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STARLIKE AND UNIFORMLY CONVEX FUNCTIONS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION

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Abstract: In this paper, we obtained some conditions on the parameters of generalized hypergeometric function and also deals with mapping properties of various subclasses of starlike and uniformly convex functions defined through a generalized hypergeometric function.

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1. Introduction

Let $\mathcal{A}$ denotes the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are *analytic* in the unit disk $\mathbb{U} = \{z : |z| < 1\}$.As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in $\mathbb{U}$. Denote by $\mathcal{T}$ [13] the subclass of $\mathcal{A}$ consisting

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of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad n = 2, 3, \ldots. \tag{2}$$

Also, for functions $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \tag{3}$$

A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \leq \alpha < 1)$, if and only if $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U})$. This function class is denoted by $\mathcal{S}^{*}(\alpha)$. We also write $\mathcal{S}^{*}(0) \equiv \mathcal{S}^{*}$, where $\mathcal{S}^{*}$ denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{U})$ is starlike with respect to the origin. A function $f \in \mathcal{A}$ is said to be convex of order α $(0 \leq \alpha < 1)$ if and only if $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U})$. This class is denoted by $\mathcal{K}(\alpha)$. Further, $\mathcal{K}(0) = \mathcal{K}$, the well-known standard class of convex functions. It is an established fact that $f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^{*}(\alpha)$.

We recall the definitions of the class of uniformly convex functions denoted by $\mathcal{UCV}$, introduced by Goodman [5, 6], studied extensively by Rønning [11, 12], and independently by Ma and Minda [7].

Definition 1.1. A function f of the form (1) is said to be uniformly convex in $\mathbb{U}$, if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in \mathbb{U}).$$

Further the class $\alpha - \mathcal{S}_P$ related to the class $\alpha - \mathcal{UCV}$ by means of the well-known Alexander equivalence between the usual classes of convex $\mathcal{K}$ and starlike $\mathcal{S}^{*}$ functions are defined and the analytic criterion for functions in these classes are given as below:

Definition 1.2. A function f of the form (1)is said to be in the class $\mathcal{S}_p(\alpha)$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha\left|\frac{zf''(z)}{f'(z)} - 1\right|, \quad (z \in \mathbb{U})$$

and $f \in \mathcal{UCV}(\alpha), \alpha \geq 0$ if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \alpha\left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in \mathbb{U})$$

Also, $\mathcal{UCT}(\alpha) = \mathcal{UCV}(\alpha) \cap \mathcal{T}$ and $\mathcal{TS}_p(\alpha) = \mathcal{S}_p(\alpha) \cap \mathcal{T}$. It is of interest to note that $\mathcal{UCV}(1) \equiv \mathcal{UCV}$ and $\equiv \mathcal{UCV}(0) \equiv \mathcal{K}$; $\mathcal{S}_p(0) \equiv \mathcal{S}^{\star}$ and $\mathcal{S}_p(1) \equiv \mathcal{S}_p$. (see[16])

we recall the following results due to Subramanian et al., [16].

Lemma 1.3. *A function $f(z)$ of the form (2) is in the class (i) $\mathcal{TS}_p(\alpha)$ if and only if*

$$\sum_{n=2}^{n=\infty} [(\alpha+1)n - \alpha]\, a_n \leq 1$$

and

(ii) $\mathcal{UCT}(\alpha)$ if and only if

$$\sum_{n=2}^{n=\infty} n\, [(\alpha+1)n - \alpha]\, a_n \leq 1.$$

For, $a_1, a_2, ..., a_p$ and $b_1, b_2, ..., b_q$ be complex numbers with $b_j \neq 0, -1, ...; j = 1, 2, ..., q$, then the generalized hypergeometric functions ${}_pF_q(z)$ is defined by

$${}_pF_q(z) = {}_p F_q(a_1, a_2, ..., a_p; b_1, b_2, ..., b_q, z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n ... (a_p)_n z^n}{(b_1)_n (b_2)_n ... (b_p)_n (1)_n}, \qquad (4)$$

$p \leq q+1$ where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1 & \text{if } n = 0 \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & \text{if } n = 1, 2, \ldots. \end{cases} \qquad (5)$$

We note that the function in the series in (4) converges absolutely in the entire complex plane for $p < q+1$, and for $p = q+1$ in the unit disc, the condition $p \leq q+1$ stated with the definition (3) will be hold true throughout this paper.We note that the function ${}_pF_q(1)$ converges whenever $\Re\left(\sum\limits_{j=1}^{q} b_j - \sum\limits_{i=1}^{p} a_j\right) > 0$.

