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CLASS OF (A, n)-POWER QUASI-NORMAL OPERATORS IN SEMI-HILBERTIAN SPACES

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Abstract: In this paper, the concept of n-power quasi-normal operators on a Hilbert space defined by Sid Ahmed in [14] is generalized when an additional semi-inner product is considered. This new concept is described by means of oblique projections. For a Hilbert space operator $T \in \mathcal{B}(\mathcal{H})$ is (A, n)-power quasi-normal operators for some positive operator A and for some positive integer n if ,

$$\left[T^n T^{\langle * \rangle_A} - T^{\langle * \rangle_A} T^n\right] T = 0, \;\; n = 1, 2, ...$$

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1. Introduction and Terminologies

A bounded linear operator T on a complex Hilbert space is quasi-normal if T and T^*T commute. The class of quasi-normal operators was first introduced and studied by A.Brown [5] in 1953. From the definition, it is easily seen that this class contains normal operators and isometries. In [14], the author introduced the class of n-power quasi-normal operators as a generalization of the class of quasi-normal operators and study some properties of such class for

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different values of the parameter n, in particular for $n = 2$ and $n = 3$, (see [14] and [15]).

The purpose of this paper is to study the class of (A, n)-power quasi-normal operators in semi-hilbertian spaces.

Along this work, $\mathcal{H}$ denotes a complex Hilbert space with inner product $\langle\,|\,\rangle$, $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$, $\mathcal{B}(\mathcal{H})^+$ is the cone of positive (semi-definite) operators of $\mathcal{B}(\mathcal{H})$ i.e., $\mathcal{B}(\mathcal{H})^+ := \{T \in \mathcal{B} : (\mathcal{H})\ \langle T\xi|\ \xi\rangle \geq 0,\ \forall \xi \in \mathcal{H}\ \}$ and $\mathcal{B}(\mathcal{H})_{cr}$ is the subset of $\mathcal{B}(\mathcal{H})$ of all operators with closed range. For every $T \in \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{N}(T), \mathcal{R}(T)$ and $\overline{\mathcal{R}(T)}$ respectively, the null space, the range and the closure of the range of T. Also the adjoint operator of T is denoted by T^* and $T^\dagger$ stands for the Moore-Penrose inverse of T. In addition, if $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ then $T_1 \geq T_2$ means that $T_1 - T_2 \in \mathcal{B}(\mathcal{H})^+$. For a closed subspace $\mathcal{S}$ of $\mathcal{H}$, $P_{\mathcal{S}}$ denotes the orthogonal projection onto $\mathcal{S}$.

Note that for $A \in \mathcal{B}(\mathcal{H})^+$, the functional

$$\langle\,|\,\rangle_A : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C},\ \langle \xi \mid \eta\rangle_A = \langle A\xi\ |\eta\rangle$$

is a semi-inner product on $\mathcal{H}$. By $\|.\|_A$, we denote the semi-norm induced by $\langle\,|\,\rangle_A$, i.e.,$\|\xi\|_A = \langle\xi|\ \xi\ \rangle_A^{\frac{1}{2}}$. Observe that $\|\xi\|_A = 0$ if and only if $\xi \in \mathcal{N}(A)$. Then $\|.\|_A$ is a norm if and only if A is an injective operator, and the semi-normed space $(\mathcal{H}, \|.\ \|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed. Moreover, $\langle\ |\ \rangle_A$ induces a semi-norm on a certain subspace of $\mathcal{B}(\mathcal{H})$, namely, on the subspace

$$\left\{\ T \in \mathcal{B}(\mathcal{H})/\ \exists\, c > 0\ :\ \|T\xi\|_A \leq c\|\xi\|_A,\ \forall\, \xi \in \mathcal{H}\ \right\}.$$

In such case, it holds

$$\begin{aligned}
\|T\|_A &= \sup_{\substack{\xi \in \overline{\mathcal{R}(A)} \\ \xi \neq 0}} \frac{\|T\xi\|_A}{\|\xi\|_A} = \sup_{\|\xi\|_A \leq 1} \|T\xi\|_A \\
&= \sup\left\{\|T\xi\|_A :\ \|\xi\|_A = 1\right\} \\
&= \inf\left\{c > 0\ : \|T\xi\|_A \leq c\|\xi\|_A,\ \ \xi \in \mathcal{H}\ \right\} < \infty.
\end{aligned}$$

