International Journal of Pure and Applied Mathematics
Volume 93 No. 1 2014, 7-21
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v93i1.2
ijpam.eu

ON CLASS PRESERVING AUTOMORPHISMS OF SOME GROUPS OF ORDER p^6

Shiv Narain[1 §], Ram Karan[2]
[1,2]Department of Mathematics
Kurukshetra University
Kurukshetra, 136 119, INDIA

Abstract: For an odd prime p there are 43 isoclinism families of groups of order p^6 given by James [8]. Let G be a group lying in these families.

We sort out those groups for which $Aut_c(G) = Inn(G)$. Let $Autcent(G)$ denotes the groups of all central automorphisms of G. We give an upper bound for $|Aut_c(G)|$ in terms of $|Aut_c(G) \cap Autcent(G)|$ and $|Aut_c(G/Z(G)|$.

AMS Subject Classification: 20D45, 20D15
Key Words: finite p-groups, inner automorphism, class preserving automorphism, central automorphism, isoclinism

1. Introduction

We consider G a finite p-group such that $|G| = p^6$ where p is an odd prime. Notations used are mostly standard, however for $x \in G$, x^G denotes the conjugacy class of the element x in G. An automorphism f is called class preserving if $f(x) \in x^G$ for all $x \in G$. The set of all class preserving automorphisms is denoted by $Aut_c(G)$ and $Inn(G)$ denotes the set of all inner automorphisms of G. $Aut_c(G)$ forms a normal subgroup of $Aut(G)$ and $Inn(G)$ forms a normal subgroup of $Aut_c(G)$. The quotient group $Aut_c(G)/Inn(G)$ is denoted by $Out_c(G)$. $Z(G)$ denotes the center of G and the Frattini subgroup

Received: August 20, 2013

§Correspondence author

of G is denoted by $\Phi(G)$. An automorphisms ϕ of G is said to be central if and only if it induces the identity automorphism on $G/Z(G)$, or equivalently, $g^{-1}\phi(g) \in Z(G)$ for all $g \in Z(G)$. The set of all central automorphisms of G, denoted by $Autcent(G)$, forms a normal subgroup of Aut(G). It is well known that $Autcent(G) = C_{Aut(G)}(Inn(G))$.

Let G be a group. A map $f : G \to G$ is called a co-cycle of G if $f(xy) = f(x)f(y)^x$, for all $x, y \in G$. Here $f(y)^x = xf(y)x^{-1}$. A co-cycle f of G is called a local co-boundary if for each $x \in G$ there exists an $a_x \in G$ such that $f(x) = {a_x}^{-1}{a_x}^x$. A co-cycle f of G is called a (global) co-boundary if there exists an $a \in G$ such that $f(x) = a^{-1}a^x$ for all $x \in G$. A group G is said to enjoy Hasse's principle if every local co-boundary is a co-boundary. Ono, T [12] proved that a group G enjoy's Hasses's principle if and only if every class preserving endomorphism is an inner automorphism. In the particular case when G is a finite group, G enjoys Hasse's principle if and only if $Aut_c(G) = Inn(G)$.

Every abelian group enjoys the Hasse's principle trivially. Ono, T. and Wada, H [13] proved that the symmetric group S_n and alternating group A_n of degree n enjoy Hasse's principle. They also proved that the dihedral group and the generalized quaternion group enjoys Hasse's principle. Further in this direction Kumar and Vermani [9] proved that a non-abelian finite p-group having a maximal subgroup which is cyclic enjoys Hasse's principle. They also proved that extra-special p-group enjoys Hasse's principle. In other notes (see [10], [11]) they showed that every p-group of order p^4 enjoys Hasse's principle. Fuma and Ninomiya [4] proved that a finite p-group having a cyclic subgroup of index p^2 enjoys Hasse's principle. Further using classification of groups of order p^5 (for an odd prime p) given by James [8], Yadav [15] showed that for an odd prime p every p-group of order p^5 except two isoclinism families enjoys Hasse's principle. We move a step ahead and using the classification given by James we try to sort out some of the groups of order p^6 (where p is an odd prime) for which $Aut_c(G) = Inn(G)$. We find an upper bound for $|Aut_c(G)|$ in terms of $|Aut_c(G) \cap Autcent(G)|$ and $|Aut_c(G/Z(G))|$.

2. Some Useful Results

For a group G and an abelian group K, Hom (G, K) represents the group of all homomorphisms from G to K. By $d(G)$ we mean the number of elements in a minimal generating set of G. A group G is flat if every finite conjugacy class is a coset of a normal subgroup. The flatness condition is weaker than that of Camina pair (see [2], [3]). Let G_f denotes the set of elements with finite conjugacy class in G. Tandra and Moran [14] showed that a group G is flat if and only if $[x, G]$ is a subgroup of G for all $x \in G_f$. A non abelian group G is said to be purely non-abelian, if it has no non-trivial abelian direct factor. It is easy to see that a group is purely non abelian if its center is contained in the Frattini subgroup. The following lemma follows from [1].

Lemma 2.1. *If G is a purely non abelian finite group, then* $|Autcent(G)| = |Hom(G/\gamma_2(G), Z(G))|$.

Lemma 2.2. *[15] If G is a finite p-group such that $Z(G) \subseteq [x, G]$, for all $x \in G - \gamma_2(G)$, then $|Aut_c(G)| \geq |Autcent(G)||G/Z_2(G)|$.*

The following lemma is proposition 14.4 of [7].

