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JOIN-MEET AND MEET-JOIN PRESERVING MAPS

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Abstract: We investigate the properties of join-meet and meet-join preserving maps in complete residuated lattice. In particular, we give their examples.

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1. Introduction

Pawlak [6,7] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. By using the concepts of lower and upper approximation operators, information systems and decision rules are investigated in complete residuated lattices [1,2,8,9]. Bělohlávek [1] developed the notion of fuzzy contexts using Galois connections with $R \in L^{X\times Y}$ on a complete residuated lattice. Zhang [10,11] introduced the fuzzy complete lattice which is defined by join and meet on fuzzy posets. It is an important mathematical tool for algebraic structure of fuzzy contexts [1,4-6]. Kim [3] show that join (resp. meet, meet join, join meet) preserving maps and upper (resp. lower, meet join, join meet) approximation maps are equivalent in complete residuated lattices.

In this paper, we investigate the properties of join-meet and meet-join preserving maps in complete residuated lattice. In particular, we give their examples.

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2. Preliminaries

Definition 1. [1,2] A structure $(L, \vee, \wedge, \odot, \to, \bot, \top)$ is called a *complete residuated lattice* iff it satisfies the following properties:

(L1) $(L, \vee, \wedge, \bot, \top)$ is a complete lattice where $\bot$ is the bottom element and $\top$ is the top element;

(L2) $(L, \odot, \top)$ is a monoid;

(L3) adjointness properties,i.e.

$$x \le y \to z \text{ iff } x \odot y \le z.$$

A map $^* : L \to L$ defined by $a^* = a \to \bot$ is called *strong negations* if $a^{**} = a$.

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \bot, & \text{otherwise.} \end{cases} \quad \top_x^*(y) = \begin{cases} \bot, & \text{if } y = x, \\ \top, & \text{otherwise.} \end{cases}$$

In this paper, we assume that $(L, \vee, \wedge, \odot, \to, ^*, \bot, \top)$ be a complete residuated lattice with a strong negation *.

Definition 2. [10,11] Let X be a set. A function $e_X : X \times X \to L$ is called:

(E1) reflexive if $e_X(x, x) = 1$ for all $x \in X$,

(E2) transitive if $e_X(x, y) \odot e_X(y, z) \le e_X(x, z)$, for all $x, y, z \in X$,

(E3) if $e_X(x, y) = e_X(y, x) = 1$, then $x = y$.

If e satisfies (E1) and (E2), (X, e_X) is a fuzzy preorder set. If e satisfies (E1), (E2) and (E3), (X, e_X) is a fuzzy partially order set (simply, fuzzy poset).

Example 3. (1) We define a function $e_L : L \times L \to L$ as $e_L(x, y) = x \to y$. Then (L, e_L) is a fuzzy poset.

(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(A, B) = \bigwedge_{x \in X}(A(x) \to B(x))$. Then (L^X, e_{L^X}) is a fuzzy poset from Lemma 10(9).

Definition 4. [10,11] Let (X, e_X) be a fuzzy poset and $A \in L^X$.

(1) A point x_0 is called a join of A, denoted by $x_0 = \sqcup A$, if it satisfies

(J1) $A(x) \le e_X(x, x_0)$,

(J2) $\bigwedge_{x \in X}(A(x) \to e_X(x, y)) \le e_X(x_0, y)$.

A point x_1 is called a meet of A, denoted by $x_1 = \sqcap A$, if it satisfies

(M1) $A(x) \le e_X(x_1, x)$,

(M2) $\bigwedge_{x \in X}(A(x) \to e_X(y, x)) \le e_X(y, x_1)$.

Remark 5. Let (X, e_X) be a fuzzy poset and $A \in L^X$.

(1) If x_0 is a join of A, then it is unique because $e_X(x_0, y) = e_X(y_0, y)$ for all $y \in X$, put $y = x_0$ or $y = y_0$, then $e_X(x_0, y_0) = e_X(y_0, x_0) = \top$ implies $x_0 = y_0$. Similarly, if a meet of A exist, then it is unique.

(2) x_0 is a join of A iff $\bigwedge_{x \in X}(A(x) \to e_X(x, y)) = e_X(x_0, y)$.

(3) x_1 is a meet of A iff $\bigwedge_{x \in X}(A(x) \to e_X(y, x)) = e_X(y, x_1)$.

Remark 6. Let (L, e_L) be a fuzzy poset and $A \in L^L$.

(1) Since x_0 is a join of A iff $\bigwedge_{x \in L}(A(x) \to e_L(x, y)) = \bigwedge_{x \in L}(A(x) \to (x \Rightarrow y)) = \bigvee_{x \in L}(x \odot A(x)) \to y = e_L(x_0, y) = x_0 \to y$, then $x_0 = \sqcup A = \bigvee_{x \in L}(x \odot A(x))$.

(2) Since x_0 is a join of A iff $\bigwedge_{x \in L}(A(x) \to e_L(x, y) = \bigwedge_{x \in L}(A(x) \to (y \to x)) = \bigwedge_{x \in L}(y \to (A(x) \to x)) = y \to \bigwedge_{x \in L}(A(x) \to x) = y \to \sqcap A$, then $\sqcap A = \bigwedge_{x \in L}(A(x) \to x)$.

Remark 7. Let (L^X, e_{L^X}) be a fuzzy poset and $\Phi \in L^{L^X}$.

(1) Since $\bigwedge_{A \in L^X}(\Phi(A) \to e_{L^X}(A, B)) = e_{L^X}(\bigvee_{A \in L^X}(\Phi(A) \odot A), B) = e_{L^X}(\sqcup \Phi, B)$, then $\sqcup \Phi = \bigvee_{A \in L^X}(\Phi(A) \odot A)$.

(2) Since $\bigwedge_{A \in L^X}(\Phi(A) \to e_{L^X}(B, A) = \bigwedge_{A \in L^X} e_{L^X}(B, (\Phi(A) \to A)) = e_{L^X}(B, \bigwedge_{A \in L^X}(\Phi(A) \to A))$, then $\sqcap \Phi = \bigwedge_{A \in L^X}(\Phi(A) \to A)$.

Definition 8. [10,11] Let (L^X, e_{L^X}) and (L^Y, e_{L^Y}) be fuzzy posets.

(1) $\mathcal{K} : L^X \to L^Y$ is a join-meet preserving map if $\mathcal{K}(\sqcup \Phi) = \sqcap \mathcal{K}^{\to}(\Phi)$ for all $\Phi \in L^{L^X}$, where $\mathcal{K}^{\to}(\Phi)(B) = \bigvee_{\mathcal{K}(A)=B} \Phi(A)$.

(2) $\mathcal{M} : L^X \to L^Y$ is a meet-join preserving map if $\mathcal{M}(\sqcap \Phi) = \sqcup \mathcal{M}^{\to}(\Phi)$ for all $\Phi \in L^{L^X}$, where $\mathcal{M}^{\to}(\Phi)(B) = \bigvee_{\mathcal{M}(A)=B} \Phi(A)$.

Theorem 9. [3] *Let X and Y be two sets. Let (L^X, e_{L^X}) and (L^Y, e_{L^Y}) be fuzzy posets. Then the following statements are equivalent:*

(1) $\mathcal{K} : L^X \to L^Y$ is a join-meet preserving map iff $\mathcal{K}(\alpha \odot A) = \alpha \to \mathcal{K}(A)$ and $\mathcal{K}(\bigvee_{i \in I} A_i) = \bigwedge_{i \in I} \mathcal{K}(A_i)$ for all $A, A_i \in L^X$, and $\alpha \in L$.

(2) $\mathcal{M} : L^X \to L^Y$ is a meet-join preserving map iff $\mathcal{M}(\alpha \to A) = \alpha \to \mathcal{J}(A)$ and $\mathcal{M}(\bigwedge_{i \in I} A_i) = \bigvee_{i \in I} \mathcal{M}(A_i)$ for all $A, A_i \in L^X$, and $\alpha \in L$.

Lemma 10. [1,2] Let $(L, \vee, \wedge, \odot, \to, ^*, \bot, \top)$ be a complete residuated lattice with a strong negation *. For each $x, y, z, x_i, y_i \in L$, the following properties hold.

(1) $\odot$ is isotone in both arguments.

(2) $\to$ is antitone in the first and isotone in the second argument.

