ON CLASS PRESERVING AUTOMORPHISMS OF SOME GROUPS OF ORDER $p^6$

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Abstract: For an odd prime $p$ there are 43 isoclinism families of groups of order $p^6$ given by James [8]. Let $G$ be a group lying in these families.

We sort out those groups for which $Aut_c(G) = Inn(G)$. Let $Autcent(G)$ denotes the groups of all central automorphisms of $G$. We give an upper bound for $|Aut_c(G)|$ in terms of $|Aut_c(G) \cap Autcent(G)|$ and $|Aut_c(G/Z(G))|$

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1. Introduction

We consider $G$ a finite p-group such that $|G| = p^6$ where $p$ is an odd prime. Notations used are mostly standard, however for $x \in G$, $x^G$ denotes the conjugacy class of the element $x$ in $G$. An automorphism $f$ is called class preserving if $f(x) \in x^G$ for all $x \in G$. The set of all class preserving automorphisms is denoted by $Aut_c(G)$ and $Inn(G)$ denotes the set of all inner automorphisms of $G$. $Aut_c(G)$ forms a normal subgroup of $Aut(G)$ and $Inn(G)$ forms a normal subgroup of $Aut_c(G)$. The quotient group $Aut_c(G)/Inn(G)$ is denoted by $Out_c(G)$. $Z(G)$ denotes the center of $G$ and the Frattini subgroup
of $G$ is denoted by $\Phi(G)$. An automorphisms $\phi$ of $G$ is said to be central if and only if it induces the identity automorphism on $G/Z(G)$, or equivalently, $g^{-1}\phi(g) \in Z(G)$ for all $g \in Z(G)$. The set of all central automorphisms of $G$, denoted by $Autcent(G)$, forms a normal subgroup of $Aut(G)$. It is well known that $Autcent(G) = C_{Aut(G)}(Inn(G))$.

Let $G$ be a group. A map $f : G \rightarrow G$ is called a co-cycle of $G$ if $f(xy) = f(x)f(y)^{x}$, for all $x, y \in G$. Here $f(y)^{x} = x f(y)x^{-1}$. A co-cycle $f$ of $G$ is called a local co-boundary if for each $x \in G$ there exists an $a_{x} \in G$ such that $f(x) = a_{x}^{-1}a_{x}^{x}$. A co-cycle $f$ of $G$ is called a (global) co-boundary if there exists an $a \in G$ such that $f(x) = a^{-1}a^{x}$ for all $x \in G$. A group $G$ is said to enjoy Hasse’s principle if every local co-boundary is a co-boundary. Ono, T [12] proved that a group $G$ enjoys Hasse’s principle if and only if every class preserving endomorphism is an inner automorphism. In the particular case when $G$ is a finite group, $G$ enjoys Hasse’s principle if and only if $Aut_{c}(G) = Inn(G)$.

Every abelian group enjoys the Hasse’s principle trivially. Ono, T. and Wada, H [13] proved that the symmetric group $S_{n}$ and alternating group $A_{n}$ of degree $n$ enjoy Hasse’s principle. They also proved that the dihedral group and the generalized quaternion group enjoys Hasse’s principle. Further in this direction Kumar and Vermani [9] proved that a non-abelian finite $p$-group having a maximal subgroup which is cyclic enjoys Hasse’s principle. They also proved that extra-special $p$-group enjoys Hasse’s principle. In other notes (see [10], [11]) they showed that every $p$-group of order $p^{4}$ enjoys Hasse’s principle. Fuma and Ninomiya [4] proved that a finite $p$-group having a cyclic subgroup of index $p^{2}$ enjoys Hasse’s principle. Further using classification of groups of order $p^{5}$ (for an odd prime $p$) given by James [8], Yadav [15] showed that for an odd prime $p$ every $p$-group of order $p^{5}$ except two isoclinism families enjoys Hasse’s principle. We move a step ahead and using the classification given by James we try to sort out some of the groups of order $p^{6}$ (where $p$ is an odd prime) for which $Aut_{c}(G) = Inn(G)$. We find an upper bound for $|Aut_{c}(G)|$ in terms of $|Aut_{c}(G) \cap Autcent(G)|$ and $|Aut_{c}(G/Z(G))|$. 

2. Some Useful Results

For a group $G$ and an abelian group $K$, $\text{Hom}(G,K)$ represents the group of all homomorphisms from $G$ to $K$. By $d(G)$ we mean the number of elements in a minimal generating set of $G$. A group $G$ is flat if every finite conjugacy class is a coset of a normal subgroup. The flatness condition is weaker than that of Camina pair (see [2], [3]). Let $G_{f}$ denotes the set of elements with finite
conjugacy class in $G$. Tandra and Moran [14] showed that a group $G$ is flat if and only if $[x,G]$ is a subgroup of $G$ for all $x \in G_f$. A non abelian group $G$ is said to be purely non-abelian, if it has no non-trivial abelian direct factor. It is easy to see that a group is purely non abelian if its center is contained in the Frattini subgroup. The following lemma follows from [1].

**Lemma 2.1.** If $G$ is a purely non abelian finite group, then $|\text{Autcent}(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))|$. 

**Lemma 2.2.** [15] If $G$ is a finite p-group such that $Z(G) \subseteq [x,G]$, for all $x \in G - \gamma_2(G)$, then $|\text{Aut}_c(G)| \geq |\text{Autcent}(G)||G/Z_2(G)|$.

The following lemma is proposition 14.4 of [7].

**Lemma 2.5.** Let $A$, $B$ and $C$ be finite abelian groups. Then:

(i) $\text{Hom}(A \times B, C) \cong \text{Hom}(A, C) \times \text{Hom}(B, C)$;

(ii) $\text{Hom}(A, B \times C) \cong \text{Hom}(A, B) \times \text{Hom}(A, C)$

**Lemma 2.6.** Let $C_m, C_n$ be cyclic groups of order $m$ and $n$ respectively. Then $\text{Hom}(C_m, C_n) \cong C_d$, where $d = \text{g.c.d.}(m, n)$.