Moreover,

$$\|T\|_A = \sup\left\{\langle T\xi|\ \eta\rangle_A;\ \xi, \eta \in \mathcal{H},\ :\ \|\xi\| \leq 1, \|\eta\| \leq 1\right\}.$$

For $\xi, \eta \in \mathcal{H}$, we say that ξ and η are A-orthogonal if $\langle \xi | \, \eta \rangle_A = 0$.
Define

$$\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) := \left\{ T \in \mathcal{B}(\mathcal{H}) : \ \|T\xi\|_A \leq c\|\xi\|_A \ \text{ for every } \ \xi \in \mathcal{H} \right\}.$$

It is easy to see that $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ is a subspace of $\mathcal{B}(\mathcal{H})$. In general if $T \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, $T^* \notin \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$.

From now on, A denotes a positive operator on $\mathcal{H}$, that is $A \in \mathcal{B}(\mathcal{H})^+$.

Definition 1.1. ([1]) For $T \in \mathcal{B}(\mathcal{H})$, an operator $S \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint of T if for every $\xi, \eta \in \mathcal{H}$

$$\langle T\xi \mid \eta \rangle_A = \langle \xi \mid S\eta \rangle_A,$$

i.e., $AS = T^*A$ or equivalently S is a solution of the equation $AX = T^*A$. We say that T is A-selfadjoint if $AT = T^*A$.

Remark 1.1. The existence of an A- adjoint operator is not guaranteed. Observe that T admits an A-adjoint if and only if the equation $AX = T^*A$ has solution. This hind of equation can be studied applying the next theorem due to Douglas (for its proof see [6], [7]).

Theorem 1.1. *Let $C, B \in \mathcal{B}(\mathcal{H})$. The following conditions are equivalents.*

(1) $\mathcal{R}(B) \subset \mathcal{R}(C)$.

(2) *There exists a positive number λ such that $BB^* \leq \lambda CC^*$.*

(3) *There exists $S \in \mathcal{B}(\mathcal{H})$ such that $CS = B$.*

If one of these conditions holds, then there exists a unique operator $D \in \mathcal{B}(\mathcal{H})$ such that $CD = B$ and $\mathcal{R}(D) \subseteq \overline{\mathcal{R}(C^*)}$ and $\mathcal{N}(D) = \mathcal{N}(B)$. Moreover

$$\|D\| = \inf \left\{ \lambda > 0 \, / BB^* \leq \lambda CC^* \right\}.$$

This solution will be called a reduced solution of the equation

$$CX = B.$$

If we denote by $\mathcal{B}_A(\mathcal{H})$ the subalgebra of $\mathcal{B}(\mathcal{H})$ of all bounded operators which admit an A-adjoint operator, then

$$\mathcal{B}_A(\mathcal{H}) = \left\{ T \in \mathcal{B}(H) : \ T^*\mathcal{R}(A) \subset \mathcal{R}(A) \right\}.$$

Furthermore, applying Douglas theorem, we can see that

$$\begin{aligned}\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) &= \left\{ T \in \mathcal{B}(\mathcal{H}) : \ T^*\mathcal{R}(A^{\frac{1}{2}}) \subset \mathcal{R}(A^{\frac{1}{2}}) \right\} \\ &= \left\{ T \in \mathcal{B}(\mathcal{H}) : \ \mathcal{R}(A^{\frac{1}{2}}T^*A^{\frac{1}{2}}) \subseteq \mathcal{R}(A) \right\}.\end{aligned}$$

The relationship between the above sets is proved in [13].

Proposition 1.1. *The following inclusion $\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ is satisfying.*

If an operator equation $BX = C$ has a solution, then it is easy to see that the distinguished solution of Douglas theorem is given by $B^\dagger C$. Therefore, given $T \in \mathcal{B}_A(\mathcal{H})$, if we denote by $T^{\langle * \rangle_A}$, the unique A-adjoint operator of T whose range is included in $\overline{\mathcal{R}(A)}$, then

$$T^{\langle * \rangle_A} = A^\dagger T^* A.$$

In view of Theorem 1.1,

$$AT^{\langle * \rangle_A} = T^*A, \ \mathcal{R}(T^{\langle * \rangle_A}) \subset \overline{\mathcal{R}(A)} \ \text{ and } \ \mathcal{N}(T^{\langle * \rangle_A}) = \mathcal{N}(T^*A).$$

Note that if S is an A-adjoint of T then $S = T^{\langle * \rangle_A} + Z$, with $Z \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{R}(Z) \subset \mathcal{N}(A)$.

