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ON GENERALIZED DERIVATIONS OF SEMIPRIME RINGS

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Abstract: Let R be a ring and S be a nonempty subset of R. A mapping $f : R \longrightarrow R$ is said to be centralizing (resp. commuting) on S if $[x, f(x)] \in Z(R)$ (resp. $[x, f(x)] = 0$) for all $x \in S$. The purpose of this paper is to generalize the classical theorem of Posner [7, Theorem 2] and to extend a result of Bell and Martindale [1, Theorem 3] for a generalized derivation of a semiprime ring R which is commuting on a left ideal of R.

1. Introduction

Throughout the paper R will denote an associative ring with centre $Z(R)$. A ring R is said to be prime (resp. semiprime) if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$ (resp. $aRa = \{0\}$ implies that $a = 0$). We shall write for any pair of elements $x, y \in R$ $[x, y]$, the commutator $xy - yx$. An additive mapping $d : R \longrightarrow R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A mapping $f : R \longrightarrow R$ is said to be skew-commuting on S if $f(x)x + xf(x) = 0$, for all $x \in S$. Many algebraist studied generalized derivation in the context of algebras on certain normed spaces (see [3] for reference). By a generalized derivation on an algebra A one usually means a map of the form $x \mapsto ax + xb$ where a and b are fixed elements in A. We prefer to call such maps generalized inner derivations for the reason they present a generalization of the concept of inner derivation (i.e. the map $x \mapsto ax - xb$). Now in a ring, let F be a generalized inner derivation given by $F(x) = ax + xb$. Notice that $F(xy) = F(x)y + xI_{\mathtt{b}}(y)$,

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where $I_b(y) = yb - by$ is an inner derivation. Motivated by these observations, Bresar [2] introduced the notion of a generalized derivation in rings. An additive mapping $F : R \longrightarrow R$ is said to be a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Hence the concept of a generalized derivation covers both the concepts, of a derivation and a left multiplier (i.e. additive map satisfying $f(xy) = f(x)y$) for all $x, y \in R$. Basic examples are derivations and generalized inner derivations. Some results on generalized derivation can be found in [3].

Throughout the paper we will make extensive use of the basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Moreover we shall require the following lemma.

2. Main Results

Theorem 1. *Let R be a semiprime ring and I be a nonzero left ideal of R. Let F be a generalized derivation on R with associated derivation d such that $Id(I) \neq (0)$. If F is commuting on I, then $[I, I]d(I) = (0)$ and there exists $0 \neq \alpha \in Z(R)$ such that $\alpha I \subseteq Z(R)$.*

For developing the proof we require the following lemma:

Lemma 1. (see [5]) *In a semiprime ring R*

(i) The centre of R contains no nonzero nilpotent elements.

(ii) R does not contain any nonzero nilpotent left ideals.

(iii) If P is a nonzero prime ideal of R and $a, b \in R$ such that $aRb \subseteq P$, then either $a \in P$ or $b \in P$.

Proof of Theorem 1. We have $[F(x), x] = 0$ for all $x \in I$. Linearization yields that

$$[F(x), y] + [F(y), x] = 0 \text{ for all } x, y \in I. \tag{1}$$

Replacing y by yx in (1), we have

$$y[F(x), x] + [F(x), y]x + [F(y), x]x + y[d(x), x] + [y, x]d(x) = 0 \text{ for all } x, y \in I. \tag{2}$$

Using (1), we have

$$y[d(x),x]+[y,x]d(x)=0 \text{ for all } x,y\in I. \tag{3}$$

Substituting ry for y in (3), we have

$$ry[d(x),x]+r[y,x]d(x)+[r,x]yd(x)=0 \text{ for all } x,y\in I,\ r\in R. \tag{4}$$