**Lemma 2.7.** [14] Let $G$ be a flat nilpotent group of order $p^n$ having nilpotency class $c$. Then $|Z(G)| > p$ if $n$ is even, and for any $k$, $1 \leq k \leq c - 1$, $|Z^{k+1}(G)/Z^k(G)| > p$.

For nilpotent groups of class $\geq 2$, we prove the following

**Lemma 2.8.** If $G$ is a nilpotent group of class $\geq 2$, then $|\text{Aut}_c(G)| \leq |\text{Aut}_c(G) \cap \text{Autcent}(G)||\text{Aut}_c(G/Z(G))|$. 

**Proof.** Observe that every class preserving automorphism $f$ in $\text{Aut}_c(G)$ induces a class preserving automorphism $\overline{f}$ in $\text{Aut}_c(G/Z(G))$. Then the map

$$\alpha : \text{Aut}_c(G) \to \text{Aut}_c(G/Z(G))$$

given by $\alpha(f) = \overline{f}$ is a homomorphism of groups with $\text{Ker}(\alpha) = \text{Aut}_c(G) \cap \text{Autcent}(G)$. Hence $\text{Aut}_c(G)/\text{Aut}_c(G) \cap \text{Autcent}(G) \cong$ a subgroup of $\text{Aut}_c(G/Z(G))$. Now it follows that

$$|\text{Aut}_c(G)| \leq |\text{Aut}_c(G) \cap \text{Autcent}(G)|.|\text{Aut}_c(G/Z(G))|.$$
Corollary 2.9. If $G$ is a nilpotent group of class $\geq 2$, then $|\text{Aut}_c(G)| \leq |\text{Autcent}(G)| \cdot |\text{Aut}_c(G/Z(G))|$. 

Note that if $\text{Aut}_c(G/Z(G)) = \text{Inn}(G/Z(G))$, then for each $\bar{f} \in \text{Aut}_c(G/Z(G))$, there exists an element $gZ \in G/Z(G)$ such that $\bar{f}(xZ) = g^{-1}xgZ$ for all $xZ \in G/Z$. Thus for each $\bar{f} \in \text{Aut}_c(G/Z(G))$, there exist some $T_g \in \text{Aut}_c(G)$ such that $\alpha(T_g) = \bar{f}$ and hence the homomorphism $\alpha$ is an epimorphism. But then $\text{Aut}_c(G) / \text{Aut}_c(G) \cap \text{Autcent}(G) \cong \text{Inn}(G/Z(G))$. Thus we have the following 

Corollary 2.10. If $G$ is a nilpotent group of class $\geq 2$ such that 

$$\text{Aut}_c(G/Z(G)) = \text{Inn}(G/Z(G)),$$

then 

$$|\text{Aut}_c(G)| = |\text{Aut}_c(G) \cap \text{Autcent}(G)| \cdot |G/Z_2(G)|. $$

Lemma 2.11. Let $G$ be a non-abelian finite $p$-group with $|Z(G)| = p$. Let 

$\{x_1, x_2, \ldots, x_r\}$ be a minimal generating set for $G$. Suppose $Z(G)$ is not contained in $[x_i, G]$, for some $i$, $1 \leq i \leq r$. Then $\text{Autcent}(G)$ is not contained in $\text{Aut}_c(G)$. 

Proof. Let $G$ be a non-abelian finite $p$-group with $|Z(G)| = p$. Then $G$ is purely non-abelian and hence $|\text{Autcent}(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))|$. Let 

$\{x_1, x_2, \ldots, x_r\}$ be a minimal generating set for $G$. Then $G/\gamma_2(G) = \langle \bar{x_1} \rangle \oplus \langle \bar{x_2} \rangle \oplus \cdots \oplus \langle \bar{x_r} \rangle$ and hence 

$$|\text{Autcent}(G)| = |\text{Hom}(\bigoplus_{i=1}^{r} <\bar{x_i}>, Z(G))| = p^r$$

Define $H_i = Z(G) \cap [x_i, G]$. Clearly $H_i$ is a subgroup of $Z(G)$. Since $|Z(G)| = p$, it follows that either $H_i = \{e\}$ or $Z(G) \subseteq [x_i, G]$. Hence if $Z(G)$ is not contained in $[x_i, G]$, for some $i$, $1 \leq i \leq r$, then $H_i = \{e\}$. Let if possible, $\text{Autcent}(G) \subseteq \text{Aut}_c(G)$. Then every central automorphism is class preserving i.e. for each $\alpha \in \text{Autcent}(G)$ and for every $x \in G$, we have $x^{-1}\alpha(x) \in Z(G) \cap [x, G]$. In particular 

$$x_i^{-1}\alpha(x_i) \in Z(G) \cap [x_i, G]$$

$$= H_i$$

$$= \{e\}$$

Thus $\alpha(x_i) = x_i$ for every $\alpha \in \text{Autcent}(G)$. But then 

$$|\text{Autcent}(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))|$$
This is a contradiction and hence the lemma holds.

**Corollary 2.12.** Let $G$ be a non-abelian finite $p$-group with $|Z(G)| = p$. Let $\{x_1, x_2, \ldots, x_r\}$ be a minimal generating set for $G$ such that $Z(G)$ is not contained in $[x_i, G]$, for some $i, 1 \leq i \leq r$. Then

$$|Z_2(G)|/|Z(G)| \leq |Aut_c(G) \cap Autcent(G)| < |Autcent(G)|$$

**Proof.** Let $G$ be a group as defined above, then by previous lemma $|Aut_c(G) \cap Autcent(G)| < |Autcent(G)|$. Since $Autcent(G) = C_{Aut(G)}(Inn(G))$, therefore $Z(Inn(G)) = Inn(G) \cap Autcent(G) \subseteq Aut_c(G) \cap Autcent(G)$. Since $Inn(G) \cong G/Z(G)$, therefore $|Z(G/Z(G))| \leq |Aut_c(G) \cap Autcent(G)|$. Hence $|Z_2(G)|/|Z(G)| \leq |Aut_c(G) \cap Autcent(G)| < |Autcent(G)|$.