(3) $x \to y = \top$ iff $x \leq y$.

(4) $x \to \top = \top$ and $\top \to x = x$.

(5) $x \odot y \leq x \wedge y$.

(6) $x \odot (\bigvee_{i \in \Gamma} y_i) = \bigvee_{i \in \Gamma}(x \odot y_i)$ and $(\bigvee_{i \in \Gamma} x_i) \odot y = \bigvee_{i \in \Gamma}(x_i \odot y)$.

(7) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.
(8) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i) \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$.
(9) $(x \to y) \odot x \leq y$ and $(y \to z) \odot (x \to y) \leq (x \to z)$.
(10) $x \to y \leq (y \to z) \to (x \to z)$.
(11) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.
(12) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ and $(x \odot y)^* = x \to y^*$.
(13) $x^* \to y^* = y \to x$ and $(x \to y)^* = x \odot y^*$.

3. Join-Meet and Meet-Join Preserving Maps

Theorem 11. *Let (L^X, e_{L^X}) be a fuzzy poset. Let $\mathcal{K}, \mathcal{K}^{-1} : L^X \to L^X$ be join-meet preserving maps such that $\mathcal{K}^{-1}(\top_x)(y) = \mathcal{K}(\top_y)(x)$ for each $x, y \in X$. Let $\mathcal{M}, \mathcal{M}^{-1} : L^X \to L^X$ be meet preserving maps such that $\mathcal{M}^{-1}(\top_x^*)(y) = \mathcal{M}(\top_y^*)(x)$ and $\mathcal{K}^*(\top_x)(y) = \mathcal{M}(\top_x^*)(y)$ for each $x, y \in X$. For $x, y \in X$, $\alpha \in L$ and $A, B \in L^X$, we have the following properties.*

(1) $\mathcal{K}(A)(y) = \bigwedge_x (A(x) \to \mathcal{K}(\top_x)(y)$ *and*

$$\mathcal{K}^{-1}(A)(y) = \bigwedge_x (A(x) \to \mathcal{K}^{-1}(\top_x)(y) = \bigwedge_x (A(x) \to \mathcal{K}(\top_y)(x))$$

(2) $\mathcal{M}(A)(y) = \bigvee_x (A^*(x) \odot \mathcal{M}(\top_x^*)(y))$ *and* $\mathcal{M}^{-1}(A)(y) = \bigvee_x (A^*(x) \odot \mathcal{M}^{-1}(\top_x^*)(y))$.
(3) $\mathcal{M}(\top) = \mathcal{M}^{-1}(\top) = \bot$ *and* $\mathcal{M}(\bot) = \mathcal{K}^{-1}(\bot) = \top$.
(4) $\mathcal{M}(A) = (\mathcal{K}(A^*))^*$ *and* $\mathcal{M}(A) = (\mathcal{M}(A^*))^*$.
(5) $\mathcal{K}(\alpha \to A) \geq \alpha \odot \mathcal{K}(A)$ *and* $\mathcal{M}(\alpha \odot A) \leq \alpha \to \mathcal{M}(A)$.
(6) $\mathcal{M}(\top_x \to \alpha)(y) = \mathcal{M}(\top_x^*)(y) \odot \alpha^* = \mathcal{K}^*(\top_x \odot \alpha^*)(y)$.
(7) $\bigwedge_{\alpha\in L}((A(y) \to \alpha) \to \alpha) = A(y)$.
(8) $\mathcal{K}(A^*) = \bigwedge_{\alpha\in L}(\mathcal{M}(A \odot \alpha^*) \to \alpha)$.
(9) $\mathcal{K}(\alpha \odot A) = \alpha \to \mathcal{K}(A) = \mathcal{M}(A^*) \to \alpha^*$.
(10) $\mathcal{M}(\alpha \odot A) \leq \mathcal{K}(A^*) \to \alpha^*$.
(11) $e_{L^X}(B, \mathcal{K}(A)) = e_{L^X}(A, \mathcal{K}^{-1}(B))$ *and*

$$e_{L^X}(\mathcal{M}(A), B) = e_{L^X}(\mathcal{M}^{-1}(B), A).$$

(12) $e_{L^X}(B, \mathcal{K}(A)) = e_{L^X}(\mathcal{M}^{-1}(B^*), A^*)$ *and*

$$e_{L^X}(\mathcal{M}(A), B) = e_{L^X}(A^*, \mathcal{K}^{-1}(B^*)).$$

(13) $e_{L^X}(A, B) \leq e_{L^X}(\mathcal{K}(B), \mathcal{K}(A))$.
(14) $e_{L^X}(A, B) \leq e_{L^X}(\mathcal{M}(B), \mathcal{M}(A))$.

Proof. (1) For $A = \bigvee_{x\in X}(A(x)\odot \top_x)$, by Theorem 9(1), we have

$$\begin{array}{ll}\mathcal{K}(A)(y) & = \mathcal{K}(\bigvee_{x\in X}(A(x)\odot \top_x))(y) = \bigwedge_{x\in X}(A(x)\to \mathcal{K}(\top_x)(y))\\ \mathcal{K}^{-1}(A)(y) & = \mathcal{K}^{-1}(\bigvee_{x\in X}(A(x)\odot \top_x))(y) = \bigwedge_{x\in X}(A(x)\to \mathcal{K}^{-1}(\top_x)(y))\\ & = \bigvee_{x\in X}(A(x)\to \mathcal{K}(\top_y)(x)).\end{array}$$

(2) For $A = \bigwedge_{x\in X}(A^*(x)\to \top^*_x)$, by Theorem 9(2), we have

$$\begin{array}{ll}\mathcal{M}(A)(y) & = \mathcal{M}(\bigwedge_{x\in X}(A^*(x)\to \top^*_x))(y) = \bigvee_{x\in X}(A^*(x)\odot \mathcal{M}(\top^*_x)(y))\\ \mathcal{M}^{-1}(A)(y) & = \bigvee_{x\in X}(A^*(x)\odot \mathcal{M}^{-1}(\top^*_x)(y))\end{array}$$

(3) $\mathcal{M}(\top)(y) = \bigvee_x(\top^*(x)\odot \mathcal{M}(\top^*_x)) = \bot$ and other cases are similarly proved.

(4) By Lemma 10 (11), we have

$$\begin{array}{l}(\mathcal{K}(A^*)(y))^* = \Big(\bigwedge_{x\in X}(A^*(x)\to \mathcal{K}(\top_x)(y)\Big)^*\\ = \bigvee_{x\in X}(A^*(x)\odot \mathcal{K}^*(\top_x)(y)) = \bigvee_{x\in X}(A^*(x)\odot \mathcal{M}(\top^*_x)(y))\\ = \mathcal{M}(\bigwedge_{x\in X}(A^*(x)\to \top^*_x)(y) = \mathcal{M}(A)(y).\end{array}$$

$$\begin{array}{l}(\mathcal{M}(A^*)(y))^* = \Big(\bigvee_{x\in X}(A(x)\odot \mathcal{M}(\top^*_x)(y))\Big)^* = \bigwedge_{x\in X}(A(x)\to \mathcal{M}^*(\top^*_x)(y))\\ = \bigwedge_{x\in X}(A(x)\to \mathcal{K}(\top_x)(y)) = \mathcal{K}(A)(y).\end{array}$$

(5) Since $\alpha\odot(\alpha\to A(x))\odot(A(x)\to \mathcal{K}(\top_x)(y)) \le A(x)\odot(A(x)\to \mathcal{K}(\top_x)(y)) \le \mathcal{K}(\top_x)(y)$ iff $\alpha\odot(A(x)\to \mathcal{K}(\top_x)(y)) \le (\alpha\to A(x))\to \mathcal{K}(\top_x)(y)$, then $\alpha\odot\mathcal{K}(A)\le \mathcal{K}(\alpha\to A)$.

Since $\alpha\odot(\alpha\to A^*)\odot\mathcal{M}(\top^*_x)(y) \le A^*\odot\mathcal{M}(\top^*_x)(y)$ iff $(\alpha\to A^*)\odot \mathcal{M}(\top^*_x)(y)\le \alpha\to A^*\odot\mathcal{M}(\top^*_x)(y)$, then $\mathcal{M}(\alpha\odot A)\le \alpha\to\mathcal{M}(A)$.