Merkes and Scott [8] used continued fractions to find sufficient conditions for the function $z_2F_1(z)$ to be in the class $\mathcal{S}^{\star}(\alpha), 0 \leq \alpha < 1$. Silverman [14] and Dixit and Pathak [2] determined sufficient conditions for the function $z_2F_1(z)$ to be in the class $\mathcal{S}^{\star}(\alpha)$and $\mathcal{UCV}(\alpha)$ respectively and various geometric properties of this function was discussed in [3, 4, 9, 14, 15]. In this paper,we we obtain sufficient condition for function $h(z)$,given by

$$h(z) = (1-\mu)\tau(z) + \mu z\tau'(z) \qquad (6)$$

where $\mu \geq 0$ and $\tau(z) = z_pF_q(z)$ belonging to the classes $\mathcal{S}_p(\alpha)$and $\mathcal{UCT}(\alpha)$.

2. Main Results

Theorem 2.1. *If $a_i > 0, (i = 1, 2, ..., p)$ and $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 2$, then a sufficient condition for the function $h(z)$ to be in the class $\mathcal{TS}_p(\alpha), 0 \leq \alpha < 1$, is that*

$$\mu(\alpha+1)\left[\frac{(a_1)_2(a_2)_2...(a_p)_2}{(b_1)_2(b_2)_2...(b_p)_2}\right] {}_pF_q(a_1+2; b_1+2, 1)$$
$$+ (\mu(\alpha+2)+\alpha+1)\left[\frac{(a_1)(a_2)...(a_p)}{(b_1)(b_2)...(b_p)}\right] {}_pF_q(a_1+1; b_1+1, 1) + {}_PF_q(1) \leq 2. \tag{7}$$

The condition (7) is necessary and sufficient for the function $h_1(z)$, defined by $h_1(z) = z\left(2 - \frac{h(z)}{z}\right)$ to be in the class $\mathcal{TS}_p(\alpha)$.

Proof. Since

$$h(z) = z + \sum_{n=2}^{\infty} (1 - \mu + \mu n) \frac{(a_1)_{n-1}(a_2)_{n-1}...(a_p)_{n-1}z^n}{(b_1)_{n-1}(b_2)_{n-1}...(b_p)_{n-1}(1)_{n-1}}.$$

According to Lemma 1.3 we need only to show that

$$\sum_{n=2}^{\infty} [n(\alpha+1) - \alpha] (1 - \mu + \mu n) \frac{(a_1)_{n-1}(a_2)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}(b_2)_{n-1}...(b_p)_{n-1}(1)_{n-1}} \leq 1. \tag{8}$$

Now from (8)

$$\sum_{n=1}^{\infty} [(n+1)(\alpha+1) - \alpha] (1 - \mu + \mu(n+1)) \frac{(a_1)_n(a_2)_n...(a_p)_n}{(b_1)_n(b_2)_n...(b_p)_n(1)_n}$$
$$= \sum_{n=1}^{\infty} \left[\mu(\alpha+1)n^2 + (\mu+\alpha+1)n + 1\right] \frac{(a_1)_n(a_2)_n...(a_p)_n}{(b_1)_n(b_2)_n...(b_p)_n(1)_n}.$$