## 3. Isoclinic Groups

The concept of isoclinic groups was introduced by P. Hall [6]. Let $X$ be a finite group and $\overline{X} = X/Z(X)$. Then commutation in $X$ gives a well-defined map $\alpha_X : \overline{X} \times \overline{X} \mapsto \gamma_2(X)$ such that $\alpha_X(xZ(X), yZ(X)) = [x, y]$ for $(x, y) \in X \times X$. Two finite groups $G$ and $H$ are called isoclinic if there exist isomorphisms

$$\theta : G/Z(G) \rightarrow H/Z(H),$$

$$\phi : \gamma_2(G) \rightarrow \gamma_2(H),$$

such that $\phi[\alpha, \beta] = [\alpha', \beta']$ for all $\alpha, \beta \in G$, where $\alpha' Z(H) = \theta(\alpha(Z(G)))$ and $\beta' Z(H) = \theta(\beta Z(G))$. The resulting pair $(\theta, \phi)$ is called an isoclinism of $G$ onto $H$.

Before we start our work we state the following theorems proved by Yadav [15].

**Theorem 3.1.** Let $G$ and $H$ be two finite non-abelian isoclinic groups. Then $Aut_c(G) \cong Aut_c(H)$.

**Theorem 3.2.** Let $G$ be a finite $p$-group of order $p^5$, where $p$ is an odd prime. Then $Out_c(G) \neq 1$ if and only if $G$ is isoclinic to the group in $\phi_7(1^5)$ and $\phi_{10}(1^5)$ the isoclinism family $\phi_7$ and $\phi_{10}$ and in these two cases $|Aut_c(G)| = p^5$. 
From the theorem 3.1, we conclude that it is sufficient to calculate $\text{Aut}_c(G)$ for only one member from each isoclinism family and from the theorem 3.2, it is clear that if $G$ be any group of order $p^5$ ($p$ is an odd prime), from the isoclinism families $\phi_1 - \phi_6$ and $\phi_8 - \phi_9$, then $\text{Aut}_c(G) = \text{Inn}(G)$.

Now we calculate $|\text{Aut}_c(G)|$ for some of the isoclinism families of groups of order $p^6$. The family $\phi_1$ corresponds to isoclinism family of abelian groups. For every abelian group we know that $\text{Aut}_c(G) = \text{Inn}(G)$.

4. Groups of Order $p^6$ Having a Cyclic Direct Factor

In this section we consider those groups of order $p^6$ which are the direct product of a group of order $p^5$ and a cyclic group of order $p$. We consider the isoclinism families $\phi_6$, $\phi_7$ and $\phi_{10}$ in this section.

Theorem 4.1. Let $G$ be the group in the isoclinism family $\phi_6$. Then $\text{Aut}_c(G) = \text{Inn}(G)$.

Proof. Let $G$ be the group $\phi_6(1^6)$. Then $G = H \times (1)$, where $H = \phi_6(1^5)$. Since $H$ is a nilpotent group of class 3 with $|Z(H)| = p^2$, therefore $G$ is a nilpotent group of class 3 having $|Z(G)| = p^3$. Clearly $\text{Aut}_c(G) = \text{Aut}_c(H)$. But $|\text{Aut}_c(H)| = |\text{Inn}(H)| = p^3$ (lemma 5.3, [15]) and hence $|\text{Aut}_c(G)| = |\text{Inn}(G)|$.

Theorem 4.2. Let $G$ be the group in the isoclinism family $\phi_7$ and $\phi_{10}$. Then $\text{Aut}_c(G) \neq \text{Inn}(G)$.

Proof. Let $G$ be the group $\phi_7(1^6)$. Then $G$ is a direct product of $\phi_7(1^5)$ and and a cyclic group of order $p$. Similarly if $G$ is the group $\phi_{10}(1^6)$, then $G$ is a direct product of $\phi_{10}(1^5)$ and and a cyclic group of order $p$. In both of the cases $|Z(G)| = p^2$. Now from theorem 3.2, it follows that $|\text{Aut}_c(G)| = p^5$. Hence $\text{Aut}_c(G) \neq \text{Inn}(G)$, since $|\text{Inn}(G)| = p^4$.

If $G$ is the group $\phi_8(42)$ in the isoclinism family $\phi_8$, then $G$ has an element of order $p^4$. Thus $\text{Aut}_c(G) = \text{Inn}(G)$ [4].

5. Nilpotent Groups of Class 2 in which $\gamma_2(G)$ is Cyclic

In this section we consider those groups of order $p^6$ which are nilpotent groups of class 2, and $\gamma_2(G)$ is cyclic. We consider the isoclinism families $\phi_2$, $\phi_5$ and $\phi_{14}$ in this section.

Theorem 5.1. Let $G$ be the group in the isoclinism family $\phi_2$. Then $\text{Aut}_c(G) = \text{Inn}(G)$. 
Then Autcent \( G \)

We study the isoclinism families \( \varphi \)

In this section we calculate Aut \( G \) subgroup of Aut therefore by theorem 2.4, \( \gamma \varphi \). Let \( \varphi \) be the group be the group in the isoclinism families \( c, \varphi \) such that \( \gamma \varphi \). We find that for these groups \( \varphi \) is a normal abelian subgroup \( Z \) show this we use the fact that if \( \gamma \varphi \). Let \( \varphi \) be the group be the group in the isoclinism families \( c, \varphi \) such that \( \gamma \varphi \).

Clearly \( \gamma_2(G) = < \alpha_2 > \). Using the conventions we note that \([\alpha_1, \alpha_2] = [\alpha, \alpha_2] = 1\). Also \([\gamma, \alpha_2] = [\gamma, \gamma \varphi] = 1\). This shows that \( \alpha_2 \in Z(G) \). But then \( \gamma_2(G) \subseteq Z(G) \). Thus \( G \) is a nilpotent group of class 2 such that \( \gamma_2(G) \) is cyclic, therefore by theorem 2.4, \( \text{Aut}_c(G) = \text{Inn}(G) \).

In the similar fashion we find that if \( G \) is a group in the isoclinism families \( \varphi_5 \) or \( \varphi_{14} \), then \( \text{Aut}_c(G) = \text{Inn}(G) \).