(6) By (4), $\mathcal{M}(\top_x\to\alpha)(z) = \mathcal{K}^*(\top_x\odot\alpha^*)(z)$ and

$$\begin{array}{l}\mathcal{M}(\top_x\to\alpha)(z) = \bigvee_{y\in X}(\mathcal{M}(\top^*_y)(z)\odot(\top_x\to\alpha)^*(y))\\ = \bigvee_{y\in X}(\mathcal{M}(\top^*_y)(z)\odot(\top_x\odot\alpha^*)(y)) = \mathcal{M}(\top^*_x)(z)\odot\alpha^*\\ = \mathcal{K}^*(\top_x)(z)\odot\alpha^* = (\alpha^*\to\mathcal{K}(\top_x)(z))^* = \mathcal{K}^*(\top_x\odot\alpha^*)(z).\end{array}$$

(7) Since $A(y)\odot(A(y)\to\alpha)\le\alpha$ iff $A(y)\le(A(y)\to\alpha)\to\alpha$, we have

$$\bigwedge_{\alpha\in L}((A(y)\to\alpha)\to\alpha)\ge A(y).$$

Put $\alpha = A(y)$. Then $\bigwedge_{\alpha\in L}((A(y)\to\alpha)\to\alpha)\le(A(y)\to A(y))\to A(y)) = A(y)$. Hence $\bigwedge_{\alpha\in L}((A(y)\to\alpha)\to\alpha) = A(y)$.

(8)

$$\begin{aligned}
&\bigwedge_{\alpha\in L}(\mathcal{M}(A\odot\alpha^*)(x)\to\alpha)\\
&=\bigwedge_{\alpha\in L}(\bigvee_{x\in X}(\mathcal{M}(\top_x^*)(y)\odot(A\to\alpha)(x))\to\alpha)\\
&=\bigwedge_{\alpha\in L}\bigwedge_{x\in X}(\mathcal{M}(\top_x^*)(y)\to((A(x)\to\alpha)\to\alpha))\\
&=\bigwedge_{x\in X}(\mathcal{K}^*(\top_x)(y)\to\bigwedge_{\alpha\in L}((A(x)\to\alpha)\to\alpha))\\
&=\bigwedge_{x\in X}(\mathcal{K}^*(\top_x)(y)\to A(x))=\bigwedge_{x\in X}(A^*(x)\to\mathcal{K}(\top_x)(y))\\
&=\mathcal{K}(A^*)(x).
\end{aligned}$$

(9)

$$\begin{aligned}
\alpha\to\mathcal{K}(A)(z) &=\mathcal{K}(\alpha\odot A)(z)\\
&=\bigwedge_{x\in X}((\alpha\odot A)(x)\to\mathcal{K}(\top_x)(z))\\
&=\bigwedge_{x\in X}(\mathcal{K}^*(\top_x)(z)\to(\alpha\odot A)^*(x))\\
&=\bigvee_{x\in X}(\mathcal{K}^*(\top_x)(z)\odot A^*(x))\to\alpha^*\\
&=\bigvee_{x\in X}(\mathcal{M}(\top_x^*)(z)\odot A(x))\to\alpha^*\\
&=\mathcal{M}(A^*)(z)\to\alpha^*
\end{aligned}$$

(10) Since $(A(x)\to\alpha^*)\odot\mathcal{M}(\top_x)(y)\odot(\mathcal{M}(\top_x)(y)\to A(x))\le(A(x)\to\alpha^*)\odot A(x)\le\alpha^*$ iff $(A(x)\to\alpha^*)\odot\mathcal{M}(\top_x)(y)\le(\mathcal{M}(\top_x)(y)\to A(x))\to\alpha^*$, we have

$$\begin{aligned}
&\mathcal{M}(A\odot\alpha)=\bigvee_{x\in X}((A\odot\alpha)^*(x)\odot\mathcal{M}(\top_x^*)(y))\\
&=\bigvee_{x\in X}((A(x)\to\alpha^*)\odot\mathcal{M}(\top_x^*)(y))\le\bigvee_{x\in X}((\mathcal{M}(\top_x^*)(y)\to A(x))\to\alpha^*)\\
&=\bigvee_{x\in X}((\mathcal{K}^*(\top_x)(y)\to A(x))\to\alpha^*)\le(\bigwedge_{x\in X}(\mathcal{K}^*(\top_x)(y)\to A(x))\to\alpha^*\\
&=(\bigwedge_{x\in X}(A^*(x)\to\mathcal{K}(\top_x)(y)))\to\alpha=\mathcal{K}(A^*)\to\alpha^*.
\end{aligned}$$

(11) By Lemma 10 (12), we have

$$\begin{aligned}
e_{L^X}(B,\mathcal{K}(A)) &=\bigwedge_{y\in X}(B(y)\to\mathcal{K}(A)(y))\\
&=\bigwedge_{y\in X}(B(y)\to\bigwedge_{x\in X}(A(x)\to\mathcal{K}(\top_x)(y)))\\
&=\bigwedge_{y\in X}\bigwedge_{x\in X}(B(y)\to(A(x)\to\mathcal{K}(\top_x)(y)))\\
&=\bigwedge_{x\in X}(A(x)\to\bigwedge_{y\in X}(B(y)\to\mathcal{K}(\top_x)(y)))\\
&=e_{L^X}(A,\mathcal{K}^{-1}(B)).
\end{aligned}$$

$$\begin{aligned}
e_{L^X}(\mathcal{M}(A),B) &=\bigwedge_{y\in X}(\mathcal{M}(A)(y)\to B(y))\\
&=\bigwedge_{y\in X}(\bigvee_{x\in X}(A^*(x)\odot\mathcal{M}(\top_x^*)(y))\to B(y))\\
&=\bigwedge_{y\in X}\bigwedge_{x\in X}(\mathcal{M}(\top_x^*)(y)\to(A^*(x)\to B(y)))\\
&=\bigwedge_{y\in X}\bigwedge_{x\in X}(\mathcal{M}(\top_x^*)(y)\to(B^*(y)\to A(x)))\\
&=\bigwedge_{x\in X}(\bigvee_{y\in X}(B^*(y)\odot\mathcal{M}(\top_x^*)(y))\to A(x))\\
&=\bigwedge_{x\in X}(\mathcal{M}^{-1}(B)(y))\to A(x))\\
&=e_{L^X}(\mathcal{M}^{-1}(B),A).
\end{aligned}$$

(12)

$$\begin{aligned} e_{L^X}(B,\mathcal{K}(A)) &= \bigwedge_{y\in X}(B(y)\to \mathcal{K}(A)(y)) \\ &= \bigwedge_{y\in X}(B(y)\to \bigwedge_{x\in X}(A(x)\to \mathcal{K}(\top_x)(y))) \\ &= \bigwedge_{y\in X}\bigwedge_{x\in X}(B(y)\to (\mathcal{K}^*(\top_x)(y)\to A^*(x))) \\ &= \bigwedge_{x\in X}(\bigvee_{y\in X}(B(y)\odot \mathcal{K}^*(\top_x)(y))\to A^*(x)) \\ &= \bigwedge_{x\in X}(\bigvee_{y\in X}(B(y)\odot \mathcal{M}^{-1}(\top_y)(x))\to A^*(x)) \\ &= e_{L^X}(\mathcal{M}^{-1}(B^*),A^*). \end{aligned}$$

$$\begin{aligned} e_{L^X}(\mathcal{M}(A),B) &= \bigwedge_{y\in X}(\mathcal{M}(A)(y)\to B(y)) \\ &= \bigwedge_{y\in X}(\bigvee_{x\in X}(A^*(x)\odot \mathcal{M}(\top^*_x)(y))\to B(y)) \\ &= \bigwedge_{y\in X}\bigwedge_{x\in X}(A^*(x)\to (\mathcal{M}(\top^*_x)(y)\to B(y))) \\ &= \bigwedge_{y\in X}\bigwedge_{x\in X}(A^*(x)\to (B^*(y)\to \mathcal{M}^*(\top^*_x)(y))) \\ &= \bigwedge_{x\in X}(A^*(x)\to \bigwedge_{y\in X}(B^*(y)\to \mathcal{K}(\top_x)(y))) \\ &= e_{L^X}(A^*,\mathcal{K}^{-1}(B)). \end{aligned}$$

(12)

$$\begin{aligned} &\mathcal{K}(A)(y)\odot e_{L^X}(B,A) = \bigwedge_{x}(A(x)\to \mathcal{K}(\top_x)(y))\odot \bigwedge_{z\in X}(B(z)\to A(z)) \\ &\le \bigwedge_{x\in X}((A(x)\to \mathcal{K}(\top_x)(y))\odot (B(x)\to A(x)) \\ &\le \bigwedge_{x\in X}((B(x)\to \mathcal{K}(\top_x)(y)) = \mathcal{K}(B)(y). \end{aligned}$$

Thus, $e_{L^X}(B,A)\le \mathcal{K}(A)(y)\to \mathcal{K}(B)(y)$. Hence $e_{L^X}(B,A)\le e_{L^X}(\mathcal{K}(A),\mathcal{K}(B))$.

