_{n-1}} \leq 1.
\]

(8)

Now from (8)

\[
\sum_{n=1}^{\infty} [(n + 1)(\alpha + 1) - \alpha] (1 - \mu + \mu(n + 1)) \frac{(a_1)_{n}(a_2)_{n}\ldots(a_p)_{n}}{(b_1)_{n}(b_2)_{n}\ldots(b_p)_{n}(1)_{n}} \\
= \sum_{n=1}^{\infty} [\mu(\alpha + 1)n^2 + (\mu + \alpha + 1)n + 1] \frac{(a_1)_{n}(a_2)_{n}\ldots(a_p)_{n}}{(b_1)_{n}(b_2)_{n}\ldots(b_p)_{n}(1)_{n}}.
\]
Using the fact that \((\lambda)_n = \lambda(\lambda + 1)_n^{-1}\) yields

\[
= \mu(\alpha + 1) \sum_{n=1}^{\infty} \frac{(a_1n_1)(a_2n_2)...(a_pn_p)}{(b_1n_1)(b_2n_2)...(b_pn_p)(1)n^{-2}}
\]

\[
+ (\mu \alpha + 2\mu + \alpha + 1) \sum_{n=1}^{\infty} \frac{(a_1n_1)(a_2n_2)...(a_pn_p)}{(b_1n_1)(b_2n_2)...(b_pn_p)(1)n^{-1}}
\]

\[
+ \sum_{n=1}^{\infty} \frac{(a_1n_1)(a_2n_2)...(a_pn_p)}{(b_1n_1)(b_2n_2)...(b_pn_p)(1)n}
\]

\[
= \mu(\alpha + 1) \left[ \frac{(a_12)(a_22)...(a_p2)}{(b_12)(b_22)...(b_p2)} \right] pF_q(a_1 + 2; b_1 + 2, 1)
\]

\[
+ (\mu(\alpha + 2) + \alpha + 1) \left[ \frac{(a_1)(a_2)...(a_p)}{(b_1)(b_2)...(b_p)} \right] pF_q(a_1 + 1; b_1 + 1, 1) + p F_q(1) - 1.
\]

But this last expression is bounded by 1 if and only if \((7)\) holds.

Since

\[
h_1(z) = z - \sum_{n=2}^{\infty} (1 - \mu + \mu n) \frac{(a_1n_1-1)(a_2n_1-1)...(a_pn_1-1)z^n}{(b_1n_1-1)(b_2n_1-1)...(b_pn_1-1)(1)n^{-1}}.
\]

\[
(9)
\]

The condition \((7)\) is also necessary for \(h_1(z)\) to be in the class \(\mathcal{T}S_p(\alpha), 0 \leq \alpha < 1\) from the Lemma 1.3

**Theorem 2.2.** If \(a_i > -1, (i = 1, 2, ..., p), b_j > 0, (j = 1, 2, ..., q), \prod_{i=1}^{n} a_i < 0\)

and \(\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 2\), then a necessary and sufficient condition for the function \(h(z)\) to be in the class \(\mathcal{T}S_p(\alpha), 0 \leq \alpha < 1\), is that

\[
\mu(\alpha + 1) \left[ \frac{(a_1+1)(a_2+1)...(a_p+1)}{(b_1+1)(b_2+1)...(b_p+1)} \right] pF_q(a_1 + 2; b_1 + 2, 1)
\]

\[
+ (\mu(\alpha + 2) + \alpha + 1) pF_q(a_1 + 1; b_1 + 1, 1) + \frac{b_1b_2...b_q}{a_1a_2...a_p} pF_q(1) \geq 0.
\]

The condition \(\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 1 - \prod_{i=1}^{n} a_i\) is necessary and sufficient for the function \(h(z)\) to be in the class \(\mathcal{T}S_0 = S^*\), a class of starlike function with negative coefficients.
Proof. Since,

\[ h(z) = z - \left| \frac{a_1a_2...a_p}{b_1b_2...b_q} \right| \sum_{n=2}^{\infty} (1 - \mu + \mu n) \frac{(a_1 + 1)_{n-2}(a_2 + 1)_{n-2}...(a_p + 1)_{n-2}z^n}{(b_1 + 1)_{n-2}(b_2 + 1)_{n-2}...(b_p + 1)_{n-2}(1)_{n-1}}. \]  \tag{11}