6. Groups of Order \( p^6 \) Having a Normal Abelian Subgroup

In this section we calculate \( \text{Aut}_c(G) \) for those groups \( G \) of order \( p^6 \) when \( G \) has a normal abelian subgroup \( H \) such that \( G/H \) is cyclic. We will study isoclinism families \( \varphi_3, \varphi_4, \varphi_9, \varphi_{16} \) and \( \varphi_{35} \) in this section.

**Theorem 6.1.** Let \( G \) be the group in the isoclinism families \( \varphi_3 \). Then \( \text{Aut}_c(G) = \text{Inn}(G) \).

**Proof.** Let \( G \) be the group \( \varphi_3(3111)_e \). Then \( G \) has the following presentation \( G = < \alpha, \alpha_1, \alpha_2, \alpha_3 > \) such that \([\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^p = \alpha_1^p = \alpha_2^p = \alpha_3^p = 1\). Here \( \gamma_2(G) = < \alpha_2, \alpha_3 > \). Let \( H = < \alpha_1, \alpha_2, \alpha_3 > \). Since \( \gamma_2(G) \subseteq H \), therefore \( H \) is a normal subgroup of \( G \). Now \([\alpha_1, \alpha_2] = [\alpha_1, \alpha_3] = [\alpha_2, \alpha_3] = 1\). This shows that \( H \) is an abelian subgroup of \( G \). Hence \( H \) is a normal abelian subgroup of \( G \) such that \( G/H = < \alpha H > \) is a cyclic group of prime order. So by theorem 2.3, \( \text{Aut}_c(G) = \text{Inn}(G) \).

On the same lines we find that if \( G \) is a group from isoclinism families \( \varphi_4, \varphi_9, \varphi_{16} \) and \( \varphi_{35} \), then \( \text{Aut}_c(G) = \text{Inn}(G) \).

7. Groups of Order \( p^6 \) in which \( \text{Autcent}(G) \subseteq \text{Aut}_c(G) \)

We study the isoclinism families \( \varphi_{24}, \varphi_{36} \) and \( \varphi_{38} \). For a group \( G \) lying in these families, we prove that for these groups \( \text{Autcent}(G) \subseteq \text{Aut}_c(G) \). To show this we use the fact that if \( Z(G) \subseteq [x, G] \) for each \( x \in G - \gamma_2(G) \), then \( \text{Autcent}(G) \subseteq \text{Aut}_c(G) \). We find that for any group \( G \) in these families \( \text{Aut}_c(G) \neq \text{Inn}(G) \).

**Theorem 7.1.** Let \( G \) be the group \( \varphi_{24}(1^6) \) in the isoclinism family \( \varphi_{24} \). Then \( |\text{Aut}_c(G)| = p^6 \) and therefore \( \text{Aut}_c(G) \neq \text{Inn}(G) \).
Proof. Let $G = \phi_{24}(1^6)$ be a group in the isoclinism family $\phi_{24}$. Then $G = <\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta>$ where $[\alpha, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_4, \alpha^p = \alpha_1^{(p)} = \beta^p = \alpha_{i+1}^p = 1, \quad i = 1, 2, 3$. Here $\gamma_2(G) = <\alpha_2, \alpha_3, \alpha_4>$ and $G$ is a nilpotent group of class 4 such that $|Z(G)| = p$. Using the presentation of the group we find that $Z(G) = <\alpha_4>$. Now it follows from § 4.1 of [8], that $G/Z(G) = \phi_3(1^5)$. Gumber and Sharma (theorem 4.2, [5]) showed that $\exp(G/Z) = p$ and $d(G) = 3$. Since $Z(G) \subseteq \gamma_2(G)$, therefore $\exp(G/\gamma_2(G)) = p$. Now any element $x \in G$ can be written as $a\alpha^l\alpha_1^m\beta^n$, where $a \in \gamma_2(G)$ and $0 \leq l, m, n \leq p-1$. Note that $x \in G - \gamma_2(G)$ unless $l = m = n = 0$. We claim that for each $x \in G - \gamma_2(G)$, $Z(G) \subseteq [x, G]$.

Let $H = <\alpha_1, \alpha_2, \alpha_3, \alpha_4>$ and $K = <\alpha_2, \alpha_3, \alpha_4, \beta>$. Then $\gamma_2(G)$ is contained in $H$ and $K$. Thus $H$ and $K$ are normal abelian subgroups of $G$. Since $\gamma_2(G)$ and $H$ commutes elementwise, therefore for each $x \in G - \gamma_2(G)$, $[x, H]$ and $[x, K]$ are subgroups of $G$ contained in $[x, G]$.

Suppose $l \neq 0$, then using the facts that $\gamma_2(G)$ is abelian and it commutes with $H$ and $K$ elementwise, we see that

$$[x, \alpha_3] = [a\alpha^l\alpha_1^m\beta^n, \alpha_3] = [\alpha^l, \alpha_3] = \alpha_4^{-l}$$

Since $p$ does not divides $l$, therefore $\alpha_4^{-l} \in [x, H]$ implies that $\alpha_4 \in [x, H]$. Hence $Z(G) \subseteq [x, H]$.

Suppose $l = m = 0$. Since $\gamma_2(G)$ and $H$ commute elementwise, therefore

$$[x, \alpha_1] = [a\beta^n, \alpha_1] = [\beta^n, \alpha_1] = \alpha_4^{-n}$$

Again using the same argument as above $Z(G) \subseteq [x, H]$.

If $l = n = 0$. Since $\gamma_2(G)$ and $K$ commute elementwise, therefore

$$[x, \beta] = [a\alpha_1^m, \beta] = [\alpha_1^m, \beta] = \alpha_4^m$$

A similar argument shows that $Z(G) \subseteq [x, K]$.