Remark 1.2. Observe that if T is A-selfadjoint, it does not mean in general that $T = T^{\langle * \rangle_A}$. In fact $T = T^{\langle * \rangle_A}$ if and only if T is A-selfadjoint and $\mathcal{R}(T) \subset \overline{\mathcal{R}(A)}$.

It is also clear that T has a unique A-adjoint (namely $T^{\langle * \rangle_A}$) if and only if A is injective. If this is the case, then we get the equality $(T^{\langle * \rangle_A})^{\langle * \rangle_A} = T$.

In the following proposition we collect some properties of $T^{\langle * \rangle_A}$. For its proof, see [1], [2] and [3].

Proposition 1.2. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then the following statements hold*

(1) $T^{\langle * \rangle_A} \in \mathcal{B}_A(\mathcal{H}), (T^{\langle * \rangle_A})^{\langle * \rangle_A} = P_{\overline{R(A)}} T P_{\overline{R(A)}}$ *and* $((T^{\langle * \rangle_A})^{\langle * \rangle_A})^{\langle * \rangle_A} = T^{\langle * \rangle_A}$.

(2) *If $S \in \mathcal{B}_A(\mathcal{H})$, then $TS \in \mathcal{B}_A(\mathcal{H})$ and $(TS)^{\langle * \rangle_A} = S^{\langle * \rangle_A} T^{\langle * \rangle_A}$.*

(3) *$T^{\langle * \rangle_A} T$ and $TT^{\langle * \rangle_A}$ are A-selfadjoint.*

(4) $\|T\|_A = \|T^{\langle * \rangle_A}\|_A = \|T^{\langle * \rangle_A} T\|_A^{\frac{1}{2}} = \|TT^{\langle * \rangle_A}\|_A^{\frac{1}{2}}$.

(5) $\|S\|_A = \|T^{\langle * \rangle_A}\|_A$ *for every* $S \in \mathcal{B}(\mathcal{H})$ *which is an* A*-adjoint of* T.

(6) *If* $S \in \mathcal{B}_A(\mathcal{H})$ *then* $\|TS\|_A = \|ST\|_A$.

Nevertheless, $T^{\langle * \rangle_A}$ *is not in general the unique* A*-adjoint of* T *that realizes the minimal norm.*

In the following definition we collect the notions of some classes of A-operators ([1], [4], [16], [19]).

Definition 1.2. Any operator $T \in \mathcal{B}(\mathcal{H})$ is

1. A-contraction if $\|T\xi\|_A \leq \|\xi\|_A$ for every $\xi \in \mathcal{H}$, or equivalently if $T^*AT \leq A$.
2. A-isometry if $T^*AT = A \iff \|T\xi\|_A = \|\xi\|_A$, $\forall\, \xi \in \mathcal{H}$.
3. A-normal if $T^*AT = TAT^* \iff \|T\xi\|_A = \|T^*\xi\|_A$, $\forall\, \xi \in \mathcal{H}$.
4. A-partial isometry if $\|T\xi\|_A = \|\xi\|_A$, $\forall\ \xi \in N(AT)^{\perp_A}$.
5. A-unitary if for any $\xi \in \mathcal{H}$, then
$$T^*AT = TAT^* = A \iff \|T^*\xi\|_A = \|T\xi\|_A = \|\xi\|_A.$$
6. A-hyponormal if
$$TAT^* \leq T^*AT \iff \|T^*\xi\|_A \leq \|T\xi\|_A,\ \forall \xi \in \mathcal{H}.$$
7. A-quasi-isometry if and only if,
$$T^*AT = T^{*2}AT^2 \iff \|T\|_A = \|T^2\|_A.$$
8. (A, m)-isometry if for every $\xi \in \mathcal{H}$, then
$$\sum_{k=0}^{m}(-1)^k \tbinom{m}{k} T^{*m-k}AT^{m-k} = 0 \Leftrightarrow \sum_{k=0}^{m}(-1)^k \tbinom{m}{k} \|T^{m-k}\xi\|_A^2 = 0.$$