Using the fact that $(\lambda)_n = \lambda(\lambda+1)_{n-1}$ yields

$$= \mu(\alpha+1)\sum_{n=1}^{\infty}\frac{(a_1)_n(a_2)_n...(a_p)_n}{(b_1)_n(b_2)_n...(b_p)_n(1)_{n-2}}$$

$$+ (\mu\alpha+2\mu+\alpha+1)\sum_{n=1}^{\infty}\frac{(a_1)_n(a_2)_n...(a_p)_n}{(b_1)_n(b_2)_n...(b_p)_n(1)_{n-1}}$$

$$+\sum_{n=1}^{\infty}\frac{(a_1)_n(a_2)_n...(a_p)_n}{(b_1)_n(b_2)_n...(b_p)_n(1)_n}$$

$$= \mu(\alpha+1)\left[\frac{(a_1)_2(a_2)_2...(a_p)_2}{(b_1)_2(b_2)_2...(b_p)_2}\right]\ {}_pF_q(a_1+2;b_1+2,1)$$
$$+ (\mu(\alpha+2)+\alpha+1)\left[\frac{(a_1)(a_2)...(a_p)}{(b_1)(b_2)...(b_p)}\right]\ {}_pF_q(a_1+1;b_1+1,1) +_P F_q(1) - 1.$$

But this last expression is bounded by 1 if and only if(7) holds.
Since

$$h_1(z) = z - \sum_{n=2}^{\infty}(1-\mu+\mu n)\,\frac{(a_1)_{n-1}(a_2)_{n-1}...(a_p)_{n-1}z^n}{(b_1)_{n-1}(b_2)_{n-1}...(b_p)_{n-1}(1)_{n-1}}. \qquad (9)$$

The condition (7) is also necessary for $h_1(z)$ to be in the class $\mathcal{TS}_p(\alpha), 0 \le \alpha < 1$ from the Lemma 1.3 □

Theorem 2.2. *If* $a_i > -1, (i = 1, 2, ..., p), b_j > 0, (j = 1, 2, ..., q), \prod_{i=1}^{n} a_i < 0$ *and* $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 2$, *then a necessary and sufficient condition for the function* $h(z)$ *to be in the class* $\mathcal{TS}_p(\alpha), 0 \le \alpha < 1$, *is that*

$$\mu(\alpha+1)\left[\frac{(a_1+1)(a_2+1)...(a_p+1)}{(b_1+1)(b_2+1)...(b_p+1)}\right]\ {}_pF_q(a_1+2;b_1+2,1)$$
$$+ (\mu(\alpha+2)+\alpha+1)\ {}_pF_q(a_1+1;b_1+1,1) + \frac{b_1b_2...b_q}{a_1a_2...a_p}{}_pF_q(1) \ge 0. \qquad (10)$$

The condition $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 1 - \prod_{i=1}^{n} a_i$ *is necessary and sufficient for the function* $h(z)$ *to be in the class* $\mathcal{TS}_0 = \mathcal{S}^{\square}$, *a class of starlike function with negative coefficients.*

Proof. Since,

$$h(z) = z - \left|\frac{a_1a_2...a_p}{b_1b_2...b_q}\right| \sum_{n=2}^{\infty}(1-\mu+\mu n)\frac{(a_1+1)_{n-2}(a_2+1)_{n-2}...(a_p+1)_{n-1}z^n}{(b_1+1)_{n-2}(b_2+1)_{n-2}...(b_p+1)_{n-2}(1)_{n-1}}. \quad (11)$$

If we show the inequality

$$\sum_{n=2}^{\infty}[n(\alpha+1)-\alpha](1-\mu+\mu n)\frac{(a_1+1)_{n-2}(a_2+1)_{n-2}...(a_p+1)_{n-2}}{(b_1+1)_{n-2}(b_2+1)_{n-2}...(b_p+1)_{n-2}(1)_{n-1}} \le \left|\frac{b_1b_2...b_q}{a_1a_2...a_p}\right| \quad (12)$$

then by Lemma 1.3, Theorem 2.2 is obtained. The left hand side of the equation (12) converges if $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 2$.From (12)

$$\sum_{n=0}^{\infty}[(n+2)(\alpha+1)-\alpha][1-\mu+\mu(n+2)]\frac{(a_1+1)_n(a_2+1)_n...(a_p+1)_n}{(b_1+1)_n(b_2+1)_n...(b_p+1)_n(1)_{n+1}}$$

$$= \mu(\alpha+1)\left[\frac{(a_1+1)(a_2+1)...(a_p+1)}{(b_1+1)(b_2+1)...(b_p+1)}\right]\sum_{n=0}^{\infty}\frac{(a_1+2)_n(a_2+2)_n...(a_p+2)_n}{(b_1+2)_n(b_2+2)_n...(b_p+2)_n(1)_n}$$