If we show the inequality

\[ \sum_{n=2}^{\infty} [n(\alpha + 1) - \alpha] (1 - \mu + \mu n) \frac{(a_1 + 1)_{n-2}(a_2 + 1)_{n-2}...(a_p + 1)_{n-2}}{(b_1 + 1)_{n-2}(b_2 + 1)_{n-2}...(b_p + 1)_{n-2}(1)_{n-1}} \leq \left| \frac{b_1b_2...b_q}{a_1a_2...a_p} \right| \]  \tag{12}

then by Lemma 1.3, Theorem 2.2 is obtained. The left hand side of the equation (12) converges if \( \sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 2. \) From (12)

\[ \sum_{n=0}^{\infty} [(n + 2)(\alpha + 1) - \alpha] [1 - \mu + \mu(n + 2)] \frac{(a_1 + 1)_n(a_2 + 1)_n...(a_p + 1)_n}{(b_1 + 1)_n(b_2 + 1)_n...(b_p + 1)_n(1)_{n+1}} \]

\[ = \mu(\alpha + 1) \left[ \frac{(a_1 + 1)(a_2 + 1)...(a_p + 1)}{(b_1 + 1)(b_2 + 1)...(b_p + 1)} \right] \sum_{n=0}^{\infty} \frac{(a_1 + 2)_n(a_2 + 2)_n...(a_p + 2)_n}{(b_1 + 2)_n(b_2 + 2)_n...(b_p + 2)_n(1)_n} \]

\[ + (\mu(\alpha + 2) + \alpha + 1) \sum_{n=0}^{\infty} \frac{(a_1 + 1)_n(a_2 + 1)_n...(a_p + 1)_n}{(b_1 + 1)_n(b_2 + 1)_n...(b_p + 1)_n(1)_n} \]

\[ + \sum_{n=1}^{\infty} \frac{(a_1 + 1)_{n-1}(a_2 + 1)_{n-1}...(a_p + 1)_{n-1}}{(b_1 + 1)_{n-1}(b_2 + 1)_{n-1}...(b_p + 1)_{n-1}(1)_{n-1}} \]

\[ = \mu(\alpha + 1) \left[ \frac{(a_1 + 1)(a_2 + 1)...(a_p + 1)}{(b_1 + 1)(b_2 + 1)...(b_p + 1)} \right] _{pF_q}(a_1 + 2; b_1 + 2, 1) \]

\[ + (\mu(\alpha + 2) + \alpha + 1) _{pF_q}(a_1 + 1; b_1 + 1, 1) + \frac{b_1b_2...b_q}{a_1a_2...a_p} \{ _{pF_q}(1) - 1 \}. \]
Hence (12) is equivalent to
\[
\mu(\alpha + 1) \left[ \frac{(a_1 + 1)(a_2 + 1)...(a_p + 1)}{(b_1 + 1)(b_2 + 1)...(b_p + 1)} \right] pF_q(a_1 + 2; b_1 + 2, 1) \\
+ (\mu(\alpha + 2) + \alpha + 1) pF_q(a_1 + 1; b_1 + 1, 1) \\
+ \frac{b_1 b_2...b_q}{a_1 a_2...a_p} pF_q(1) \leq \left| \frac{b_1 b_2...b_q}{a_1 a_2...a_p} \right| + \frac{b_1 b_2...b_q}{a_1 a_2...a_p} = 0. \tag{13}
\]
Thus (13) is valid if and only if
\[
\mu(\alpha + 1) \left[ \frac{(a_1 + 1)(a_2 + 1)...(a_p + 1)}{(b_1 + 1)(b_2 + 1)...(b_p + 1)} \right] pF_q(a_1 + 2; b_1 + 2, 1) \\
+ (\mu(\alpha + 2) + \alpha + 1) pF_q(a_1 + 1; b_1 + 1, 1) + \frac{b_1 b_2...b_q}{a_1 a_2...a_p} pF_q(1) \geq 0.
\]
This is equivalent to (10). This completes the proof of the Theorem 2.2. \hfill \square

**Theorem 2.3.** If \( a_i > 0, i = 1, 2, ..., p, \) and \( \sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 3, \) then a sufficient condition for the function \( h(z) \) to be in the class \( \mathcal{UCT}(\alpha), 0 \leq \alpha < 1, \) is that
\[
\mu(\alpha + 1) \left[ \frac{(a_1 + 1)(a_2 + 1)...(a_p + 1)}{(b_1 + 1)(b_2 + 1)...(b_p + 1)} \right] pF_q(a_1 + 3; b_1 + 3, 1) \\
+ (4\mu\alpha + 5\mu + \alpha + 1) \left[ \frac{(a_1 + 1)(a_2 + 1)...(a_p + 1)}{(b_1 + 1)(b_2 + 1)...(b_p + 1)} \right] pF_q(a_1 + 2; b_1 + 2, 1) \\
+ (2\mu\alpha + 4\mu + \alpha + 3) \frac{a_1 a_2...a_p}{b_1 b_2...b_q} pF_q(a_1 + 1; b_1 + 1, 1) + pF_q(1) \leq 2. \tag{14}
\]
The condition (14) is a necessary and sufficient condition for the function \( h_1(z) \) defined by (9) to be in the class \( \mathcal{UCT}(\alpha). \)