(13)

$$\begin{aligned} &\mathcal{M}^*(\top^*_x)(y)\odot A^*(x))\odot e_{L^X}(B,A) \\ &\le \mathcal{M}^*(\top^*_x)(y)\odot A^*(x))\odot (B(x)\to A(x)) \\ &= \mathcal{M}^*(\top^*_x)(y)\odot A^*(x))\odot (A^*(x)\to B^*)\le \mathcal{M}^*(\top^*_x)(y)\odot B^*(x). \end{aligned}$$

Thus,$e_{L^X}(B,A)\le \mathcal{M}^*(\top^*_x)(y)\odot A^*(x))\to \mathcal{M}^*(\top^*_x)(y)\odot B^*(x)$. Hence

$$e_{L^X}(A,B)\le e_{L^X}(\mathcal{M}(B),\mathcal{M}(A)).$$

□

Theorem 12. *Let (L^X, e_{L^X}) be a fuzzy poset. Let $\mathcal{K},\mathcal{K}^{-1}: L^X\to L^X$ be join-meet preserving maps such that $\mathcal{K}^{-1}(\top_x)(y)=\mathcal{K}(\top_y)(x)$ for all $x,y\in X$. Let $\mathcal{M},\mathcal{M}^{-1}: L^X\to L^X$ be meet-join preserving maps such that $\mathcal{M}^{-1}(\top^*_x)(y)=\mathcal{M}(\top^*_y)(x)$ for all $x,y\in X$. For $x,y,z\in X$ and $A\in L^X$, we have the following properties.*

(1) If $\top_x\le \mathcal{M}(\top^*_x)$, then $A^*\le \mathcal{M}(A)$ and $A^*\le \mathcal{M}^{-1}(A)$.

(2) *If* $\mathcal{K}(\top_x) \leq \top^*_x$, *then* $\mathcal{K}(A) \leq A^*$ *and* $\mathcal{K}^{-1}(A) \leq A^*$.

(3) $\bigvee_{y\in X} \mathcal{K}^*(\top_x)(y)\odot\mathcal{K}^*(\top_y)(z) \leq \mathcal{K}^*(\top_x)(z)$ *for all* $x, z \in X$ *iff* $\mathcal{K}(\mathcal{K}^*(\top_x)) \geq \mathcal{K}(\top_x)$ *iff* $\mathcal{K}^{-1}(\mathcal{K}(\top_x)) \geq \mathcal{K}^*(\top_x)$ *iff* $\mathcal{K}(\mathcal{K}^*(A)) \geq \mathcal{K}(A)$ *iff* $\mathcal{K}^{-1}(\mathcal{K}(A)) \geq \mathcal{K}^*(A)$.

(4) $\bigvee_{x\in X} \mathcal{K}^*(\top_x)(y)\odot\mathcal{K}^*(\top_x)(z) \leq \mathcal{K}^*(\top_y)(z)$ *for all* $x, z \in X$ *iff* $\mathcal{K}(\mathcal{K}^{-1*}(\top_y)) \geq \mathcal{K}(\top_y)$ *iff* $\mathcal{K}^{-1}(\mathcal{K}(\top_y)) \geq \mathcal{K}^{-1*}(\top_y)$ *iff* $\mathcal{K}(\mathcal{K}^{-1*}(A)) \leq \mathcal{K}(A)$ *iff* $\mathcal{K}^{-1}(\mathcal{K}(A)) \leq \mathcal{K}^{-1*}(A)$.

(5) $\bigvee_{x\in X} \mathcal{K}(\top_y)(x)\odot\mathcal{K}(\top_z)(x) \leq \mathcal{K}(\top_y)(z)$ *for all* $x, z \in X$ *iff* $\mathcal{K}^{-1}(\mathcal{K}^*(\top_y)) \geq \mathcal{K}(\top_y)$ *iff* $\mathcal{K}(\mathcal{K}(\top_y)) \geq \mathcal{K}^*(\top_y)$ *iff* $\mathcal{K}^{-1}(\mathcal{K}^*(A)) \geq \mathcal{K}(A)$ *iff* $\mathcal{K}(\mathcal{K}(A)) \geq \mathcal{K}^*(A)$.

(6) $\bigvee_{y\in X} \mathcal{M}(\top^*_x)(y)\odot\mathcal{M}(\top^*_y)(z) \leq \mathcal{M}(\top^*_x)(z)$ *for all* $y, z \in X$ *iff* $\mathcal{M}(\mathcal{M}^*(\top^*_x)) \leq \mathcal{M}(\top^*_x)$ *iff* $\mathcal{M}^{-1}(\mathcal{M}^{-1*}(\top^*_x)) \leq \mathcal{M}^{-1}(\top^*_x)$ *iff* $\mathcal{M}(\mathcal{M}^*(A)) \leq \mathcal{M}(A)$ *iff* $\mathcal{M}^{-1}(\mathcal{M}^{-1*}(A)) \leq \mathcal{M}^{-1}(A)$.

(7) $\bigvee_{x\in X} \mathcal{M}(\top^*_x)(y) \odot \mathcal{M}(\top^*_x)(z) \leq \mathcal{M}(\top^*_y)(z)$ *for all* $x, y \in X$ *iff* $\mathcal{M}(\mathcal{M}^{-1*}(\top^*_x)) \leq \mathcal{M}(\top^*_x)$ *iff* $\mathcal{M}(\mathcal{M}^{-1*}(\top^*_z)) \leq \mathcal{M}^{-1}(\top^*_z)$ *iff* $\mathcal{M}(\mathcal{M}^{-1*}(A)) \leq \mathcal{M}(A)$ *iff* $\mathcal{M}(\mathcal{M}^{-1*}(A)) \leq \mathcal{M}^{-1}(A)$.

(8) $\bigvee_{x\in X} \mathcal{M}(\top^*_y)(x) \odot \mathcal{M}(\top^*_z)(x) \leq \mathcal{M}(\top^*_y)(z)$ *for all* $y, z \in X$ *iff* $\mathcal{M}^{-1}(\mathcal{M}^*(\top^*_z)) \leq \mathcal{M}^{-1}(\top^*_z)$ *iff* $\mathcal{M}^{-1}(\mathcal{M}^*(\top^*_y)) \leq \mathcal{M}(\top^*_y)$ *iff* $\mathcal{M}^{-1}(\mathcal{M}^*(A)) \leq \mathcal{M}^{-1}(A)$ *iff* $\mathcal{M}^{-1}(\mathcal{M}^*(A)) \leq \mathcal{M}(A)$.

(9) *If* $\mathcal{K}^*(\top_x)(y) = \mathcal{M}(\top^*_x)(y)$ *for all* $x, y \in X$, *then* $\bigvee_{y\in X} \mathcal{K}^*(\top_x)(y) \odot \mathcal{K}^*(\top_y)(z) \leq \mathcal{K}^*(\top_x)(z)$ *for all* $x, z \in X$ *iff* $\mathcal{M}(\mathcal{K}(\top_x)) \leq \mathcal{K}^*(\top_x)$ *iff* $\mathcal{M}^{-1}(\mathcal{K}^{-1}(\top_z)) \leq \mathcal{K}^{-1*}(\top_z)$ *iff* $\mathcal{M}(\mathcal{K}(A)) \leq \mathcal{K}^*(A)$ *iff* $\mathcal{M}^{-1}(\mathcal{K}^{-1}(A)) \leq \mathcal{K}^{-1*}(A)$.