$$+ (\mu(\alpha+2)+\alpha+1)\sum_{n=0}^{\infty}\frac{(a_1+1)_n(a_2+1)_n...(a_p+1)_n}{(b_1+1)_n(b_2+1)_n...(b_p+1)_n(1)_n}$$

$$+ \sum_{n=1}^{\infty}\frac{(a_1+1)_{n-1}(a_2+1)_{n-1}...(a_p+1)_{n-1}}{(b_1+1)_{n-1}(b_2+1)_{n-1}...(b_p+1)_{n-1}(1)_n}$$

$$= \mu(\alpha+1)\left[\frac{(a_1+1)(a_2+1)...(a_p+1)}{(b_1+1)(b_2+1)...(b_p+1)}\right]{}_pF_q(a_1+2;b_1+2,1)$$

$$+ (\mu(\alpha+2)+\alpha+1)\ {}_pF_q(a_1+1;b_1+1,1) + \frac{b_1b_2...b_q}{a_1a_2...a_p}\{{}_pF_q(1)-1\}.$$

Hence (12) is equivalent to

$$\mu(\alpha+1)\left[\frac{(a_1+1)(a_2+1)...(a_p+1)}{(b_1+1)(b_2+1)...(b_p+1)}\right]{}_pF_q(a_1+2;b_1+2,1)$$

$$+(\mu(\alpha+2)+\alpha+1)\,{}_pF_q(a_1+1;b_1+1,1) \tag{13}$$

$$+\frac{b_1b_2...b_q}{a_1a_2...a_p}{}_pF_q(1)\leq\left|\frac{b_1b_2...b_q}{a_1a_2...a_p}\right|+\frac{b_1b_2...b_q}{a_1a_2...a_p}=0.$$

Thus (13) is valid if and only if

$$\mu(\alpha+1)\left[\frac{(a_1+1)(a_2+1)...(a_p+1)}{(b_1+1)(b_2+1)...(b_p+1)}\right]{}_pF_q(a_1+2;b_1+2,1)$$

$$+(\mu(\alpha+2)+\alpha+1)\ {}_pF_q(a_1+1;b_1+1,1)+\frac{b_1b_2...b_q}{a_1a_2...a_p}{}_pF_q(1)\geq 0.$$

This is equivalent to (10). This completes the proof of the Theorem 2.2. □

Theorem 2.3. *If $a_i>0, i=1,2,...,p,$ and $\sum_{j=1}^{q}b_j>\sum_{i=1}^{p}a_i+3,$ then a sufficient condition for the function $h(z)$ to be in the class $\mathcal{UCT}(\alpha), 0\leq\alpha<1,$ is that*

$$\begin{aligned}&\mu(\alpha+1)\left[\frac{(a_1)_3(a_2)_3...(a_p)_3}{(b_1)_3(b_2)_3...(b_q)_3}\right]{}_pF_q(a_1+3;b_1+3,1)\\&+(4\mu\alpha+5\mu+\alpha+1)\left[\frac{(a_1)_2(a_2)_2...(a_p)_2}{(b_1)_2(b_2)_2...(b_q)_2}\right]{}_pF_q(a_1+2;b_1+2,1)\\&+(2\mu\alpha+4\mu+\alpha+3)\,\frac{a_1a_2...a_p}{b_1b_2...b_q}\,{}_pF_q(a_1+1;b_1+1,1)+{}_pF_q(1)\leq 2.\end{aligned} \tag{14}$$

The condition (14) is a necessary and sufficient condition for the function $h_1(z)$ defined by (9) to be in the class $\mathcal{UCT}(\alpha)$.