Lemma 2.3. *Let G be a finite group and H be an abelian normal subgroup of G such that G/H is cyclic. Then $Out_c(G) = 1$ i.e. $Aut_c(G) = Inn(G)$.*

Theorem 2.4. *[15] Let G be a finite p-group of class 2 such that $\gamma_2(G)$ is cyclic. Then $Out_c(G) = 1$.*

Lemma 2.5. *Let A, B and C be finite abelian groups. Then:*

(i) $Hom(A \times B, C) \cong Hom(A, C) \times Hom(B, C)$;

(ii) $Hom(A, B \times C) \cong Hom(A, B) \times Hom(A, C)$

Lemma 2.6. *Let C_m, C_n be cyclic groups of order m and n respectively. Then $Hom(C_m, C_n) \cong C_d$, where $d = g.c.d.(m, n)$.*

Lemma 2.7. *[14] Let G be a flat nilpotent group of order p^n having nilpotency class c. Then $|Z(G)| > p$ if n is even, and for any k, $1 \leq k \leq c - 1$, $|Z^{k+1}(G)/Z^k(G)| > p$.*

For nilpotent groups of class ≥ 2, we prove the following

Lemma 2.8. *If G is a nilpotent group of class ≥ 2, then $|Aut_c(G)| \leq |Aut_c(G) \cap Autcent(G)||Aut_c(G/Z(G))|$.*

Proof. Observe that every class preserving automorphism f in $Aut_c(G)$ induces a class preserving automorphism $\overline{f}$ in $Aut_c(G/Z(G))$. Then the map

$$\alpha : Aut_c(G) \to Aut_c(G/Z(G))$$

given by $\alpha(f) = \overline{f}$ is a homomorphism of groups with $Ker(\alpha) = Aut_c(G) \cap Autcent(G)$. Hence $Aut_c(G)/Aut_c(G) \cap Autcent(G) \cong$ a subgroup of $Aut_c(G/Z(G))$. Now it follows that

$$|Aut_c(G)| \leq |Aut_c(G) \cap Autcent(G)|.|Aut_c(G/Z(G))|.$$

Corollary 2.9. *If G is a nilpotent group of class ≥ 2, then $|Aut_c(G)| \leq |Autcent(G)|.|Aut_c(G/Z(G))|$.*

Note that if $Aut_c(G/Z(G)) = Inn(G/Z(G))$, then for each $\overline{f} \in Aut_c(G/Z(G))$, there exists an element $gZ \in G/Z(G)$ such that $\overline{f}(xZ) = g^{-1}xgZ$ for all $xZ \in G/Z$. Thus for each $\overline{f} \in Aut_c(G/Z(G))$, there exist some $T_g \in Aut_c(G)$ such that $\alpha(T_g) = \overline{f}$ and hence the homomorphism α is an epimorphism. But then $Aut_c(G)/Aut_c(G) \cap Autcent(G) \cong Inn(G/Z(G))$. Thus we have the following

Corollary 2.10. *If G is a nilpotent group of class ≥ 2 such that*

$$Aut_c(G/Z(G)) = Inn(G/Z(G)),$$

then

$$|Aut_c(G)| = |Aut_c(G) \cap Autcent(G)|.|G/Z_2(G)|.$$

Lemma 2.11. *Let G be a non-abelian finite p-group with $|Z(G)| = p$. Let $\{x_1, x_2, ...x_r\}$ be a minimal generating set for G. Suppose $Z(G)$ is not contained in $[x_i, G]$, for some i, $1 \leq i \leq r$. Then $Autcent(G)$ is not contained in $Aut_c(G)$.*

Proof. Let G be a non- abelian finite p-group with $|Z(G)| = p$. Then G is purely non-abelian and hence $|Autcent(G)| = |Hom(G/\gamma_2(G), Z(G))|$. Let $\{x_1, x_2, ...x_r\}$ be a minimal generating set for G. Then $G/\gamma_2(G) =< \overline{x_1} > \bigoplus < \overline{x_2} > \bigoplus \cdots < \overline{x_r} >$ and hence

$$\begin{aligned} |Autcent(G)| &= |Hom(\bigoplus_{i=1}^{r} \sum < \overline{x_i} >, Z(G))| \\ &= p^r \end{aligned}$$

Define $H_i = Z(G) \cap [x_i, G]$. Clearly H_i is a subgroup of $Z(G)$. Since $|Z(G)| = p$, it follows that either $H_i = \{e\}$ or $Z(G) \subseteq [x_i, G]$. Hence if $Z(G)$ is not contained in $[x_i, G]$, for some i, $1 \leq i \leq r$, then $H_i = \{e\}$. Let if possible, $Autcent(G) \subseteq Aut_c(G)$. Then every central automorphism is class preserving i.e. for each $\alpha \in Autcent(G)$ and for every $x \in G$, we have $x^{-1}\alpha(x) \in Z(G) \cap [x, G]$. In particular

$$\begin{aligned} x_i^{-1}\alpha(x_i) &\in Z(G) \cap [x_i, G] \\ &= H_i \\ &= \{e\} \end{aligned}$$

Thus $\alpha(x_i) = x_i$ for every $\alpha \in Autcent(G)$. But then

$$|Autcent(G)| = |Hom(G/\gamma_2(G), Z(G))|$$

$$
\begin{aligned}
&= |Hom(\bigoplus \sum_{j \boxminus i} < \overline{x_j} >, Z(G))| \\
&= p^k \qquad where\ k < r
\end{aligned}
$$

This is a contradiction and hence the lemma holds.

Corollary 2.12. *Let G be a non-abelian finite p-group with $|Z(G)| = p$. Let $\{x_1, x_2, ...x_r\}$ be a minimal generating set for G such that $Z(G)$ is not contained in $[x_i, G]$, for some i,$1 \le i \le r$. Then*

$$|Z_2(G)|/|Z(G)| \le |Aut_c(G) \cap Autcent(G)| < |Autcent(G)|$$

Proof. Let G be a group as defined above, then by previous lemma $|Aut_c(G) \cap Autcent(G)| < |Autcent(G)|$. Since $Autcent(G) = C_{Aut(G)}(Inn(G))$, therefore $Z(Inn(G)) = Inn(G) \cap Autcent(G) \subseteq Aut_c(G) \cap Autcent(G)$. Since $Inn(G) \cong G/Z(G)$, therefore $|Z(G/Z(G))| \le |Aut_c(G) \cap Autcent(G)|$. Hence $|Z_2(G)|/|Z(G)| \le |Aut_c(G) \cap Autcent(G)| < |Autcent(G)|$.