**Proof.** Taking \( h(z) \) defined by (6), in view of (ii) of Lemma 1.3, we need only to show that
\[
\sum_{n=2}^{\infty} n [n(\alpha + 1) - \alpha] (1 - \mu + \mu n) \frac{(a_1 + 1)(a_2 + 1)...(a_p + 1)}{(b_1 + 1)(b_2 + 1)...(b_p + 1)} \leq 1. \tag{15}
\]
The left hand side of (15) converges if \( \sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 3\). From (15)

\[
\sum_{n=0}^{\infty} (n + 2)[(n + 2)(\alpha + 1) - \alpha] (1 - \mu + \mu(n + 2)) \frac{(a_1)_{n+1}(a_2)_{n+1} \ldots (a_p)_{n+1}}{(b_1)_{n+1}(b_2)_{n+1} \ldots (b_p)_{n+1}(1)_{n+1}}
\]

\[
= \mu(\alpha + 1) \sum_{n=0}^{\infty} \frac{(a_1)_{n+3}(a_2)_{n+3} \ldots (a_p)_{n+3}}{(b_1)_{n+3}(b_2)_{n+3} \ldots (b_p)_{n+3}(1)_{n+1}}
\]

\[
+ (4\mu\alpha + 5\mu + \alpha + 1) \sum_{n=0}^{\infty} \frac{(a_1)_{n+2}(a_2)_{n+2} \ldots (a_p)_{n+2}}{(b_1)_{n+2}(b_2)_{n+2} \ldots (b_p)_{n+2}(1)_{n+1}}
\]

\[
+ (2\mu\alpha + 4\mu + \alpha + 3) \sum_{n=0}^{\infty} \frac{(a_1)_{n+1}(a_2)_{n+1} \ldots (a_p)_{n+1}}{(b_1)_{n+1}(b_2)_{n+1} \ldots (b_p)_{n+1}(1)_{n+1}} + \sum_{n=1}^{\infty} \frac{(a_1)_{n}(a_2)_{n} \ldots (a_p)_{n}}{(b_1)_{n}(b_2)_{n} \ldots (b_p)_{n}(1)_{n}}.
\]

Since \((a)_{n+k} = (a)_k(a+k)_n\), we may write (16) as

\[
= \mu(\alpha + 1) \left[ \frac{(a_1)_3(a_2)_3 \ldots (a_p)_3}{(b_1)_3(b_2)_3 \ldots (b_q)_3} \right] \sum_{n=0}^{\infty} \frac{(a_1 + 3)_n(a_2 + 3)_n \ldots (a_p + 3)_n}{(b_1 + 3)_n(b_2 + 3)_n \ldots (b_p + 3)_n(1)_n}
\]

\[
+ (4\mu\alpha + 5\mu + \alpha + 1) \left[ \frac{(a_1)_2(a_2)_2 \ldots (a_p)_2}{(b_1)_2(b_2)_2 \ldots (b_q)_2} \right] \sum_{n=0}^{\infty} \frac{(a_1 + 2)_n(a_2 + 2)_n \ldots (a_p + 2)_n}{(b_1 + 2)_n(b_2 + 2)_n \ldots (b_p + 2)_n(1)_n}
\]

\[
+ (2\mu\alpha + 4\mu + \alpha + 3) \frac{a_1a_2 \ldots a_p}{b_1b_2 \ldots b_q} \sum_{n=0}^{\infty} \frac{(a_1 + 1)_n(a_2 + 1)_n \ldots (a_p + 1)_n}{(b_1 + 1)_n(b_2 + 1)_n \ldots (b_p + 1)_n(1)_n}
\]

\[
+ \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \ldots (a_p)_n}{(b_1)_n(b_2)_n \ldots (b_p)_n(1)_n} - 1.
\]

The last expression is bounded by 1 if and only if (14) holds, hence the proof of the Theorem 2.3.

\[\square\]