Proof. Taking $h(z)$ defined by (6), in view of (ii) of Lemma 1.3, we need only to show that

$$\sum_{n=2}^{\infty}n\left[n(\alpha+1)-\alpha\right](1-\mu+\mu n)\,\frac{(a_1)_{n-1}(a_2)_{n-1}...(a_p)_{n-1}}{(b_1)_{n-1}(b_2)_{n-1}...(b_p)_{n-1}(1)_{n-1}}\leq 1. \tag{15}$$

The left hand side of (15) converges if $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 3$. From (15)

$$\sum_{n=0}^{\infty}(n+2)\left[(n+2)(\alpha+1)-\alpha\right](1-\mu+\mu(n+2))\,\frac{(a_1)_{n+1}(a_2)_{n+1}...(a_p)_{n+1}}{(b_1)_{n+1}(b_2)_{n+1}...(b_p)_{n+1}(1)_{n+1}}$$
$$= \mu(\alpha+1)\sum_{n=0}^{\infty}\frac{(a_1)_{n+3}(a_2)_{n+3}...(a_p)_{n+3}}{(b_1)_{n+3}(b_2)_{n+3}...(b_p)_{n+3}(1)_n}$$
$$+ (4\mu\alpha+5\mu+\alpha+1)\sum_{n=0}^{\infty}\frac{(a_1)_{n+2}(a_2)_{n+2}...(a_p)_{n+2}}{(b_1)_{n+2}(b_2)_{n+2}...(b_p)_{n+2}(1)_n}$$
$$+ (2\mu\alpha+4\mu+\alpha+3)\sum_{n=0}^{\infty}\frac{(a_1)_{n+1}(a_2)_{n+1}...(a_p)_{n+1}}{(b_1)_{n+1}(b_2)_{n+1}...(b_p)_{n+1}(1)_n} + \sum_{n=1}^{\infty}\frac{(a_1)_n(a_2)_n...(a_p)_n}{(b_1)_n(b_2)_n...(b_p)_n(1)_n}. \tag{16}$$

Since $(a)_{n+k} = (a)_k(a+k)_n$, we may write (16) as

$$= \mu(\alpha+1)\left[\frac{(a_1)_3(a_2)_3...(a_p)_3}{(b_1)_3(b_2)_3...(b_q)_3}\right]\sum_{n=0}^{\infty}\frac{(a_1+3)_n(a_2+3)_n...(a_p+3)_n}{(b_1+3)_n(b_2+3)_n...(b_p+3)_n(1)_n}$$
$$+ (4\mu\alpha+5\mu+\alpha+1)\left[\frac{(a_1)_2(a_2)_2...(a_p)_2}{(b_1)_2(b_2)_2...(b_q)_2}\right]\sum_{n=0}^{\infty}\frac{(a_1+2)_n(a_2+2)_n...(a_p+2)_n}{(b_1+2)_n(b_2+2)_n...(b_p+2)_n(1)_n}$$
$$+ (2\mu\alpha+4\mu+\alpha+3)\,\frac{a_1a_2...a_p}{b_1b_2...b_q}\sum_{n=0}^{\infty}\frac{(a_1+1)_n(a_2+1)_n...(a_p+1)_n}{(b_1+1)_n(b_2+1)_n...(b_p+1)_n(1)_n}$$
$$+ \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n...(a_p)_n}{(b_1)_n(b_2)_n...(b_p)_n(1)_n} - 1.$$

The last expression is bounded by 1 if and only if (14) holds,hence the proof of the Theorem 2.3. □

Theorem 2.4. *If* $a_i > -1, (i = 1, 2, ..., p)$, $\prod_{i=1}^{n} a_i < 0$ *and* $\sum_{j=1}^{q} b_j > \sum_{i=1}^{p} a_i + 3$, *then a necessary and sufficient condition for the function* $h(z)$ *to be in the class*

$\mathcal{UCT}(\alpha), 0 \leq \alpha < 1$, *is that*

$$\mu(\alpha+1)\left[\frac{(a_1+1)_2(a_2+1)_2...(a_p+1)_2}{(b_1+1)_2(b_2+1)_2...(b_p+1)_2}\right] {}_pF_q(a_1+3;b_1+3,1)$$
$$+(4\mu\alpha+5\mu+\alpha+1)\left[\frac{(a_1+1)(a_2+1)...(a_p+1)}{(b_1+1)(b_2+1)...(b_p+1)}\right] {}_pF_q(a_1+2;b_1+2,1)$$
$$+(2\mu\alpha+4\mu+\alpha+3)\, {}_pF_q(a_1+1;b_1+1,1)+\left[\frac{b_1b_2...b_q}{a_1a_2...a_p}\right] {}_pF_q(1) \geq 0. \tag{17}$$