3. Isoclinic Groups

The concept of isoclinic groups was introduced by P.Hall [6]. Let X be a finite group and $\overline{X} = X/Z(X)$. Then commutation in X gives a well-defined map $\alpha_X : \overline{X} \times \overline{X} \mapsto \gamma_2(X)$ such that $\alpha_X(xZ(X), yZ(X)) = [x, y]$ for $(x, y) \in X \times X$. Two finite groups G and H are called isoclinic if there exist isomorphisms

$$\theta : G/Z(G) \to H/Z(H),$$

$$\phi : \gamma_2(G) \to \gamma_2(H),$$

such that $\phi[\alpha, \beta] = [\alpha', \beta']$ for all $\alpha, \beta \in G$, where $\alpha' Z(H) = \theta(\alpha(Z(G))$ and $\beta' Z(H) = \theta(\beta Z(G))$. The resulting pair (θ, ϕ) is called an isoclinism of G onto H.

Before we start our work we state the following theorems proved by Yadav [15].

Theorem 3.1. *Let G and H be two finite non-abelian isoclinic groups. Then $Aut_c(G) \cong Aut_c(H)$.*

Theorem 3.2. *Let G be a finite p-group of order p^5, where p is an odd prime. Then $Out_c(G) \neq 1$ if and only if G is isoclinic to the group in $\phi_7(1^5)$ and $\phi_{10}(1^5)$ the isoclinism family ϕ_7 and ϕ_{10} and in these two cases $|Aut_c(G)| = p^5$.*

From the theorem 3.1, we conclude that it is sufficient to calculate $Aut_c(G)$ for only one member from each isoclinism family and from the theorem 3.2, it is clear that if G be any group of order p^5 (p is an odd prime), from the isoclinism families $\phi_1 - \phi_6$ and $\phi_8 - \phi_9$, then $Aut_c(G) = Inn(G)$.

Now we calculate $|Aut_c(G)|$ for some of the isoclinism families of groups of order p^6. The family ϕ_1 corresponds to isoclinism family of abelian groups. For every abelian group we know that $Aut_c(G) = Inn(G)$.

4. Groups of Order p^6 Having a Cyclic Direct Factor

In this section we consider those groups of order p^6 which are the direct product of a group of order p^5 and a cyclic group of order p. We consider the isoclinism families ϕ_6, ϕ_7 and ϕ_{10} in this section.

Theorem 4.1. *Let G be the group in the isoclinism familiy ϕ_6. Then $Aut_c(G) = Inn(G)$.*

Proof. Let G be the group $\phi_6(1^6)$. Then $G = H \times (1)$, where $H = \phi_6(1^5)$. Since H is a nilpotent group of class 3 with $|Z(H)| = p^2$, therefore G is a nilpotent group of class 3 having $|Z(G)| = p^3$. Clearly $Aut_c(G) = Aut_c(H)$. But $|Aut_c(H)| = |Inn(H)| = p^3$(lemma 5.3,[15]) and hence $|Aut_c(G)| = |Inn(G)|$.

Theorem 4.2. *Let G be the group in the isoclinism familiy ϕ_7 and ϕ_{10}. Then $Aut_c(G) \neq Inn(G)$.*

Proof. Let G be the group $\phi_7(1^6)$. Then G is a direct product of $\phi_7(1^5)$ and and a cyclic group of order p. Similarly if G is the group $\phi_{10}(1^6)$, then G is a direct product of $\phi_{10}(1^5)$ and and a cyclic group of order p. In both of the cases $|Z(G)| = p^2$. Now from theorem 3.2, it follows that $|Aut_c(G)| = p^5$. Hence $Aut_c(G) \neq Inn(G)$, since $|Inn(G)| = p^4$.

If G is the group $\phi_8(42)$ in the isoclinism family ϕ_8, then G has an element of order p^4. Thus $Aut_c(G) = Inn(G)$ [4].

5. Nilpotent Groups of Class 2 in which $\gamma_2(G)$ is Cyclic

In this section we consider those groups of order p^6 which are nilpotent groups of class 2, and $\gamma_2(G)$ is cyclic. We consider the isoclinism families ϕ_2, ϕ_5 and ϕ_{14} in this section.

Theorem 5.1. *Let G be the group in the isoclinism familiy ϕ_2. Then $Aut_c(G) = Inn(G)$.*

Proof. Let G be the group $\phi_2(411)_b$. Then G has the following presentation $G =< \alpha, \alpha_1, \alpha_2, \gamma >$ such that $[\alpha, \alpha_1] = \alpha_2 = \gamma^{p^3}$, $\alpha^p = \alpha_1^p = \alpha_2^p = 1$.

Clearly $\gamma_2(G) =< \alpha_2 >$. Using the conventions we note that $[\alpha_1, \alpha_2] = [\alpha, \alpha_2] = 1$. Also $[\gamma, \alpha_2] = [\gamma, \gamma^{p^3}] = 1$. This shows that $\alpha_2 \in Z(G)$. But then $\gamma_2(G) \subseteq Z(G)$. Thus G is a nilpotent group of class 2 such that $\gamma_2(G)$ is cyclic, therefore by theorem 2.4, $Aut_c(G) = Inn(G)$.

In the similar fashion we find that if G is a group in the isoclinism families ϕ_5 or ϕ_{14}, then $Aut_c(G) = Inn(G)$.

6. Groups of Order p^6 Having a Normal Abelian Subgroup

In this section we calculate $Aut_c(G)$ for those groups G of order p^6 when G has a normal abelian subgroup H such that G/H is cyclic. We will study isoclinism families ϕ_3, ϕ_4,ϕ_9, ϕ_{16} and ϕ_{35} in this section.