Thus in each case discussed above either $Z(G) \subseteq [x, H] \subseteq [x, G]$ or $Z(G) \subseteq [x, K] \subseteq [x, G]$ for each $x \in G - \gamma_2(G)$. But then $\text{Autcent}(G) \subseteq Aut_e(G)$. We now calculate $|\text{Autcent}(G)|$. Since $|Z(G)| = p$, therefore $G$ is purely non-abelian. Now it follows that

$$|\text{Autcent}(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))| = p^3$$

Now $G/Z(G) = \phi_3(1^5) = \phi_3(1^4) \times (1)$ where

$$\phi_3(1^4) = <\alpha, \alpha_1, \alpha_2, \alpha_3|[\alpha, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^{(p)} = \alpha_3^p = 1 > \quad i = 1, 2.$$
Let $L = \phi_3(1^4)$. Then $L$ is a non-abelian group of order $p^4$ with maximal class. Now it follows that $|Z(L)| = p$. But then $|Z(G/Z(G))| = p^2$ and hence $|Z_2(G)| = p^3$. By theorem 3.2, it follows that \( \text{Aut}_c(G/Z(G)) = \text{Inn}(G/Z(G)) \). Hence by Corollary 2.10, we have

\[
|\text{Aut}_c(G)| = |\text{Aut}_c(G) \cap \text{Autcent}(G)||G/Z_2(G)|
\]

\[
= |\text{Autcent}(G)||G/Z_2(G)|
\]

\[
= p^6
\]

Hence $|\text{Aut}_c(G)| = p^6$. Also $\text{Aut}_c(G) \neq \text{Inn}(G)$ as $|\text{Inn}(G)| = |G/Z(G)| = p^5$.

Let $G$ be the group $\phi_{36}(1^6)$ or $\phi_{38}(1^6)$ in the isoclinism families $\phi_{36}$ or $\phi_{38}$. Then both $G$ and $G/Z(G)$ are nilpotent groups of maximal class. In these cases $G/Z(G)$ is $\phi_9(1^5)$ and $\phi_{10}(1^5)$ respectively. Now using the theorem 3.2 and a similar argument as in the above theorem, we find that $\text{Aut}_c(G) \neq \text{Inn}(G)$.

8. Groups of Order $p^6$ in which $\text{Autcent}(G)$ is Minimum i.e. \( \text{Autcent}(G) = Z(\text{Inn}(G)) \)

In this section we prove that if $G$ is a group from the isoclinism families $\phi_{25}$, $\phi_{26}$, $\phi_{28}$, $\phi_{29}$ and $\phi_{40}-\phi_{43}$, then $\text{Aut}_c(G) = \text{Inn}(G)$. To prove this we need the following theorem proved by Gumber and Sharma [5].

**Theorem 8.1.** If $G$ is a group of order $p^6$ with nilpotency class 3 or 4, then $\text{Autcent}(G) = Z(\text{Inn}(G))$ if and only if $G$ is isomorphic to one of the group in the isoclinism families $\phi_{25}$, $\phi_{26}$, $\phi_{28}$, $\phi_{29}$ and $\phi_{40}-\phi_{43}$.

**Theorem 8.2.** Let $G$ be the group in the isoclinism families $\phi_{25}$, $\phi_{26}$, $\phi_{28}$, $\phi_{29}$ and $\phi_{40}-\phi_{43}$. Then $\text{Aut}_c(G) = \text{Inn}(G)$.

**Proof.** Let $G$ be any group in the isoclinism families $\phi_{25}$, $\phi_{26}$, $\phi_{28}$, $\phi_{29}$ and $\phi_{40}-\phi_{43}$. Then form § 4.1 of [8], it follows that $G/Z(G)$ is one of the following groups $\phi_3(221)_{b_1}$, $\phi_3(221)_{b_2}$, $\phi_6(1^5)$, $\phi_6(221)b_{1/2^{(p-1)}}$ and $\phi_6(221)d_0$. In either case we have $\text{Aut}_c(G/Z(G)) = \text{Inn}(G/Z(G))$ (by theorem 3.2) and hence

\[
|\text{Aut}_c(G)| = |\text{Aut}_c(G) \cap \text{Autcent}(G)||G/Z_2(G)|
\]

\[
\leq |\text{Autcent}(G)||G/Z_2(G)|
\]

But for these groups (by Theorem 8.1),

\[
|\text{Autcent}(G)| = |Z(\text{Inn}(G)| = |Z_2(G)|/|Z(G)|.
\]
Thus

$$|\text{Aut}_c(G)| \leq |\text{Autcent}(G)| \cdot |G/Z_2(G)| = \frac{|Z_2(G)|}{|Z(G)|} \cdot \frac{|G|}{|Z_2(G)|} = |\text{Inn}(G)|$$

Hence if $G$ is isomorphic to one of the groups in the isoclinism families $\phi_{25}, \phi_{26}, \phi_{28}, \phi_{29}$ and $\phi_{40}-\phi_{43}$. Then $\text{Aut}_c(G) = \text{Inn}(G)$.

9. Nilpotent Groups of Class 3

In this section we consider the isoclinism families $\phi_{31} - \phi_{34}$ and $\phi_{22}$. The group $G$ lying in any one of these families is a nilpotent group of class 3. Using the lemma 2.7 and the definition of flat groups we find that $\text{Aut}_c(G) = \text{Inn}(G)$.

**Theorem 9.1.** Let $G$ be the group $\phi_{31}(1^6)$ in the isoclinism family $\phi_{31}$. Then $\text{Aut}_c(G) = \text{Inn}(G)$.