(10) *If* $\mathcal{K}^*(\top_x)(y) = \mathcal{M}(\top^*_x)(y)$ *for all* $x, y \in X$, *then* $\bigvee_{x\in X} \mathcal{K}^*(\top_x)(y) \odot \mathcal{K}^*(\top_x)(z) \leq \mathcal{K}^*(\top_y)(z)$ *for all* $x, z \in X$ *iff* $\mathcal{M}(\mathcal{K}^{-1}(\top_y)) \leq \mathcal{K}^*(\top_y)$ *iff* $\mathcal{M}(\mathcal{K}^{-1}(\top_z)) \leq \mathcal{K}^{-1*}(\top_z)$ *iff* $\mathcal{M}(\mathcal{K}^{-1}(A)) \leq \mathcal{K}^*(A)$ *iff* $\mathcal{M}(\mathcal{K}^{-1}(A)) \leq \mathcal{K}^{-1*}(A)$.

(11) *If* $\mathcal{K}^*(\top_x)(y) = \mathcal{M}(\top^*_x)(y)$ *for all* $x, y \in X$, *then* $\bigvee_{x\in X} \mathcal{K}^*(\top_y)(x) \odot \mathcal{K}^*(\top_z)(x) \leq \mathcal{K}^*(\top_y)(z)$ *for all* $x, z \in X$ *iff* $\mathcal{M}^{-1}(\mathcal{K}(\top_y)) \leq \mathcal{K}^*(\top_y)$ *iff* $\mathcal{M}^{-1}(\mathcal{K}(\top_z)) \leq \mathcal{K}^{-1*}(\top_z)$ *iff* $\mathcal{M}^{-1}(\mathcal{K}(A)) \leq \mathcal{K}^*(A)$ *iff* $\mathcal{M}^{-1}(\mathcal{K}(A)) \leq \mathcal{K}^{-1*}(A)$.

Proof. (1) For $A = \bigwedge_{x\in X}(A^*(x) \to \top^*_x)$, we have

$$\begin{aligned}\mathcal{M}(A)(y) &= \mathcal{M}(\textstyle\bigwedge_{x\in X}(A^*(x) \to \top^*_x))(y) = \bigvee_{x\in X}(A^*(x) \odot \mathcal{M}(\top^*_x)(y)) \\ &\geq \textstyle\bigvee_{x\in X}(A^*(x) \odot \top^*_x(y)) = A^*(y).\end{aligned}$$

Similarly, we have $\mathcal{M}^{-1}(A) \geq A^*$ for each $A \in L^X$.

(2) For $A = \bigvee_{x\in X}(A(x) \odot \top_x)$, we have

$$\begin{aligned}\mathcal{K}(A)(y) &= \mathcal{K}(\textstyle\bigvee_{x\in X}(A(x) \odot \top_x))(y) = \bigwedge_{x\in X}(A(x) \to \mathcal{K}(\top_x)(y)) \\ &\leq \textstyle\bigwedge_{x\in X}(A(x) \to \top^*_x(y)) = A^*(y).\end{aligned}$$

Similarly, we have $\mathcal{K}^{-1}(A) \geq A^*$ for each $A \in L^X$.

(3) Since $\bigvee_{y\in X}\mathcal{K}^*(\top_x)(y)\odot\mathcal{K}^*(\top_y)(z)\le\mathcal{K}^*(\top_x)(z)$ for all $x,z\in X$, then $\bigwedge_{y\in X}\mathcal{K}^*(\top_x)(y)\to\mathcal{K}(\top_y)(z)=\mathcal{K}(\bigvee_{y\in X}\mathcal{K}^*(\top_x)(y)\odot\top_y)=\mathcal{K}(\mathcal{K}^*(\top_x))(z)\ge\mathcal{K}(\top_x)(z)$ for all $x,z\in X$.

Since $\bigvee_{y\in X}\mathcal{K}^*(\top_x)(y)\odot\mathcal{K}^*(\top_y)(z)\le\mathcal{K}^*(\top_x)(z)$ for all $x,z\in X$, then $\mathcal{K}^*(\top_x)(y)\le\mathcal{K}^*(\top_y)(z)\to\mathcal{K}^*(\top_x)(z)$.

$\mathcal{K}^*(\top_x)(y)\le\bigwedge_{y\in X}(\mathcal{K}^*(\top_y)(z)\to\mathcal{K}^*(\top_x)(z))=\bigwedge_{y\in X}(\mathcal{K}(\top_x)(z)\to\mathcal{K}(\top_y)(z))=\bigwedge_{y\in X}(\mathcal{K}(\top_x)(z)\to\mathcal{K}^{-1}(\top_z)(y))=\mathcal{K}^{-1}(\mathcal{K}(\top_x))(y)$.

$$\begin{aligned}\mathcal{K}(\mathcal{K}^*(A))(z) &= \mathcal{K}(\bigvee_{y\in X}(\mathcal{K}^*(A)(y)\odot\top_y)(z)=\bigwedge_{y\in X}(\mathcal{K}^*(A)(y)\to\mathcal{K}(\top_x)(z)\\ &= \bigwedge_{y\in X}((\bigwedge_{x\in X}(A(x)\to\mathcal{K}(\top_x)(y)))^*\to\mathcal{K}(\top_y)(z)\\ &= \bigwedge_{y\in X}(\bigvee_{x\in X}(A(x)\odot\mathcal{K}^*(\top_x)(y))\to\mathcal{K}(\top_y)(z)\\ &= \bigwedge_{y\in X}\bigwedge_{x\in X}(\mathcal{K}^*(\top_x)(y)\odot\mathcal{K}^*(\top_y)(z)\to A^*(x))\\ &\ge \bigwedge_{y\in X}(\mathcal{K}^*(\top_x)(z)\to A^*(x))\\ &= \mathcal{K}(A)(z).\end{aligned}$$

$$\begin{aligned}\mathcal{K}^{-1}(\mathcal{K}(A))(z) &= \mathcal{K}^{-1}(\bigvee_{y\in X}(\mathcal{K}(A)(y)\odot\top_y)(z)=\bigwedge_{y\in X}(\mathcal{K}(A)(y)\\ &\qquad\to\mathcal{K}^{-1}(\top_y)(z))\\ &= \bigwedge_{y\in X}((\bigwedge_{x\in X}(A(x)\to\mathcal{K}(\top_x)(y)))\to\mathcal{K}^{-1}(\top_y)(z)\\ &= \bigwedge_{y\in X}(\mathcal{K}^{-1*}(\top_y)(z)\to(\bigvee_{x\in X}(A(x)\odot\mathcal{K}^*(\top_x)(y))))\end{aligned}$$

Since

$$\begin{aligned}&\mathcal{K}^*(\top_x)(z)\odot\mathcal{K}^*(\top_z)(y)\odot(\mathcal{K}^*(\top_x)(y)\to A^*(x))\\ &\le\mathcal{K}^*(\top_x)(y)\odot(\mathcal{K}^*(\top_x)(y)\to A^*(x))\le A^*(x)\\ &\text{iff }\ \mathcal{K}^*(\top_z)(y)\odot(\mathcal{K}^*(\top_x)(y)\to A^*(x))\le\mathcal{K}^*(\top_x)(z)\to A^*(x)\end{aligned}$$

$$\begin{aligned}\Big(\mathcal{K}^{-1}(\mathcal{K}(A))(z)\Big)^* &= \Big(\bigwedge_{y\in X}(\mathcal{K}^{-1*}(\top_y)(z)\to(\bigvee_{x\in X}(A(x)\odot\mathcal{K}^*(\top_x)(y))))\Big)^*\\ &= \bigvee_{y\in X}(\mathcal{K}^{-1*}(\top_y)(z)\odot\bigwedge_{x\in X}(\mathcal{K}^*(\top_x)(y)\to A^*(x)))\\ &\le \bigvee_{y\in X}\bigvee_{x\in X}(\mathcal{K}^{-1*}(\top_y)(z)\odot(\mathcal{K}^*(\top_x)(y)\to A^*(x)))\\ &\le \bigvee_{x\in X}(\mathcal{K}^*(\top_x)(z)\to A^*(x))=\mathcal{K}(A)(z)\end{aligned}$$

Other cases are similarly proved.