Theorem 6.1. *Let G be the group in the isoclinism families ϕ_3. Then $Aut_c(G) = Inn(G)$.*

Proof. Let G be the group $\phi_3(3111)_e$. Then G has the following presentation $G =< \alpha, \alpha_1, \alpha_2, \alpha_3 >$ such that $[\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3$, $\alpha^p = \alpha_1^{p^3} = \alpha_2^p = \alpha_3^p = 1$. Here $\gamma_2(G) =< \alpha_2, \alpha_3 >$. Let $H =< \alpha_1, \alpha_2, \alpha_3 >$. Since $\gamma_2(G) \subseteq H$, therefore H is a normal subgroup of G. Now $[\alpha_1, \alpha_2] = [\alpha_1, \alpha_3] = [\alpha_2, \alpha_3] = 1$. This shows that H is an abelian subgroup of G. Hence H is a normal abelian subgroup of G such that $G/H =< \alpha H >$ is a cyclic group of prime order. So by theorem 2.3, $Aut_c(G) = Inn(G)$.

On the same lines we find that if G is a group from isoclinism families ϕ_4,ϕ_9, ϕ_{16} and ϕ_{35}, then $Aut_c(G) = Inn(G)$.

7. Groups of Order p^6 in which $Autcent(G) \subseteq Aut_c(G)$

We study the isoclinism families ϕ_{24}, ϕ_{36} and ϕ_{38}. For a group G lying in these families, we prove that for these groups $Autcent(G) \subseteq Aut_c(G)$. To show this we use the fact that if $Z(G) \subseteq [x, G]$ for each $x \in G - \gamma_2(G)$, then $Autcent(G) \subseteq Aut_c(G)$. We find that for any group G in these families $Aut_c(G) \neq Inn(G)$.

Theorem 7.1. *Let G be the group $\phi_{24}(1^6)$ in the isoclinism familiy ϕ_{24}. Then $|Aut_c(G)| = p^6$ and therefore $Aut_c(G) \neq Inn(G)$.*

Proof. Let $G = \phi_{24}(1^6)$ be a group in the isoclinism family ϕ_{24}. Then $G =< \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta >$ where $[\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_4$, $\alpha^p = \alpha_1^{(p)} = \beta^p = \alpha_{i+1}^{(p)} = 1, \quad i = 1, 2, 3$. Here $\gamma_2(G) =< \alpha_2, \alpha_3, \alpha_4 >$ and G is a nilpotent group of class 4 such that $|Z(G)| = p$. Using the presentation of the group we find that $Z(G) =< \alpha_4 >$. Now it follows from § 4.1 of [8], that $G/Z(G) = \phi_3(1^5)$. Gumber and Sharma (theorem 4.2, [5]) showed that exp $(G/Z(G)) = p$ and $d(G) = 3$. Since $Z(G) \subseteq \gamma_2(G)$, therefore exp $(G/\gamma_2(G)) = p$. Now any element $x \in G$ can be written as $a\alpha^l\alpha_1^m\beta^n$, where $a \in \gamma_2(G)$ and $0 \leq l, m, n \leq p-1$. Note that $x \in G - \gamma_2(G)$ unless l=m=n=0. We claim that for each $x \in G - \gamma_2(G)$, $Z(G) \subseteq [x, G]$.

Let $H =< \alpha_1, \alpha_2, \alpha_3, \alpha_4 >$ and $K =< \alpha_2, \alpha_3, \alpha_4, \beta >$. Then $\gamma_2(G)$ is contained in H and K. Thus H and K are normal abelian subgroups of G. Since $\gamma_2(G)$ and H commutes elementwise, therefore for each $x \in G - \gamma_2(G)$, $[x, H]$ and $[x, K]$ are subgroups of G contained in $[x, G]$.

Suppose $l \neq 0$, then using the facts that $\gamma_2(G)$ is abelian and it commutes with H and K elemenwise, we see that

$$[x, \alpha_3] = [a\alpha^l\alpha_1^m\beta^n, \alpha_3] = [\alpha^l, \alpha_3] = \alpha_4^{-l}$$

Since p does not divides l, therefore $\alpha_4^{-l} \in [x, H]$ implies that $\alpha_4 \in [x, H]$. Hence $Z(G) \subseteq [x, H]$.

Suppose $l = m = 0$. Since $\gamma_2(G)$ and H commute elementwise, therefore

$$[x, \alpha_1] = [a\beta^n, \alpha_1] = [\beta^n, \alpha_1] = \alpha_4^{-n}$$

Again using the same argument as above $Z(G) \subseteq [x, H]$.

If $l = n = 0$. Since $\gamma_2(G)$ and K commute elementwise, therefore

$$[x, \beta] = [a\alpha_1^m, \beta] = [\alpha_1^m, \beta] = \alpha_4^m$$

A similar argument shows that $Z(G) \subseteq [x, K]$.

Thus in each case discussed above either $Z(G) \subseteq [x, H] \subseteq [x, G]$ or $Z(G) \subseteq [x, K] \subseteq [x, G]$ for each $x \in G - \gamma_2(G)$. But then $Autcent(G) \subseteq Aut_c(G)$. We now calculate $|Autcent(G)|$. Since $|Z(G)| = p$, therefore G is purely non-abelian. Now it follows that

$$|Autcent(G)| = |Hom(G/\gamma_2(G), Z(G))| = p^3$$

Now $G/Z(G) = \phi_3(1^5) = \phi_3(1^4) \times (1)$ where

$$\phi_3(1^4) =< \alpha, \alpha_1, \alpha_2, \alpha_3 | [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_i^{(p)} = \alpha_3^p = 1 > \quad i = 1, 2.$$

Let $L = \phi_3(1^4)$. Then L is a non-abelian group of order p^4 with maximal class. Now it follows that $|Z(L)| = p$. But then $|Z(G/Z(G))| = p^2$ and hence $|Z_2(G)| = p^3$. By theorem 3.2, it follows that $Aut_c(G/Z(G)) = Inn(G/Z(G))$. Hence by Corollary 2.10, we have

$$\begin{aligned} |Aut_c(G)| &= |Aut_c(G) \cap Autcent(G)||G/Z_2(G)| \\ &= |Autcent(G)||G/Z_2(G)| \\ &= p^6 \end{aligned}$$

Hence $|Aut_c(G)| = p^6$. Also $Aut_c(G) \neq Inn(G)$ as $|Inn(G)| = |G/Z(G)| = p^5$.