**Proof.** Let $G$ be the group $\phi_{31}(1^6)$. Then $G = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \rangle$ where $[\alpha_i, \alpha] = \beta_i, [\alpha_1, \beta_1] = \gamma, [\alpha_2, \beta_2] = \gamma, \alpha^p = \alpha_i^p = \beta_i^p = \gamma^p = 1, \quad i = 1, 2$. Note that $\gamma_2(G) = \langle \beta_1, \beta_2, \gamma \rangle$ and $G$ is a nilpotent group of class 3. It follows that $\gamma_2(G) \subseteq Z_2(G)$. Now $G/\gamma_2(G)$ is an elementary abelian group of order $p^3$. Now from § 4.1 of [8], we find that $|Z(G)| = p$ and $G/Z(G) = \phi_4(1^5)$. Hence by theorem 3.2 and Corollary 2.10, it follows that $|\text{Aut}_c(G)| = |\text{Aut}_c(G) \cap \text{Autcent}(G)|, |G/Z_2(G)|$. Now we calculate $|\text{Aut}_c(G) \cap \text{Autcent}(G)|$. Since $|Z(G)| = p$ and $|G| = p^6$, therefore by lemma 2.7 $G$ can not be flat, But $G/Z(G)$ being a nilpotent group of class 2,is flat. This shows that there exists some $x_1 \in G - Z(G)$ such that $Z(G)$ is not contained in $[x_1, G]$. Since $|Z(G)| = p$, therefore $x_1 \in G - \gamma_2(G)$. Now $\{x_1\}$ can be extended to a minimal generating set say $\{x_1, x_2, x_3\}$ of $G$. But then by Corollary 2.12 $|\text{Aut}_c(G) \cap \text{Autcent}(G)| < |\text{Autcent}(G)|$, i.e. $|\text{Aut}_c(G) \cap \text{Autcent}(G)| \leq p^2$. Now $G$ has nilpotency class 3, therefore $\gamma_2(G) \subseteq Z_2(G)$. Now

$$|\text{Aut}_c(G)| = |\text{Aut}_c(G) \cap \text{Autcent}(G)|, |G/Z_2(G)| \leq p^2 \cdot p^3$$

Thus $|\text{Aut}_c(G)| \leq p^5$. Since $|\text{Aut}_c(G)| \geq |\text{Inn}(G)| = |G/Z(G)| = p^5$, therefore $\text{Aut}_c(G) = \text{Inn}(G)$.

If $G$ is a group in isoclinism families $\phi_{32} - \phi_{34}$ and $\phi_{22}$, then $G$ is a nilpotent group of class 3 with $|Z(G)| = p$. Using the presentations of these groups and the similar arguments as in the above we find that $\text{Aut}_c(G) = \text{Inn}(G)$. 

10.

**Theorem 10.1.** Let $G$ be a nilpotent group of class $\geq 2$. Then $\text{Aut}_c(G) \cap \text{Autcent}(G) \cong \text{Hom}_c(G/\gamma_2(G), Z(G))$.

**Proof.** Let $G$ be a nilpotent group of class $\geq 2$. Let $f \in \text{Aut}_c(G) \cap \text{Autcent}(G)$. Then for each $x \in G$, $x^{-1}f(x) \in Z(G) \cap [x, G]$. Thus we can define a map $\alpha_f : G \to Z(G)$ by $\alpha_f(x) = x^{-1}f(x)$. Clearly $\alpha_f$ is a homomorphism of $G$ into $Z(G)$. Since $Z(G)$ is abelian, therefore $\alpha_f$ sends elements of $\gamma_2(G)$ to 1. Therefore we have a homomorphism $\overline{\alpha_f} : G/\gamma_2(G) \to Z(G)$ given by $\overline{\alpha_f}(x\gamma_2(G)) = x^{-1}f(x)$. Now the map $f \mapsto \overline{\alpha_f}$ is a homomorphism of the group $\text{Aut}_c(G) \cap \text{Autcent}(G)$ into $\text{Hom}(G/\gamma_2(G), Z(G))$. We denote this map by $\psi$. Now it is fairly easy to see that $\psi$ is a monomorphism of groups. We denote

$$\{f \in \text{Hom}(G/\gamma_2(G), Z(G)) : f(x\gamma_2(G)) \in Z(G) \cap [x, G] \forall x \in G\}$$

by $\text{Hom}_c(G/\gamma_2(G), Z(G))$. Now it is clear that if $f \in \text{Aut}_c(G) \cap \text{Autcent}(G)$, then $\overline{\alpha_f} \in \text{Hom}_c(G/\gamma_2(G), Z(G))$. On the other hand let

$$f \in \text{Hom}_c(G/\gamma_2(G), Z(G)),$$

then we can define a map $\phi : G \to G$ by $\phi(x) = xf(x\gamma_2(G))$. Since $f(x\gamma_2(G)) \in Z(G) \cap [x, G]$, therefore for each $x \in G$ there exists some $a \in G$ such that $f(x\gamma_2(G)) = [x, a]$. But then $\phi(x) = xf(x\gamma_2(G)) = x[x, a] = a^{-1}xa$. Thus $\phi$ is a class preserving homomorphism. Since $G$ is finite, $\phi \in \text{Aut}_c(G)$. Also we see that $x^{-1}\phi(x) = f(x\gamma_2(G)) \in Z(G)$. Thus $\phi \in \text{Autcent}(G)$ and hence $\phi \in \text{Aut}_c(G) \cap \text{Autcent}(G)$. Now $\psi(\phi) = \overline{\alpha_\phi}(x\gamma_2(G)) = x^{-1}\phi(x) = x^{-1}a^{-1}xa = f(x\gamma_2(G))$. Hence

$$\psi(\phi) = f.$$

Thus $\psi : \text{Aut}_c(G) \cap \text{Autcent}(G) \to \text{Hom}_c(G/\gamma_2(G), Z(G))$ is an isomorphism.

**Theorem 10.2.** Let $G$ be the group $\phi_{17}(1^6)$ in the isoclinism family $\phi_{17}$. Then $\text{Aut}_c(G) \not\cong \text{Inn}(G)$.

**Proof.** Let $G$ be the group $\phi_{17}(1^6)$. Then

$$G = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \rangle,$$

where $[\alpha_i, \alpha] = \alpha_{i+1}$, $[\beta, \alpha_1] = \gamma$, $\alpha^p = \alpha_1^p = \beta^p = \alpha^p_{i+1} = \gamma^p = 1$, $i = 1, 2$. Here $\gamma_2(G) = \langle \alpha_2, \alpha_3, \gamma \rangle$ and $G$ is a nilpotent group of class 3. From § 4.1 of [8], we find that $|Z(G)| = p^2$. We observe that $\alpha_3, \gamma \in Z(G)$. Since $\alpha_3$ and $\gamma$ are the elements of order $p$, therefore $Z(G) = \langle \alpha_3, \gamma \rangle$. Now $G/Z(G)$ is a non-abelian group of order $p^4$ with class 2 and every group of class 2 is flat. Hence
by lemma 2.7, $|Z_2(G)|/|Z(G)| \geq p^2$. But we know that for a non-abelian group $G$ of order $p^4$, $|Z(G)| \leq p^2$. Therefore $|Z_2(G)|/|Z(G)| \leq p^2$. Thus $|Z_2(G)| = p^4$. Now $G/\gamma_2(G)$ is an elementary abelian $p$-group of order $p^3$. Since every group of order $p^4$ enjoys Hasse’s principle [10]. Therefore $\text{Aut}_c(G/Z(G)) = \text{Inn}(G/Z(G))$. Now by Corollary 2.10, it follows that $|\text{Aut}_c(G)| = |\text{Aut}_c(G) \cap \text{Autcent}(G)|.|G/Z_2(G)|$.