**Theorem 2.4.** If \(a_i > -1, (i = 1, 2, \ldots, p)\), \(\prod_{i=1}^{n} a_i < 0\) and \(\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 3\), then a necessary and sufficient condition for the function \(h(z)\) to be in the class...
\( \mathcal{UCT}(\alpha), 0 \leq \alpha < 1, \) is that

\[
\mu(\alpha + 1) \left[ \frac{(a_1 + 1)(a_2 + 1)\ldots(a_p + 1)}{(b_1 + 1)(b_2 + 1)\ldots(b_p + 1)} \right] pF_q(a_1 + 3; b_1 + 3, 1) \\
+ (4\mu \alpha + 5\mu + \alpha + 1) \left[ \frac{(a_1 + 1)(a_2 + 1)\ldots(a_p + 1)}{(b_1 + 1)(b_2 + 1)(b_p + 1)} \right] pF_q(a_1 + 2; b_1 + 2, 1) \\
+ (2\mu \alpha + 4\mu + \alpha + 3) pF_q(a_1 + 1; b_1 + 1, 1) + \left[ \frac{b_1 b_2 \ldots b_q}{a_1 a_2 \ldots a_p} \right] pF_q(1) \geq 0.
\]

(17)

**Proof.** Taking \( h(z) \) defined by (11), in view of (ii) of Lemma 1.3, we must show that

\[
\sum_{n=2}^{\infty} n [n(\alpha + 1) - \alpha] (1 - \mu + \mu n) \frac{(a_1 + 1)(a_2 + 1)\ldots(a_p + 1)}{(b_1 + 1)(b_2 + 1)\ldots(b_p + 1)} \geq \left| \frac{b_1 b_2 \ldots b_q}{a_1 a_2 \ldots a_p} \right|.
\]

(18)

The left side of (18) converges if \( \sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 3 \). Now from (18)

\[
\sum_{n=0}^{\infty} (n + 2) [(n + 2)(\alpha + 1) - \alpha] (1 - \mu + \mu(n + 2)) \frac{(a_1 + 1)(a_2 + 1)\ldots(a_p + 1)}{(b_1 + 1)(b_2 + 1)\ldots(b_p + 1)} \\
= \mu(\alpha + 1) \sum_{n=0}^{\infty} \frac{(a_1 + 1)(a_2 + 1)\ldots(a_p + 1)}{(b_1 + 1)(b_2 + 1)\ldots(b_p + 1)} \\
+ (4\mu \alpha + 5\mu + \alpha + 1) \sum_{n=0}^{\infty} \frac{(a_1 + 1)(a_2 + 1)\ldots(a_p + 1)}{(b_1 + 1)(b_2 + 1)\ldots(b_p + 1)} \\
+ (2\mu \alpha + 4\mu + \alpha + 3) \sum_{n=0}^{\infty} \frac{(a_1 + 1)(a_2 + 1)\ldots(a_p + 1)}{(b_1 + 1)(b_2 + 1)\ldots(b_p + 1)} \\
+ \left[ \frac{b_1 b_2 \ldots b_q}{a_1 a_2 \ldots a_p} \right] \sum_{n=0}^{\infty} \frac{(a_1 + 1)(a_2 + 1)\ldots(a_p + 1)}{(b_1 + 1)(b_2 + 1)\ldots(b_p + 1)}
\]
\begin{align*}
&= \mu(\alpha + 1) \left[ \frac{(a_1 + 1)(a_2 + 1)...(a_p + 1)}{(b_1 + 1)(b_2 + 1)...(b_p + 1)} \right] \sum_{n=0}^{\infty} \frac{(a_1 + 3)_n(a_2 + 3)_n...(a_p + 3)_n}{(b_1 + 3)_n(b_2 + 3)_n...(b_p + 3)_n(1)_n} \\
&+ (4\mu\alpha + 5\mu + \alpha + 1) \left[ \frac{(a_1 + 1)(a_2 + 1)...(a_p + 1)}{(b_1 + 1)(b_2 + 1)...(b_p + 1)} \right] \sum_{n=0}^{\infty} \frac{(a_1 + 2)_n(a_2 + 2)_n...(a_p + 2)_n}{(b_1 + 2)_n(b_2 + 2)_n...(b_p + 2)_n(1)_n} \\
&+ (2\mu\alpha + 4\mu + \alpha + 3) \sum_{n=0}^{\infty} \frac{(a_1 + 1)_n(a_2 + 1)_n...(a_p + 1)_n}{(b_1 + 1)_n(b_2 + 1)_n...(b_p + 1)_n(1)_n} \\
&+ \left[ \frac{b_1 b_2...b_q}{a_1 a_2...a_p} \right] \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n...(a_p)_n}{(b_1)_n(b_2)_n...(b_p)_n(1)_n} - \left[ \frac{b_1 b_2...b_q}{a_1 a_2...a_p} \right].
\end{align*}

This last expression is bounded by \( \left| \frac{b_1 b_2...b_q}{a_1 a_2...a_p} \right| \) if and only if (17) holds. Thus the proof of the Theorem 2.4 is completed.

\textbf{Concluding Remak:} If \( p = 2, q = 1 \), the results reduces to Dixit and Pathak[4] result.

\section*{References}