Let G be the group $\phi_{36}(1^6)$ or $\phi_{38}(1^6)$ in the isoclinism families ϕ_{36} or ϕ_{38}. Then both G and $G/Z(G)$ are nilpotent groups of maximal class. In these cases $G/Z(G)$ is $\phi_9(1^5)$ and $\phi_{10}(1^5)$ respectively. Now using the theorem 3.2 and a similar argument as in the above theorem, we find that $Aut_c(G) \neq Inn(G)$.

8. Groups of Order p^6 in which $Autcent(G)$ is Minimum i.e. $Autcent(G) = Z(Inn(G))$

In this section we prove that if G is a group from the isoclinism families ϕ_{25}, ϕ_{26}, ϕ_{28}, ϕ_{29} and ϕ_{40}-ϕ_{43}, then $Aut_c(G) = Inn(G)$. To prove this we need the following theorem proved by Gumber and Sharma [5].

Theorem 8.1. *If G is a group of order p^6 with nilpotency class 3 or 4, then $Autcent(G) = Z(Inn(G))$ if and only if G is isomorphic to one of the group in the isoclinism families ϕ_{25}, ϕ_{26}, ϕ_{28},ϕ_{29} and ϕ_{40}-ϕ_{43}.*

Theorem 8.2. *Let G be the group in the isoclinism families ϕ_{25}, ϕ_{26}, ϕ_{28}, ϕ_{29} and ϕ_{40}-ϕ_{43}. Then $Aut_c(G) = Inn(G)$.*

Proof. Let G be any group in the isoclinism families ϕ_{25}, ϕ_{26}, ϕ_{28}, ϕ_{29} and ϕ_{40}-ϕ_{43}. Then form § 4.1 of [8], it follows that $G/Z(G)$ is one of the following groups $\phi_3(221)_{b_1}$, $\phi_3(221)_{b_\nu}$, $\phi_6(1^5)$, $\phi_6(221)b_{1/2(p-1)}$ and $\phi_6(221)d_0$. In either case we have $Aut_c(G/Z(G)) = Inn(G/Z(G))$ (by theorem 3.2) and hence

$$|Aut_c(G)| = |Aut_c(G) \cap Autcent(G)|.|G/Z_2(G)|$$

$$\leq |Autcent(G)|.|G/Z_2(G)|$$

But for these groups (by Theorem 8.1),

$$|Autcent(G)| = |Z(Inn(G)| = |Z_2(G)|/|Z(G)|.$$

Thus

$$|Aut_c(G)| \leq |Autcent(G)|.|G/Z_2(G)| = \frac{|Z_2(G)|}{|Z(G)|} \cdot \frac{|G|}{|Z_2(G)|} = |Inn(G)|$$

Hence if G is isomorphic to one of the groups in the isoclinism families ϕ_{25}, ϕ_{26}, ϕ_{28}, ϕ_{29} and ϕ_{40}-ϕ_{43}. Then $Aut_c(G) = Inn(G)$.

9. Nilpotent Groups of Class 3

In this section we consider the isoclinism families $\phi_{31} - \phi_{34}$ and ϕ_{22}. The group G lying in any one of these families is a nilpotent group of class 3. Using the lemma 2.7 and the definition of flat groups we find that $Aut_c(G) = Inn(G)$.

Theorem 9.1. *Let G be the group $\phi_{31}(1^6)$ in the isoclinism familiy ϕ_{31}. Then $Aut_c(G) = Inn(G)$.*

Proof. Let G be the group $\phi_{31}(1^6)$. Then $G =< \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma >$ where $[\alpha_i, \alpha] = \beta_i, [\alpha_1, \beta_1] = \gamma$, $[\alpha_2, \beta_2] = \gamma, \alpha^p = \alpha_i^p = \beta_i^p = \gamma^p = 1, \quad i = 1, 2$. Note that $\gamma_2(G) =< \beta_1, \beta_2, \gamma >$ and G is a nilpotent group of class 3. It follows that $\gamma_2(G) \subseteq Z_2(G)$. Now $G/\gamma_2(G)$ is an elementary abelian group of order p^3. Now from § 4.1 of [8], we find that $|Z(G)| = p$ and $G/Z(G) = \phi_4(1^5)$. Hence by theorem 3.2 and Corollary 2.10, it follows that $|Aut_c(G)| = |Aut_c(G) \cap Autcent(G)|.|G/Z_2(G)|$. Now we calculate $|Aut_c(G) \cap Autcent(G)|$. Since $|Z(G)| = p$ and $|G| = p^6$, therefore by lemma 2.7 G can not be flat, But $G/Z(G)$ being a nilpotent group of class 2,is flat. This shows that there exists some $x_1 \in G - Z(G)$ such that $Z(G)$ is not contained in $[x_1, G]$. Since $|Z(G)| = p$, therefore $x_1 \in G - \gamma_2(G)$. Now $\{x_1\}$ can be extended to a minimal generating set say $\{x_1, x_2, x_3\}$ of G. But then by Corollary 2.12 $|Aut_c(G) \cap Autcent(G)| < |Autcent(G)|$. i.e. $|Aut_c(G) \cap Autcent(G)| \leq p^2$. Now G has nilpotency class 3, therefore $\gamma_2(G) \subseteq Z_2(G)$. Now

$$|Aut_c(G)| = |Aut_c(G) \cap Autcent(G)|.|G/Z_2(G)| \leq p^2.p^3$$

Thus $|Aut_c(G)| \leq p^5$. Since $|Aut_c(G)| \geq |Inn(G)| = |G/Z(G)| = p^5$, therefore $Aut_c(G) = Inn(G)$.