(4) Since $\bigvee_{x\in X}\mathcal{K}^*(\top_x)(y)\odot\mathcal{K}^*(\top_x)(z)\le\mathcal{K}^*(\top_x)(z)$ for all $x,z\in X$, then $\bigwedge_{x\in X}\mathcal{K}^*(\top_x)(y)\to\mathcal{K}(\top_x)(z)=\mathcal{K}(\bigvee_{x\in X}\mathcal{K}^{-1*}(\top_y)(x)\odot\top_x)(z)=\mathcal{K}(\mathcal{K}^{-1*}(\top_y))(z)\ge\mathcal{K}(\top_y)(z)$ for all $y,z\in X$.

Since $\bigvee_{y\in X}\mathcal{K}^*(\top_x)(y)\odot\mathcal{K}^*(\top_x)(z)\le\mathcal{K}^*(\top_y)(z)$ for all $x,z\in X$, then $\mathcal{K}^*(\top_x)(y)\le\mathcal{K}^*(\top_x)(z)\to\mathcal{K}^*(\top_y)(z)$.

$\mathcal{K}^*(\top_x)(y)\le\bigwedge_{z\in X}(\mathcal{K}^*(\top_x)(z)\to\mathcal{K}^*(\top_y)(z))=\bigwedge_{y\in X}(\mathcal{K}(\top_y)(z)$

$\to \mathcal{K}(\top_x)(z)) = \bigwedge_{y\in X}(\mathcal{K}(\top_y)(z) \to \mathcal{K}^{-1}(\top_z)(x)) = \mathcal{K}^{-1}(\mathcal{K}(\top_y))(x)$.

$$\begin{aligned}
\mathcal{K}(\mathcal{K}^{-1*}(A))(z) &= \mathcal{K}(\textstyle\bigvee_{y\in X}(\mathcal{K}^{-1*}(A)(y)\odot \top_y)(z) = \bigwedge_{y\in X}(\mathcal{K}^{-1*}(A)(y) \\
&\quad \to \mathcal{K}(\top_x)(z) \\
&= \textstyle\bigwedge_{y\in X}((\bigwedge_{x\in X}(A(x)\to \mathcal{K}^{-1}(\top_x)(y)))^* \to \mathcal{K}(\top_y)(z) \\
&= \textstyle\bigwedge_{y\in X}(\bigvee_{x\in X}(A(x)\odot \mathcal{K}^{-1*}(\top_x)(y)) \to \mathcal{K}(\top_y)(z) \\
&= \textstyle\bigwedge_{x\in X}\bigwedge_{y\in X}(\mathcal{K}^*(\top_y)(x)\odot \mathcal{K}^*(\top_y)(z)\to A^*(x)) \\
&\geq \textstyle\bigwedge_{x\in X}(\mathcal{K}^*(\top_x)(z)\to A^*(x)) \\
&= \mathcal{K}(A)(z).
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}^{-1}(\mathcal{K}(A))(z) &= \mathcal{K}^{-1}(\textstyle\bigvee_{y\in X}(\mathcal{K}(A)(y)\odot \top_y)(z) = \bigwedge_{y\in X}(\mathcal{K}(A)(y) \\
&\quad \to \mathcal{K}^{-1}(\top_y)(z)) \\
&= \textstyle\bigwedge_{y\in X}((\bigwedge_{x\in X}(A(x)\to \mathcal{K}(\top_x)(y))) \to \mathcal{K}^{-1}(\top_y)(z) \\
&= \textstyle\bigwedge_{y\in X}(\mathcal{K}^{-1*}(\top_y)(z)\to (\bigvee_{x\in X}(A(x)\odot \mathcal{K}^*(\top_x)(y))))
\end{aligned}$$

Since

$$\begin{aligned}
&\mathcal{K}^*(\top_z)(x)\odot \mathcal{K}^*(\top_z)(y)\odot (\mathcal{K}^*(\top_x)(y)\to A^*(x)) \\
&\leq \mathcal{K}^*(\top_x)(y)\odot (\mathcal{K}^*(\top_x)(y)\to A^*(x)) \leq A^*(x) \\
&\text{iff } \mathcal{K}^*(\top_z)(y)\odot (\mathcal{K}^*(\top_x)(y)\to A^*(x)) \leq \mathcal{K}^*(\top_z)(x)\to A^*(x)
\end{aligned}$$

$$\begin{aligned}
\Big(\mathcal{K}^{-1}(\mathcal{K}(A))(z)\Big)^* &= \Big(\textstyle\bigwedge_{y\in X}(\mathcal{K}^{-1*}(\top_y)(z)\to (\bigvee_{x\in X}(A(x)\odot \mathcal{K}^*(\top_x)(y))))\Big)^* \\
&= \textstyle\bigvee_{y\in X}(\mathcal{K}^{-1*}(\top_y)(z)\odot \bigwedge_{x\in X}(\mathcal{K}^*(\top_x)(y)\to A^*(x))) \\
&\leq \textstyle\bigvee_{y\in X}\bigvee_{x\in X}(\mathcal{K}^{-1*}(\top_y)(z)\odot (\mathcal{K}^*(\top_x)(y)\to A^*(x))) \\
&\leq \textstyle\bigvee_{x\in X}(\mathcal{K}^*(\top_z)(x)\to A^*(x)) = \mathcal{K}^{-1}(A)(z)
\end{aligned}$$

Other cases are similarly proved.
(5) It is similarly proved as (4).
(6) For $\mathcal{M}(\top^*_x) = \bigwedge_{y\in X}(\mathcal{M}^*(\top^*_x)(y)\to \top^*_y)$, we have

$$\begin{aligned}
&\textstyle\bigvee_{y\in X}\mathcal{M}(\top^*_x)(y)\odot \mathcal{M}(\top^*_y)(z) \leq \mathcal{M}(\top^*_x)(z) \\
&\text{iff } \mathcal{M}(\textstyle\bigwedge_{y\in X}(\mathcal{M}(\top^*_x)(y)\to \top^*_y)(z) \leq \mathcal{M}(\top^*_x)(z) \\
&\text{iff } \mathcal{M}(\mathcal{M}^*(\top^*_x))(z) \leq \mathcal{M}(\top^*_x)(z)
\end{aligned}$$

$$\begin{aligned}
&\textstyle\bigvee_{y\in X}\mathcal{M}(\top^*_x)(y)\odot \mathcal{M}(\top^*_y)(z) \leq \mathcal{M}(\top^*_x)(z) \\
&\text{iff } \textstyle\bigvee_{y\in X}\mathcal{M}^{-1}(\top^*_y)(x)\odot \mathcal{M}^{-1}(\top^*_z)(y) \leq \mathcal{M}(\top^*_x)(z) \\
&\text{iff } \mathcal{M}^{-1}(\textstyle\bigwedge_{y\in X}(\mathcal{M}^{-1}(\top^*_z)(y)\to \top^*_y)(x) \leq \mathcal{M}^{-1}(\top^*_z)(x) \\
&\text{iff } \mathcal{M}^{-1}(\mathcal{M}^{-1*}(\top^*_z))(x) \leq \mathcal{M}^{-1}(\top^*_z)(x)
\end{aligned}$$

$$
\begin{array}{ll}
\mathcal{M}(\mathcal{M}^*(A))(z) & = \mathcal{M}(\bigwedge_{y\in X}(\mathcal{M}(A)(y) \to \top_y^*)) \\
& = \bigvee_{y\in X}(\mathcal{M}(A)(y) \odot \mathcal{M}(\top_y^*)(z)) \\
& = \bigvee_{y\in X}(\bigvee_{x\in X}(A^*(x) \odot \mathcal{M}(\top_x^*)(y)) \odot \mathcal{M}(\top_y^*)(z)) \\
& \leq \bigvee_{x\in X}(A^*(x) \odot \mathcal{M}(\top_x^*)(z)) = \mathcal{M}(A)(z))
\end{array}
$$