Proof. Taking $h(z)$ defined by (11), in view of (ii) of Lemma 1.3, we must show that

$$\sum_{n=2}^{\infty} n\left[n(\alpha+1)-\alpha\right](1-\mu+\mu n)\frac{(a_1+1)_{n-2}(a_2+1)_{n-2}...(a_p+1)_{n-2}}{(b_1+1)_{n-2}(b_2+1)_{n-2}...(b_p+1)_{n-2}(1)_{n-1}}$$
$$\leq \left|\frac{b_1b_2...b_q}{a_1a_2...a_p}\right|. \tag{18}$$

The left side of (18) converges if $\sum\limits_{j=1}^{q} b_j > \sum\limits_{i=1}^{p} a_i + 3$. Now from (18)

$$\sum_{n=0}^{\infty}(n+2)\left[(n+2)(\alpha+1)-\alpha\right](1-\mu+\mu(n+2))\frac{(a_1+1)_n(a_2+1)_n...(a_p+1)_n}{(b_1+1)_n(b_2+1)_n...(b_p+1)_n(1)_{n+1}}$$
$$=\mu(\alpha+1)\sum_{n=0}^{\infty}\frac{(a_1+1)_{n+2}(a_2+1)_{n+2}...(a_p+1)_{n+2}}{(b_1+1)_{n+2}(b_2+1)_{n+2}...(b_p+1)_{n+2}(1)_n}$$
$$+(4\mu\alpha+5\mu+\alpha+1)\sum_{n=0}^{\infty}\frac{(a_1+1)_{n+1}(a_2+1)_{n+1}...(a_p+1)_{n+1}}{(b_1+1)_{n+1}(b_2+1)_{n+1}...(b_p+1)_{n+1}(1)_n}$$
$$+(2\mu\alpha+4\mu+\alpha+3)\sum_{n=0}^{\infty}\frac{(a_1+1)_n(a_2+1)_n...(a_p+1)_n}{(b_1+1)_n(b_2+1)_n...(b_p+1)_n(1)_n}$$
$$+\left[\frac{b_1b_2...b_q}{a_1a_2...a_p}\right]\sum_{n=0}^{\infty}\frac{(a_1)_{n+1}(a_2)_{n+1}...(a_p)_{n+1}}{(b_1)_{n+1}(b_2)_{n+1}...(b_p)_{n+1}(1)_{n+1}}$$

$$= \mu(\alpha+1)\left[\frac{(a_1+1)_2(a_2+1)_2...(a_p+1)_2}{(b_1+1)_2(b_2+1)_2...(b_p+1)_2}\right]\sum_{n=0}^{\infty}\frac{(a_1+3)_n(a_2+3)_n...(a_p+3)_n}{(b_1+3)_n(b_2+3)_n...(b_p+3)_n(1)_n}$$

$$+(4\mu\alpha+5\mu+\alpha+1)\left[\frac{(a_1+1)(a_2+1)...(a_p+1)}{(b_1+1)(b_2+1)...(b_p+1)}\right]\sum_{n=0}^{\infty}\frac{(a_1+2)_n(a_2+2)_n...(a_p+2)_n}{(b_1+2)_n(b_2+2)_n...(b_p+2)_n(1)_n}$$

$$+(2\mu\alpha+4\mu+\alpha+3)\sum_{n=0}^{\infty}\frac{(a_1+1)_n(a_2+1)_n...(a_p+1)_n}{(b_1+1)_n(b_2+1)_n...(b_p+1)_n(1)_n}$$

$$+\left[\frac{b_1b_2...b_q}{a_1a_2...a_p}\right]\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n...(a_p)_n}{(b_1)_n(b_2)_n...(b_p)_n(1)_n}-\left[\frac{b_1b_2...b_q}{a_1a_2...a_p}\right].$$

This last expression is bounded by $\left|\frac{b_1b_2...b_q}{a_1a_2...a_p}\right|$ if and only if (17) holds. Thus the proof of the Theorem 2.4 is completed. □

Concluding Remak: If $p = 2, q = 1$, the results reduces to Dixit and Pathak[4] result.

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