If G is a group in isoclnism families $\phi_{32} - \phi_{34}$ and ϕ_{22}, then G is a nilpotent group of class 3 with $|Z(G)| = p$. Using the presentations of these groups and the similar arguments as in the above theorem we find that $Aut_c(G) = Inn(G)$.

10.

Theorem 10.1. *Let G be a nilpotent group of class ≥ 2. Then $Aut_c(G) \cap Autcent(G) \cong Hom_c(G/\gamma_2(G), Z(G))$.*

Proof. Let G be a nilpotent group of class ≥ 2. Let $f \in Aut_c(G) \cap Autcent(G)$. Then for each $x \in G$, $x^{-1}f(x) \in Z(G) \cap [x, G]$. Thus we can define a map $\alpha_f : G \to Z(G)$ by $\alpha_f(x) = x^{-1}f(x)$. Clearly α_f is a homomorphism of G into $Z(G)$. Since $Z(G)$ is abelian, therefore α_f sends elements of $\gamma_2(G)$ to 1. Therefore we have a homomorphism $\overline{\alpha_f} : G/\gamma_2(G) \to Z(G)$ given by $\overline{\alpha_f}(x\gamma_2(G)) = x^{-1}f(x)$. Now the map $f \mapsto \overline{\alpha_f}$ is a homomorphism of the group $Aut_c(G) \cap Autcent(G)$ into $Hom(G/\gamma_2(G), Z(G))$. We denote this map by ψ. Now it is fairly easy to see that ψ is a monomorphism of groups. We denote

$$\{f \in Hom(G/\gamma_2(G), Z(G)) : f(x\gamma_2(G) \in Z(G) \cap [x, G] \ \forall x \in G\}$$

by $Hom_c(G/\gamma_2(G), Z(G))$. Now it is clear that if $f \in Aut_c(G) \cap Autcent(G)$, then $\overline{\alpha_f} \in Hom_c(G/\gamma_2(G), Z(G))$. On the other hand let

$$f \in Hom_c(G/\gamma_2(G), Z(G)),$$

then we can define a map $\phi : G \to G$ by $\phi(x) = xf(x\gamma_2(G))$. Since $f(x\gamma_2(G)) \in Z(G) \cap [x, G]$, therefore for each $x \in G$ there exists some $a \in G$ such that $f(x\gamma_2(G)) = [x, a]$. But then $\phi(x) = xf(x\gamma_2(G)) = x[x, a] = a^{-1}xa$. Thus ϕ is a class preserving homomorphism. Since G is finite, $\phi \in Aut_c(G)$. Also we see that $x^{-1}\phi(x) = f(x\gamma_2(G)) \in Z(G)$. Thus $\phi \in Autcent(G)$ and hence $\phi \in Aut_c(G) \cap Autcent(G)$. Now $\psi(\phi) = \overline{\alpha_\phi}(x\gamma_2(G)) = x^{-1}\phi(x) = x^{-1}a^{-1}xa = f(x\gamma_2(G))$. Hence

$$\psi(\phi) = f.$$

Thus $\psi : Aut_c(G) \cap Autcent(G) \to Hom_c(G/\gamma_2(G), Z(G))$ is an isomorphism.

Theorem 10.2. *Let G be the group $\phi_{17}(1^6)$ in the isoclinism familiy ϕ_{17}. Then $Aut_c(G) \neq Inn(G)$.*

Proof. Let G be the group $\phi_{17}(1^6)$. Then

$$G =< \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma >,$$

where $[\alpha_i, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \gamma$, $\alpha^p = \alpha_1^p = \beta^p = \alpha_{i+1}^p = \gamma^p = 1, \quad i = 1, 2$. Here $\gamma_2(G) =< \alpha_2, \alpha_3, \gamma >$ and G is a nilpotent group of class 3. From § 4.1 of [8], we find that $|Z(G)| = p^2$. We observe that $\alpha_3, \gamma \in Z(G)$. Since α_3 and γ are the elements of order p, therefore $Z(G) =< \alpha_3, \gamma >$. Now $G/Z(G)$ is a non-abelian group of order p^4 with class 2 and every group of class 2 is flat. Hence by lemma 2.7, $|Z_2(G)|/|Z(G)| \geq p^2$. But we know that for a non-abelian group G of order p^4, $|Z(G)| \leq p^2$. Therefore $|Z_2(G)|/|Z(G)| \leq p^2$. Thus $|Z_2(G)| = p^4$. Now $G/\gamma_2(G)$ is an elementary abelian p-group of order p^3. Since every group of order p^4 enjoys Hasse's principle [10]. Therefore $Aut_c(G/Z(G)) = Inn(G/Z(G))$. Now by Corollary 2.10, it follows that $|Aut_c(G)| = |Aut_c(G) \cap Autcent(G)|.|G/Z_2(G)|$.

Let $Z_1 = [\alpha, G] \cap Z(G)$, $Z_2 = [\alpha_1, G] \cap Z(G)$, $Z_3 = [\beta, G] \cap Z(G)$. Then each Z_i is a central subgroup of G. Now $\alpha_3 = [\alpha_2, \alpha]$ implies that $\alpha_3 \in [\alpha, G] \cap Z(G) = Z_1$. Similarly $\gamma = [\beta, \alpha_1]$ implies that $\gamma \in [\alpha_1, G] \cap Z(G) = Z_2$ and $\gamma \in [\beta, G] \cap Z(G) = Z_3$. Hence in each case $|Z_i| \geq p$. Now

$$\begin{aligned} |Aut_c(G) \cap Autcent(G)| &= |Hom_c(G/\gamma_2(G), Z(G))| \\ &= |Hom_c((<\overline{\alpha}> \oplus <\overline{\alpha_1}> \oplus <\overline{\beta}>), Z(G))| \\ &\geq p.p.p = p^3 \end{aligned}$$

Hence

$$\begin{aligned} |Aut_c(G)| &= |Aut_c(G) \cap Autcent(G)|.|G/Z_2(G)| \\ &\geq \frac{p^3 p^6}{p^4} = p^5 \end{aligned}$$

Thus $Aut_c(G) \neq Inn(G)$ as $|Inn(G)| = |G/Z(G)| = p^4$.