Let $Z_1 = [\alpha, G] \cap Z(G)$, $Z_2 = [\alpha_1, G] \cap Z(G)$, $Z_3 = [\beta, G] \cap Z(G)$. Then each $Z_i$ is a central subgroup of $G$. Now $\alpha_3 = [\alpha_2, \alpha]$ implies that $\alpha_3 \in [\alpha, G] \cap Z(G) = Z_1$. Similarly $\gamma = [\beta, \alpha_1]$ implies that $\gamma \in [\alpha_1, G] \cap Z(G) = Z_2$ and $\gamma \in [\beta, G] \cap Z(G) = Z_3$. Hence in each case $|Z_i| \geq p$. Now

$$|\text{Aut}_c(G) \cap \text{Autcent}(G)| = |\text{Hom}_c(G/\gamma_2(G), Z(G))|$$
$$= |\text{Hom}_c((<\alpha> \oplus <\alpha_1> \oplus <\beta>), Z(G))|$$
$$\geq p.p.p = p^3$$

Hence

$$|\text{Aut}_c(G)| = |\text{Aut}_c(G) \cap \text{Autcent}(G)|.|G/Z_2(G)|$$
$$\geq \frac{p^3 p^6}{p^4} = p^5$$

Thus $\text{Aut}_c(G) \neq \text{Inn}(G)$ as $|\text{Inn}(G)| = |G/Z(G)| = p^4$.

Let $G$ be the group in the isoclinism families $\phi_{18}$-$\phi_{21}$. Then in each case $G$ is a nilpotent group of class 3. We observe that $Z(G)$ and $G/\gamma_2(G)$ are elementary abelian $p$-groups of order $p^2$ and $p^3$ respectively and $|Z_2(G)| = p^4$. Using a similar argument as in the above theorem we find that in each case there are three central subgroups $Z_i$ of order $\geq p$. Hence $|\text{Aut}_c(G) \cap \text{Autcent}(G)| \geq p^3$. Since a group of order $p^4$ enjoys Hasse’s principle, from this we deduce that $\text{Aut}_c(G) \neq \text{Inn}(G)$.

If $G$ is a group in the isoclinism family $\phi_{23}$, then $G$ is a nilpotent group of class 4. We observe that both $Z(G)$ and $G/\gamma_2(G)$ are elementary abelian $p$-groups of order $p^2$. Now $G/Z(G)$ is a group of order $p^4$ having nilpotency class 3. Now it follows that $|Z_2(G)| = p^3$. There are two central subgroups $Z_i$ of $G$ such that $|Z_i| \geq p$. This implies that $|\text{Aut}_c(G) \cap \text{Autcent}(G)| \geq p^2$. Using the Corollary 2.10, we find that $\text{Aut}_c(G) \neq \text{Inn}(G)$.

If $G$ is a group in the isoclinism family $\phi_{27}$, then $G$ is a nilpotent group of class 4. We observe that $|Z(G)| = p$ and $G/\gamma_2(G)$ is an elementary abelian $p$-groups of order $p^3$. Using the presentation of $G/Z(G)$ from James [8], we find that $|Z_2(G)| = p^3$. Again there are three central subgroups $Z_i$ each of order
11. Nilpotent Groups of Class 2 in which $\gamma_2(G)$ is Not Cyclic

Let $G$ be nilpotent group of class 2. Then $\text{Aut}_c(G) \subseteq \text{Autcent}(G)$. Hence for a nilpotent group of class 2, $\text{Aut}_c(G) \cap \text{Autcent}(G) = \text{Aut}_c(G)$. In this section we discuss the isoclinism families $\phi_{11}, \phi_{12}, \phi_{13}$ and $\phi_{15}$. We find that a group $G$, lying in any of these isoclinism family is a nilpotent group of class 2. and from § of [8], we observe that $|x^G| \leq p^2$ for each $x \in G - Z(G)$. We first consider the isoclinism family $\phi_{11}$ and prove the following

**Theorem 11.1.** Let $G$ be the group in the isoclinism families $\phi_{11}$. Then $|\text{Aut}_c(G)| = p^6$ and therefore $\text{Aut}_c(G) \neq \text{Inn}(G)$.

**Proof.** Suppose $G$ be the group $\phi_{11}(1^6)$, then $G = \langle \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \rangle$ where $[\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha^p = \beta^p = 1$, $i = 1, 2, 3$. Here $\gamma_2(G) = \langle \beta_1, \beta_2, \beta_3 \rangle$. Since $[\alpha_i, \alpha_j] \neq 1$ and $[\alpha_i, \beta_j] = [\beta_i, \beta_j] = 1$ for $1 \leq i, j \leq 3$, therefore $\beta_1, \beta_2, \beta_3 \in Z(G)$. Since $\gamma_2(G) \subseteq Z(G)$ and $G$ is non-abelian, $G$ is a nilpotent group of class 2. Hence $\text{Aut}_c(G) = \text{Aut}_c(G) \cap \text{Autcent}(G)$.

From § 4.1 of [8], we find that $|Z(G)| = p^3$. Since $\beta^p_1 = \beta^p_2 = \beta^p_3 = 1$, $Z(G) = \langle \beta_1, \beta_2, \beta_3 \rangle$. Now $G/\gamma_2(G) = \langle \overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3} \rangle$ such that $\overline{\alpha^p_1} = \overline{\alpha^p_2} = \overline{\alpha^p_3} = 1$.