(7) For $\mathcal{M}(\top_x^*) = \bigwedge_{y\in X}(\mathcal{M}^*(\top_x^*)(y) \to \top_y^*)$, we have

$$
\begin{array}{l}
\bigvee_{y\in X} \mathcal{M}(\top_x^*)(y) \odot \mathcal{M}(\top_x^*)(z) \leq \mathcal{M}(\top_y^*)(z) \\
\text{iff } \mathcal{M}(\bigwedge_{x\in X}(\mathcal{M}^{-1}(\top_y^*)(z) \to \top_x^*)(z) \leq \mathcal{M}(\top_y^*)(z) \\
\text{iff } \mathcal{M}(\mathcal{M}^{-1*}(\top_y^*))(z) \leq \mathcal{M}(\top_y^*)(z)
\end{array}
$$

$$
\begin{array}{l}
\bigvee_{x\in X} \mathcal{M}(\top_x^*)(y) \odot \mathcal{M}(\top_x^*)(z) \leq \mathcal{M}(\top_y^*)(z) \\
\text{iff } \bigvee_{x\in X} \mathcal{M}(\top_x^*)(y) \odot \mathcal{M}^{-1}(\top_z^*)(x) \leq \mathcal{M}(\top_y^*)(z) \\
\text{iff } \mathcal{M}(\bigwedge_{x\in X}(\mathcal{M}^{-1}(\top_z^*)(x) \to \top_x^*)(y) \leq \mathcal{M}^{-1}(\top_z^*)(y) \\
\text{iff } \mathcal{M}(\mathcal{M}^{-1*}(\top_z^*))(y) \leq \mathcal{M}^{-1}(\top_z^*)(y)
\end{array}
$$

$$
\begin{array}{ll}
\mathcal{M}(\mathcal{M}^{-1*}(A))(z) & = \mathcal{M}(\bigwedge_{y\in X}(\mathcal{M}^{-1}(A)(y) \to \top_y^*)) \\
& = \bigvee_{y\in X}(\mathcal{M}^{-1}(A)(y) \odot \mathcal{M}(\top_y^*)(z)) \\
& = \bigvee_{y\in X}(\bigvee_{x\in X}(A^*(x) \odot \mathcal{M}^{-1}(\top_x^*)(y)) \odot \mathcal{M}(\top_y^*)(z)) \\
& \leq \bigvee_{x\in X}(A^*(x) \odot \mathcal{M}(\top_x^*)(z)) = \mathcal{M}(A)(z))
\end{array}
$$

(8) It is similarly proved.

(9) Since $\bigvee_{y\in X} \mathcal{K}^*(\top_x)(y)\odot\mathcal{M}(\top_y^*)(z) = \mathcal{M}(\bigwedge_{y\in X}(\mathcal{K}^*(\top_x)(y) \to \top_y))(z) = \mathcal{M}(\mathcal{K}(\top_x))(z)$, we have

$$
\begin{array}{l}
\bigvee_{y\in X} \mathcal{K}^*(\top_x)(y) \odot \mathcal{K}^*(\top_y)(z) \leq \mathcal{K}^*(\top_x)(z) \\
\text{iff } \bigvee_{y\in X} \mathcal{K}^*(\top_x)(y) \odot \mathcal{M}(\top_y^*)(z) \leq \mathcal{K}^*(\top_x)(z) \\
\text{iff } \mathcal{M}(\mathcal{K}(\top_x))(z) \leq \mathcal{K}^*(\top_x)(z).
\end{array}
$$

Since $\bigvee_{y\in X} \mathcal{K}^{-1*}(\top_z)(y)\odot\mathcal{M}^{-1}(\top_y^*)(x) = \mathcal{M}^{-1}(\bigwedge_{y\in X}(\mathcal{K}^{-1*}(\top_z)(y) \to \top_y^*))(x) = \mathcal{M}^{-1}(\mathcal{K}^{-1}(\top_z))(x)$, we have

$$
\begin{array}{l}
\bigvee_{y\in X} \mathcal{K}^*(\top_x)(y) \odot \mathcal{K}^*(\top_y)(z) \leq \mathcal{K}^*(\top_x)(z) \\
\text{iff } \bigvee_{y\in X} \mathcal{K}^{-1*}(\top_z)(y) \odot \mathcal{K}^{-1*}(\top_y)(x) \leq \mathcal{K}^*(\top_x)(z) \\
\text{iff } \bigvee_{y\in X} \mathcal{K}^*(\top_x)(y) \odot \mathcal{M}(\top_y^*)(z) \leq \mathcal{K}^{-1*}(\top_z)(x) \\
\text{iff } \mathcal{M}^{-1}(\mathcal{K}^{-1}(\top_z))(x) \leq \mathcal{K}^{-1*}(\top_z)(x).
\end{array}
$$

$$\begin{aligned}\mathcal{M}(\mathcal{K}(A))(z) &= \mathcal{M}(\bigwedge_{y\in X}(\mathcal{K}^*(A)(y)\to \top_y^*))(z)\\ &= \bigvee_{y\in X}(\mathcal{K}^*(A)(y)\odot \mathcal{M}(\top_y^*)(z))\\ &= \bigvee_{y\in X}(\bigvee_{x\in X}(A(x)\odot \mathcal{K}^*(\top_x)(y))\odot \mathcal{K}^*(\top_y)(z))\\ &\le \bigvee_{x\in X}(A(x)\odot \mathcal{K}^*(\top_x)(z)) = \mathcal{K}^*(A)(z)).\end{aligned}$$

(10)

$$\begin{aligned}&\bigvee_{y\in X}\mathcal{K}^*(\top_x)(y)\odot \mathcal{K}^*(\top_x)(z)\le \mathcal{K}^*(\top_y)(z)\\ &\text{iff } \bigvee_{y\in X}\mathcal{K}^{-1*}(\top_y)(x)\odot \mathcal{M}(\top_x^*)(y)\le \mathcal{K}^*(\top_y)(z)\\ &\text{iff } \mathcal{M}(\mathcal{K}^{-1}(\top_y))(z)\le \mathcal{K}^*(\top_y)(z).\end{aligned}$$

$$\begin{aligned}&\bigvee_{y\in X}\mathcal{K}^*(\top_x)(y)\odot \mathcal{K}^*(\top_x)(z)\le \mathcal{K}^*(\top_y)(z)\\ &\text{iff } \bigvee_{y\in X}\mathcal{K}^{-1*}(\top_z)(x)\odot \mathcal{M}(\top_x^*)(y)\le \mathcal{K}^*(\top_y)(z)\\ &\text{iff } \mathcal{M}(\mathcal{K}^{-1}(\top_z))(y)\le \mathcal{K}^{-1*}(\top_z)(y).\end{aligned}$$

Other cases and (11) are similarly proved.

□

Example 13. Let $(L=[0,1],\odot,\to,^*)$ be a complete residuated lattice with the law of double negation defined by

$$x\odot y=(x+y-1)\vee 0,\ \ x\to y=(1-x+y)\wedge 1,\ x^*=1-x.$$

Let $X=\{x,y,z\}$ and $A,B\in L^X$ as follows:

$$A(x)=0.9, A(y)=0.8, A(z)=0.3,\ \ B(x)=0.3, A(y)=0.7, A(z)=0.8$$

Define $\mathcal{K}^*(1_x)(y)=\mathcal{M}(1_x^*)(y)$ as follows

$$\begin{pmatrix} \mathcal{K}^*(1_x)(x)=1 & \mathcal{K}^*(1_x)(y)=0.8 & \mathcal{K}^*(1_x)(z)=0.6\\ \mathcal{K}^*(1_y)(x)=0.7 & \mathcal{K}^*(1_y)(y)=1 & \mathcal{K}^*(1_y)(z)=0.3\\ \mathcal{K}^*(1_z)(x)=0.5 & \mathcal{K}^*(1_z)(y)=0.6 & \mathcal{K}^*(1_z)(z)=1 \end{pmatrix}$$

(1) We have $\bigvee_{y\in X}(\mathcal{K}^*(1_x)(y)\odot \mathcal{K}^*(1_y)(z)=\mathcal{K}^*(1_x)(z)$ and $1_x\le \mathcal{K}^*(1_x)$ for all $x,y\in X$. Since $\mathcal{K}(A)(y)=\bigwedge_{x\in X}(A(x)\to \mathcal{K}(1_x)(y))$ and $\mathcal{M}(A)(y)=\bigvee_{x\in X}(\mathcal{M}(1_x^*)(y)\odot A^*(x))$, we have

$$\mathcal{K}(A)=(0.1,0.2,0.5),\ \mathcal{K}(B)=(0.6,0.3,0.2),$$

$$\mathcal{K}(A^*)=(0.8,0.7,0.3),\ \mathcal{K}(B^*)=(0.3,0.5,0.7),$$

$$\mathcal{M}(A)=(0.2,0.3,0.7),\ \mathcal{M}(B)=(0.7,0.5,0.3),$$

$$\mathcal{M}(A^*) = (0.9, 0.8, 0.5),\ \mathcal{M}(B^*) = (0.4, 0.7, 0.8).$$

Furthermore, by Theorem 12(3,6,9), $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A)$, $\mathcal{M}(\mathcal{M}^*(A)) = \mathcal{M}(A)$ and $\mathcal{M}(\mathcal{K}(A)) = \mathcal{K}^*(A)$, for all $A \in X$. For $\mathcal{K}^*(1_x) = (1, 0.8, 0.6)$, by Theorem 12(2,6), $\mathcal{K}(\mathcal{K}^*(1_x) = \mathcal{K}(1_x) = (0, 0.2, 0.4)$. For $\mathcal{M}(1_x^*) = (1, 0.8, 0.6)$, by Theorem 12(2,6), $\mathcal{M}(\mathcal{M}^*(1_x^*) = \mathcal{M}(1_x^*)$.