Let G be the group in the isoclinism families ϕ_{18}-ϕ_{21}. Then in each case G is a nilpotent group of class 3. We observe that $Z(G)$ and $G/\gamma_2(G)$ are elementary abelian p-groups of order p^2 and p^3 respectively and $|Z_2(G)| = p^4$. Using a similar argument as in the above theorem we find that in each case there are three central subgroups Z_i of order $\geq p$. Hence $|Aut_c(G) \cap Autcent(G)| \geq p^3$. Since a group of order p^4 enjoys Hasse's principle, from this we deduce that $Aut_c(G) \neq Inn(G)$.

If G is a group in the isoclinism family ϕ_{23}, then G is a nilpotent group of class 4. We observe that both $Z(G)$ and $G/\gamma_2(G)$ are elementary abelian p-groups of order p^2. Now $G/Z(G)$ is a group of order p^4 having nilpotency class 3. Now it follows that $|Z_2(G)| = p^3$. There are two central subgroups Z_i of G such that $|Z_i| \geq p$. This implies that $|Aut_c(G) \cap Autcent(G)| \geq p^2$. Using the Corollary 2.10, we find that $Aut_c(G) \neq Inn(G)$.

If G is a group in the isoclinism family ϕ_{27}, then G is a nilpotent group of class 4. We observe that $|Z(G)| = p$ and $G/\gamma_2(G)$ is an elementary abelian p-groups of order p^3. Using the presentation of $G/Z(G)$ from James [8], we find that $|Z_2(G)| = p^3$. Again there are three central subgroups Z_i each of order

p. This implies that $|Aut_c(G) \cap Autcent(G)| = p^3$. From this we deduce that $|Aut_c(G)| = p^6$ and therefore $Aut_c(G) \neq Inn(G)$.

11. Nilpotent Groups of Class 2 in which $\gamma_2(G)$ is Not Cyclic

Let G be nilpotent group of class 2. Then $Aut_c(G) \subseteq Autcent(G)$. Hence for a nilpotent group of class 2, $Aut_c(G) \cap Autcent(G) = Aut_c(G)$. In this section we discuss the isoclinism families ϕ_{11},ϕ_{12},ϕ_{13} and ϕ_{15}. We find that a group G, lying in any of these isoclinism family is a nilpotent group of class 2. and from § of [8], we observe that $|x^G| \leq p^2$ for each $x \in G - Z(G)$. We first consider the isoclinism family ϕ_{11} and prove the following

Theorem 11.1. *Let G be the group in the isoclinism families ϕ_{11}. Then $|Aut_c(G)| = p^6$ and therefore $Aut_c(G) \neq Inn(G)$.*

Proof. Suppose G be the group $\phi_{11}(1^6)$, then $G =< \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 >$ where $[\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2$, $\alpha_i^p = \beta_i^p = 1, \quad i = 1, 2, 3$. Here $\gamma_2(G) =< \beta_1, \beta_2, \beta_3 >$. Since $[\alpha_i, \alpha_j] \neq 1$ and $[\alpha_i, \beta_j] = [\beta_i, \beta_j] = 1$ for $1 \leq i, j \leq 3$, therefore $\beta_1, \beta_2, \beta_3 \in Z(G)$. Since $\gamma_2(G) \subseteq Z(G)$ and G is non-abelian, G is a nilpotent group of class 2. Hence $Aut_c(G) = Aut_c(G) \cap Autcent(G)$. From § 4.1 of [8], we find that $|Z(G)| = p^3$. Since $\beta_1^p = \beta_2^p = \beta_3^p = 1$, $Z(G) =< \beta_1, \beta_2, \beta_3 >$. Now $G/\gamma_2(G) =< \overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3} >$ such that $\overline{\alpha_1}^p = \overline{\alpha_2}^p = \overline{\alpha_3}^p = 1$.

Let $Z_i = [\alpha_i, G] \cap Z(G)$. Then each Z_i is a central subgroup of G. Since $[\alpha_i, G] \subseteq \gamma_2(G) \subseteq Z(G)$, therefore $|Z_i| = |[\alpha_i, G]| = |\alpha_i^G|$. Since for each $x \in G - Z(G)$, $|x^G| \leq p^2$, therefore $|\alpha_i^G| \leq p^2$ i.e. $|Z_i| \leq p^2$. Now $\beta_2 = [\alpha_3, \alpha_1]$ and $\beta_3 = [\alpha_1, \alpha_2]$, this implies that $\beta_2, \beta_3 \in [\alpha_1, G] \cap Z(G) = Z_1$. Hence $|Z_1| \geq p^2$. Thus $|Z_1| = p^2$ and $Z_1 =< \beta_2, \beta_3 >$. Similarly $\beta_1, \beta_3 \in [\alpha_2, G] \cap Z(G) = Z_2$ and hence $Z_2 =< \beta_1, \beta_3 >$. Also $\beta_1, \beta_2 \in [\alpha_3, G] \cap Z(G) = Z_3$ and hence $Z_3 =< \beta_1, \beta_2 >$. Now

$$\begin{aligned} |Aut_c(G)| &= |Aut_c(G) \cap Autcent(G)| \\ &= |Hom_c(G/\gamma_2(G), Z(G))| \\ &= |Hom_c(< \overline{\alpha_1} > \oplus < \overline{\alpha_2} > \oplus < \overline{\alpha_3} >, Z(G))| \\ &= p^2.p^2.p^2 = p^6 \end{aligned}$$

Hence $Aut_c(G) \neq Inn(G)$ as $Inn(G) = |G/Z(G)| = p^3$.

In the similar fashion, we prove that if G is a group in the isoclinism families ϕ_{13} and ϕ_{15}, then $|Aut_c(G)|$ is p^6 and p^8 respectively and hence in both cases $Aut_c(G) \neq Inn(G)$.