Let $Z_i = [\alpha_i, G] \cap Z(G)$. Then each $Z_i$ is a central subgroup of $G$. Since $[\alpha_i, G] \subseteq \gamma_2(G) \subseteq Z(G)$, therefore $|Z_i| = |[\alpha_i, G]| = |\alpha^G_i|$. Since for each $x \in G - Z(G)$, $|x^G| \leq p^2$, therefore $|\alpha^G_i| \leq p^2$ i.e. $|Z_i| \leq p^2$. Now $\beta_2 = [\alpha_3, \alpha_1]$ and $\beta_3 = [\alpha_1, \alpha_2]$, this implies that $\beta_2, \beta_3 \in [\alpha_1, G] \cap Z(G) = Z_1$. Hence $|Z_1| \geq p^2$. Thus $|Z_1| = p^2$ and $Z_1 = \langle \beta_2, \beta_3 \rangle$. Similarly $\beta_1, \beta_3 \in [\alpha_2, G] \cap Z(G) = Z_2$ and hence $Z_2 = \langle \beta_1, \beta_3 \rangle$. Also $\beta_1, \beta_2 \in [\alpha_3, G] \cap Z(G) = Z_3$ and hence $Z_3 = \langle \beta_1, \beta_2 \rangle$. Now

$$|\text{Aut}_c(G)| = |\text{Aut}_c(G) \cap \text{Autcent}(G)| = |\text{Hom}_c(G/\gamma_2(G), Z(G))| = |\text{Hom}_c(<\alpha_1^G > \oplus <\alpha_2^G > \oplus <\alpha_3^G >, Z(G))| = p^2 \cdot p^2 \cdot p^2 = p^6$$

Hence $\text{Aut}_c(G) \neq \text{Inn}(G)$ as $\text{Inn}(G) = |G/Z(G)| = p^3$.

In the similar fashion, we prove that if $G$ is a group in the isoclinism families $\phi_{13}$ and $\phi_{15}$, then $|\text{Aut}_c(G)| = p^6$ and $p^8$ respectively and hence in both cases $\text{Aut}_c(G) \neq \text{Inn}(G)$. 

Theorem 11.2. Let $G$ be the group in the isoclinism families $\phi_{12}$. Then $\text{Aut}_c(G) = \text{Inn}(G)$.

Proof. Suppose $G$ be the group $\phi_{12}(2211)_i$. Then

$$G = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma_1, \gamma_2 \rangle,$$

where $[\alpha_1, \alpha_3] = \gamma_1$, $[\alpha_2, \alpha_4] = \gamma_2$, $\alpha_1^p = \gamma_1$,

$$\alpha_2^p = \gamma_1 \gamma_2, \alpha_3^p = \gamma_2, \alpha_4^p = \gamma_i^p = 1, \quad i = 1, 2.$$

Here $\gamma_2(G) = \langle \gamma_1, \gamma_2 \rangle$. For $1 \leq i \leq 4$ and $1 \leq j \leq 2$, $[\alpha_i, \gamma_j] = [\gamma_1, \gamma_2] = 1$. This implies that $\gamma_1, \gamma_2 \in Z(G)$. Thus $\gamma_2(G) \leq Z(G)$ and hence $G$ is a nilpotent group of class 2. Now $\text{Aut}_c(G) = \text{Aut}_c(G) \cap \text{Autcent}(G)$. From § 4.1 of [8], we find that $|Z(G)| = p^2$. Since $\gamma_1^p = \gamma_2^p = 1$, therefore $Z(G) = \langle \gamma_1, \gamma_2 \rangle$. Let $Z_i = [\alpha_i, G] \cap Z(G)$. Then each $Z_i$ is a central subgroup of $G$. Since $[\alpha_i, G] \subseteq \gamma_2(G) \subseteq Z(G)$, therefore $|Z_i| = |[\alpha_i, G]| = |\gamma_i^G|$. Since for each $x \in G - Z(G)$, $|x^G| \leq p^2$, therefore $|\alpha_i^G| \leq p^2$ i.e. $|Z_i| \leq p^2$. Since $\alpha_1$ commutes with $\alpha_2, \alpha_4, \gamma_1, \gamma_2$ and $[\alpha_1, \alpha_3] = \gamma_1$, therefore $[\alpha_1, G] = \langle \gamma_1 \rangle$. But then $Z_1 = \langle \gamma_1 \rangle$. Similarly $\alpha_2$ commutes with $\alpha_1, \alpha_3, \gamma_1, \gamma_2$ and $[\alpha_2, \alpha_4] = \gamma_2$, therefore $[\alpha_2, G] = \langle \gamma_2 \rangle$. But then $Z_2 = \langle \gamma_2 \rangle$. In the same fashion we find that $Z_3 = [\alpha_3, G] \cap Z(G) = \langle \gamma_1 \rangle$ and $Z_4 = [\alpha_4, G] \cap Z(G) = \langle \gamma_2 \rangle$. Thus $|Z_i| = p$ for each $i$, $1 \leq i \leq 4$. Since $\gamma_2(G) \subseteq Z(G)$ and $|\gamma_2(G)| = |Z(G)| = p^2$, therefore $\gamma_2(G) = Z(G)$. Here $G/\gamma_2(G) = \langle \alpha_1 \alpha_2, \alpha_3, \alpha_4 \rangle$ such that $\alpha_1^p = \alpha_2^p = \alpha_3^p = \alpha_4^p = 1$. Now

$$|\text{Aut}_c(G)| = |\text{Aut}_c(G) \cap \text{Autcent}(G)| = \prod_{i=1}^{4} |\text{Hom}_c(<\alpha_i^>, Z(G))| = p^4$$

Thus $\text{Aut}_c(G) = \text{Inn}(G)$ as $|\text{Inn}(G)| = |G/Z(G)| = p^4$.

Remark: Out of 43 isoclinism families, unfortunately we are unable to find relation between $\text{Aut}_c(G)$ and $\text{Inn}(G)$ for a group $G$ lying in the isoclinism families $\phi_{30}, \phi_{37}$ and $\phi_{39}$.

References