(2) We obtain $\mathcal{K}^{-1*}(1_x)(y) = \mathcal{M}^{-1}(1_y^*)(x) = \mathcal{K}^*(1_y)(x)$ as follows

$$\begin{pmatrix} \mathcal{K}^{-1*}(1_x)(x) = 1 & \mathcal{K}^{-1*}(1_x)(y) = 0.7 & \mathcal{K}^{-1*}(1_x)(z) = 0.5 \\ \mathcal{K}^{-1*}(1_y)(x) = 0.8 & \mathcal{K}^{-1*}(1_y)(y) = 1 & \mathcal{K}^{-1*}(1_y)(z) = 0.6 \\ \mathcal{K}^{-1*}(1_z)(x) = 0.6 & \mathcal{K}^{-1*}(1_z)(y) = 0.3 & \mathcal{K}^{-1*}(1_z)(z) = 1 \end{pmatrix}$$

We have $\bigvee_{y\in X}(\mathcal{K}^{-1*}(1_x)(y) \odot \mathcal{K}^{-1*}(1_y)(z) = \mathcal{K}^{-1*}(1_x)(z)$ and $1_x \leq \mathcal{K}^{-1*}(1_x)$ for all $x, y \in X$.

$$\mathcal{K}^{-1}(A) = (0.1, 0.2, 0.6),\ \mathcal{K}^{-1}(B) = (0.5, 0.3, 0.2).$$

$$\mathcal{K}^{-1}(A^*) = (0.7, 0.8, 0.3),\ \mathcal{K}^{-1}(B^*) = (0.3, 0.6, 0.8),$$

$$\mathcal{M}^{-1}(A) = (0.3, 0.2, 0.7),\ \mathcal{M}^{-1}(B) = (0.7, 0.4, 0.2),$$

$$\mathcal{M}^{-1}(A^*) = (0.9, 0.8, 0.4),\ \mathcal{M}^{-1}(B^*) = (0.5, 0.7, 0.8).$$

Furthermore, by Theorem 12(3,6,9), $\mathcal{K}^{-1}(\mathcal{K}(A)) = \mathcal{K}^*(A)$, $\mathcal{M}^{-1}(\mathcal{M}^{-1*}(A)) = \mathcal{M}^{-1}(A)$ and $\mathcal{M}^{-1}(\mathcal{K}^{-1}(A)) = \mathcal{K}^{-1*}(A)$, for all $A \in X$. For $\mathcal{K}^*(1_x) = (1, 0.8, 0.6)$, by Theorem 12(2,6), $\mathcal{K}^{-1}(\mathcal{K}(1_x) = \mathcal{K}^*(1_x)$. For $\mathcal{M}^{-1}(1_x^*) = (1, 0.7, 0.5)$, by Theorem 12(2,6), $\mathcal{M}^{-1}(\mathcal{M}^{-1*}(1_x^*) = \mathcal{M}^{-1}(1_x^*)$.

(3) Since $0.6 = \bigvee_{x\in X}(\mathcal{K}^*(1_x)(y) \odot \mathcal{K}^*(1_x)(z) \not\leq \mathcal{K}^*(1_y)(z) = 0.3$, then

$$(0.3, 0, 0.7) = \mathcal{K}(1_y) \not\leq \mathcal{K}(\mathcal{K}^{-1*}(1_y)) = (0.2, 0, 0.4)$$

$$(0.8, 1, 0.6) = \mathcal{K}^{-1*}(1_y) \not\leq \mathcal{K}^{-1}(\mathcal{K}(1_y)) = (0.7, 1, 0.3)$$

$$(0.7, 1, 0.3) = \mathcal{K}^*(1_y) \not\geq \mathcal{M}(\mathcal{K}^{-1}(1_y)) = (0.8, 1, 0.6)$$

$$(0.7, 1, 0.3) = \mathcal{M}(1_y^*) \not\geq \mathcal{M}(\mathcal{M}^{-1}(1_y^*)) = (0.8, 1, 0.6).$$

(4) Since $0.6 = \bigvee_{x\in X}(\mathcal{K}(1_y)(x) \odot \mathcal{H}(1_z)(x) \not\leq \mathcal{K}(1_y)(z) = 0.3$, then

$$(0.3, 0, 0.7) = \mathcal{K}(1_y) \not\leq \mathcal{K}^{-}(\mathcal{K}^*(1_y)) = (0.2, 0, 0.4)$$

$$(0.7, 1, 0.3) = \mathcal{K}^*(1_y) \not\leq \mathcal{K}(\mathcal{K}(1_y)) = (0.7, 0.7, 0.3)$$

$$(0.7, 1, 0.3) = \mathcal{K}^*(1_y) \not\geq \mathcal{M}^{-1}(\mathcal{K}(1_y)) = (0.8, 1, 0.6)$$

$$(0.7, 1, 0.3) = \mathcal{M}(1_y^*) \not\geq \mathcal{M}^{-1}(\mathcal{M}^*(1_y^*)) = (0.8, 1, 0.6).$$

References

[1] R. Bělohlávek,*Fuzzy Relational Systems*, Kluwer Academic Publishers, New York , (2002).

[2] P. Hájek, *Metamathematices of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht (1998).

[3] Y.C. Kim, L-join meet approximation operators to appear *International Journal of Pure and Applied Mathematics.* **3(5)** (2013), 1193-1210.

[4] H. Lai, D. Zhang, Fuzzy preorder and fuzzy topology, *Fuzzy Sets and Systems,* **157** (2006), 1865-1885, **doi:** 10.1016/j.fss.2006.02.013.

[5] H. Lai, D. Zhang, Concept lattices of fuzzy contexts: Formal concept analysis vs. rough set theory, *Int. J. Approx. Reasoning,* **50** (2009), 695-707, **doi:** 10.1016/j.ijar.2008.12.002.

[6] Z. Pawlak, Rough sets, *Int. J. Comput. Inf. Sci.*, **11** (1982), 341-356.

[7] Z. Pawlak, Rough probability, *Bull. Pol. Acad. Sci. Math.*, **32**(1984), 607-615.

[8] A. M. Radzikowska, E.E. Kerre, A comparative study of fuzy rough sets, *Fuzzy Sets and Systems*, **126**(2002), 137-155, **doi:** 10.1016/so165-0114(01)00032-x.

[9] Y.H. She, G.J. Wang, An axiomatic approach of fuzzy rough sets based on residuated lattices, *Computers and Mathematics with Applications*, **58** (2009), 189-201, **doi:** 10.1016/j.camwa.2009.03.100.

[10] Q. Y. Zhang, L. Fan, Continuity in quantitive domains, *Fuzzy Sets and Systems*, **154** (2005), 118-131, **doi:** 10.1016/j.fss.2005.01.007.

[11] Q. Y. Zhang, W. X. Xie, Fuzzy complete lattices, *Fuzzy Sets and Systems*, **160** (2009), 2275-2291, **doi:** 10.1016/j.fss.2008.12.001.

[12] Zhen Ming Ma, Bao Qing Hu, Topological and lattice structures of L-fuzzy rough set determined by lower and upper sets, *Information Sciences*, **218**(2013), 194-204, **doi:** 10.1016/j.ins.2012.06.029.