Theorem 11.2. *Let G be the group in the isoclinism families ϕ_{12}. Then $Aut_c(G) = Inn(G)$.*

Proof. Suppose G be the group $\phi_{12}(2211)_i$. Then

$$G =< \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma_1, \gamma_2 >,$$

where $[\alpha_1, \alpha_3] = \gamma_1$, $[\alpha_2, \alpha_4] = \gamma_2, \alpha_1^p = \gamma_1$,

$$\alpha_2^p = \gamma_1\gamma_2, \alpha_3^p = \gamma_2, \alpha_4^p = \gamma_i^p = 1, \quad i = 1, 2.$$

Here $\gamma_2(G) =< \gamma_1, \gamma_2 >$. For $1 \leq i \leq 4$ and $1 \leq j \leq 2$, $[\alpha_i, \gamma_j] = [\gamma_1, \gamma_2] = 1$. This implies that $\gamma_1, \gamma_2 \in Z(G)$. Thus $\gamma_2(G) \subseteq Z(G)$ and hence G is a nilpotent group of class 2. Now $Aut_c(G) = Aut_c(G) \cap Autcent(G)$. From § 4.1 of [8], we find that $|Z(G)| = p^2$. Since $\gamma_1^p = \gamma_2^p = 1$, therefore $Z(G) =< \gamma_1, \gamma_2 >$. Let $Z_i = [\alpha_i, G] \cap Z(G)$. Then each Z_i is a central subgroup of G. Since $[\alpha_i, G] \subseteq \gamma_2(G) \subseteq Z(G)$, therefore $|Z_i| = |[\alpha_i, G]| = |\alpha_i^G|$. Since for each $x \in G - Z(G)$, $|x^G| \leq p^2$, therefore $|\alpha_i^G| \leq p^2$ i.e. $|Z_i| \leq p^2$. Since α_1 commutes with $\alpha_2, \alpha_4, \gamma_1, \gamma_2$ and $[\alpha_1, \alpha_3] = \gamma_1$, therefore $[\alpha_1, G] =< \gamma_1 >$. But then $Z_1 =< \gamma_1 >$. Similarly α_2 commutes with $\alpha_1, \alpha_3, \gamma_1, \gamma_2$ and $[\alpha_2, \alpha_4] = \gamma_2$,therefore $[\alpha_2, G] =< \gamma_2 >$. But then $Z_2 =< \gamma_2 >$. In the same fashion we find that $Z_3 = [\alpha_3, G] \cap Z(G) =< \gamma_1 >$ and $Z_4 = [\alpha_4, G] \cap Z(G) =< \gamma_2 >$. Thus $|Z_i| = p$ for each i, $1 \leq i \leq 4$. Since $\gamma_2(G) \subseteq Z(G)$ and $|\gamma_2(G)| = |Z(G)| = p^2$, therefore $\gamma_2(G) = Z(G)$. Here $G/\gamma_2(G) =< \overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}, \overline{\alpha_4} >$ such that $\overline{\alpha_1}^p = \overline{\alpha_2}^p = \overline{\alpha_3}^p = \overline{\alpha_4}^p = 1$. Now

$$\begin{aligned} |Aut_c(G)| &= |Aut_c(G) \cap Autcent(G)| \\ &= |Hom_c(G/\gamma_2(G), Z(G))| \\ &= \prod_{i=1}^{4} |Hom_c(< \overline{\alpha_i} >, Z(G))| \\ &= p.p.p.p = p^4 \end{aligned}$$

Thus $Aut_c(G) = Inn(G)$ as $|Inn(G)| = |G/Z(G)| = p^4$.

Remark: Out of 43 isoclinism families, unfortunately we are unable to find relation between $Aut_c(G)$ and $Inn(G)$ for a group G lying in the isoclinism families ϕ_{30}, ϕ_{37} and ϕ_{39}.

References

[1] J.E. Adney, T.Yen, *Automorphisms of a p-group*, Illinois J. Math, **9** (1965),137-143.

[2] A.R. Camina, *Some conditions which almost characterize Frobenious groups*, Israel journal of Mathematics, **31** (1978), 153-160.

[3] D.Rex, C. M. Scoppola, *On Camina groups of prime power order*, Journal of algebra, **181**(1996),787-802.

[4] M. Fuma,Y. Ninomiya, *Hasse's principle for finite p-groups with cyclic subgroups of index* p^2, Math J.Okayama University, **46** (2004), 31-38.

[5] D.Gumber, Mahak Sharma, *On central automorphisms of finite p-groups*, Archiv der mathematike, **9**(2011), 152-182.

[6] P. Hall,*The classification of prime power groups*, J.fur die reine und angewandte Mathematik,**182** (1940), 130141.

[7] M. Hertweck, *Contributions to the integral representation theory of groups*, Habilitationsschrift (University of Stuttgart),(2004)

[8] R. James, *The groups of order* p^6 *(p an odd prime)*, Math. comp., **34** (1980), 613-637.

[9] M. Kumar, L. R. Vermani, *Hasse's principle for extra-special p-groups*, Proc. Japan Acad., **76 A** (2000), 123-125.

[10] M. Kumar, L. R. Vermani, *Hasse's principle for groups of order* p^4, Proc. Japan Acad., **77 A**(2001), 95-98.

[11] M. Kumar, L. R. Vermani,*On automorphisms of some p-groups*, Proc. Japan Acad., **78 A** (2002), 46-50.

[12] T. Ono, *On Shafarevich Tate set for profinite groups*, Proc. Japan Acad., **75 A** (1999),97-98.

[13] T. Ono, H. Wada,*Hasse's Principle for symmetric and alternating groups*, Proc.Japan Acad, **75 A** (1999), 61-62.

[14] H. Tandra, W. Moran,*Flatness conditions on finite p-groups*, (Comm. Algebra), **32**(2004), 2215-2224.

[15] M. K. Yadav,*On automorphisms of some p-groups*, Proc. Indian Acad. Sci., **118**(2008), 1-11.