Application of (3), yields that

$$[r,x]yd(x)=0 \text{ for all } x,y\in I,\ r\in R. \tag{5}$$

Replacing y by ry in (5), we get

$$[r,x]ryd(x)=0 \text{ for all } x,y\in I,\ r\in R. \tag{6}$$

By (5), we have $[x,R]Id(x)=0$ for all $x\in I$. Take the family $\{P_\alpha\}$ of prime ideals of R such that $\cap P_\alpha=(0)$. Let x_1 be a fixed element of I. Thus for any $x_2\in I$, (5) implies that

$$(0)=[x_1+x_2,R]Id(x_1+x_2)=[x_1,R]Id(x_2)+[x_2,R]Id(x_1). \tag{7}$$

Moreover, for any P_α, by (6) and Lemma 1 (iii), either $Id(x_1)\subseteq P_\alpha$, or $[x_1,R]\subseteq P_\alpha$. In case $Id(x_1)\subseteq P_\alpha$, then a fortiori $[x_2,R]Id(x_1)\subseteq P_\alpha$. On the other hand, if $[x_1,R]\subseteq P_\alpha$, then by (7) it follows again $[x_2,R]Id(x_1)\subseteq P_\alpha$. Therefore in any case $[I,R]Id(I)\subseteq P_\alpha$, for any α. This implies that

$$[I,R]Id(I)\subseteq \cap P_\alpha=(0). \tag{8}$$

In particular we get $[I,I]RId(I)=(0)$ and from this we also have

$$[I,I]d(I)R[I,I]d(I)=(0).$$

Hence, by the semiprimeness of R we get

$$[I,I]d(I)=(0). \tag{9}$$

Moreover, for any $r,s\in R$, $x,y,z\in I$ and by (9) it follows

$$0=[rx,s]yd(z)=[r,s]xyd(z) \tag{10}$$

and replacing y with ty in (10), for any $t\in R$, we get $[r,s]xtyd(z)=0$, that is $[R,R]IRId(I)=(0)$. Again by the semiprimeness of R, it follows

$[R, R]Id(I) = (0)$ and a fortiori $[R, R]RId(I) = (0)$. This last implies easily that $[Id(I), R]R[Id(I), R] = (0)$, that is $Id(I) \subseteq Z(R)$.

Therefore for $x, y, z \in I$, we have $xd(z)y + xzd(y) = xd(zy) \in Z(R)$, and since $xd(zy) \in Z(R)$, it follows that also $xd(z)y \in Z(R)$, for any $x, y, z \in I$. Moreover, by $Id(I) \neq (0)$, there exist $x_0, z_0 \in I$ such that $0 \neq x_0 d(z_0) = \alpha \in Z(R)$. Hence, for all $y \in I$, we get $\alpha y \in Z(R)$, that is $\alpha I \subseteq Z(R)$.

Theorem 2. *Let R be a semiprime ring and I be a nonzero left ideal of R. Let F be a generalized derivation on R with associated derivation d such that $Id(I) \neq (0)$. If F is skew-commuting on I, then $[I, I]d(I) = (0)$ and there exists $0 \neq \alpha \in Z(R)$ such that $\alpha I \subseteq Z(R)$.*

Proof. We have $F(x)x + xF(x) = 0$ for all $x \in I$. Linearization yields that

$$F(x)y + F(y)x + yF(x) + xF(y) = 0 \quad \text{for all } x, y \in I. \tag{11}$$

Replacing y by yx in (11), we have

$$(F(x)y + F(y)x + xF(y))x + yd(x)x + xyd(x) + yxF(x) = 0 \quad \text{for all } x, y \in I. \tag{12}$$

Comparing (11) and (12), we get

$$-yF(x)x + yxF(x) + yd(x)x + xyd(x) = 0 \quad \text{for all } x, y \in I. \tag{13}$$

This implies that

$$y[x, F(x)] + yd(x)x + xyd(x) = 0 \quad \text{for all } x, y \in I. \tag{14}$$

Replacing y by ry in (14), we obtain

$$r(y[F(x), x] + yd(x)x) + xryd(x) = 0 \quad \text{for all } x, y \in I, r \in R. \tag{15}$$

From (14) and (15), we have

$$-rxyd(x) + xryd(x) = [x, r]yd(x) = 0 \quad \text{for all } x, y \in I, r \in R. \tag{16}$$

This implies that $[x, r]Ryd(x) = (0)$ for all $x, y \in I$ and $r \in R$. Arguing in the similar manner as we have done in the proof of above theorem, we get the required result.

An immediate consequence of the above theorems is the following corollary:

Corollary 1. *Let R be a prime ring and I be a nonzero left ideal of R. Let F be a generalized derivation on R with associated derivation d. If either F is commuting or skew commuting on I, then R is commutative.*

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