Remark 1.3. If $T \in \mathcal{B}_A(\mathcal{H})$, then

$$T \text{ is an } (A, m)-\text{isometry } \iff \sum_{k=0}^{m}(-1)^k \binom{m}{k} \left(T^{\langle * \rangle_A}\right)^{m-k} T^{m-k} = 0.$$

For $m = 2$, we get

$$T^{*2}AT^2-2T^*AT+A=0 \Longleftrightarrow \left(T^{\langle * \rangle_A}\right)^2T^2-2T^{\langle * \rangle_A}T+P_{\overline{\mathcal{R}(A)}}=0$$

Proposition 1.3. ([1]) *If $T \in \mathcal{B}_A(\mathcal{H})$, then T is an A-isometry if and only if*

$$T^{\langle * \rangle_A}T = P_{\overline{\mathcal{R}(A)}}.$$

Proposition 1.4. ([4] *Theorem 2.5*) *Let $T \in \mathcal{B}_A(\mathcal{H})$ with closed range. Then the following statements are equivalent*

1. *$TT^{\langle * \rangle_A}T = T$.*

2. *$T^{\langle * \rangle_A}T$ is a projection (i.e., T is an A-partial isometry).*

2. (A, n)-Power Quasi-Normal Operators

As an extension of the classes of A-quasi-normal operators and A-normal operators ([16], [17]), the following definition describes the class of operators that we will study in this paper.

Definition 2.1. An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be (A, n)-power quasi-normal operator for a positive integer n, if

$$T^nT^{\langle * \rangle_A}T = T^{\langle * \rangle_A}TT^n.$$

We denote the set of all (A, n)-power Quasi-normal operators by $[nQN]_A$.

This class includes the class of A-quasi-normal operator and A-normal operator.

Remark 2.1. Clearly if $n = 1$, then $(A, 1)$- power quasi-normal operator is precisely A- quasi-normal operator.

In the following theorem, we collect some properties of the class $[nQN]_A$.

Theorem 2.1. *Let $T \in \mathcal{B}_A(\mathcal{H})$. If $T \in [nQN]_A$, then*

1. *T is in the class $[2nQN]_A$.*

2. *If T has a dense range in $\mathcal{H}$, then $T^nT^{\langle * \rangle_A} = T^{\langle * \rangle_A}T^n$.*

3. *If S is in the class $[nQN]_A$ such that $[T,\ S] = [T,\ S^{\langle * \rangle_A}] = [T^{\langle * \rangle_A}, S] = 0$, then TS is in the class $[nQN]_A$.*

4. *If S is in the class $[nQN]_A$ such that $ST = TS = T^{\langle * \rangle_A}S = S^{\langle * \rangle_A}T = ST^{\langle * \rangle_A} = TS^{\langle * \rangle_A} = 0$, then $S+T$ is in the class $[nQN]_A$.*

Proof.

1. Since $T \in [nQN]_A$, then
$$T^nT^{\langle * \rangle_A}T = T^{\langle * \rangle_A}TT^n. \tag{2.1}$$
Multiplying (2.1) from left by T^n, we obtain
$$T^{2n}T^{\langle * \rangle_A}T = T^{\langle * \rangle_A}TT^{2n}.$$
Thus T is in the class $[2nQN]_A$.

2. Let T be in the class $[nQN]_A$, and $\eta \in \mathcal{R}(T) : \eta = T\xi,\ \xi \in \mathcal{H}$. Then
$$\begin{aligned}(T^nT^{\langle * \rangle_A} - T^{\langle * \rangle_A}T^n)\eta &= (T^nT^{\langle * \rangle_A} - T^{\langle * \rangle_A}T^n)T\xi \\ &= (T^nT^{\langle * \rangle_A}T - T^{\langle * \rangle_A}T^{n+1})\xi \\ &= 0.\end{aligned}$$
Thus,
$$(T^nT^{\langle * \rangle_A} - T^{\langle * \rangle_A}T^n) = 0 \ \ \text{on} \mathcal{R}(T).$$
Hence,
$$T^nT^{\langle * \rangle_A} = T^{\langle * \rangle_A}T^n \ \ \text{on} \mathcal{H}.$$

3.
$$\begin{aligned}(TS)^n(TS)^{\langle * \rangle_A}TS &= T^nS^nS^{\langle * \rangle_A}T^{\langle * \rangle_A}TS \\ &= T^nT^{\langle * \rangle_A}TS^nS^{\langle * \rangle_A}S \\ &= T^{\langle * \rangle_A}T^{n+1}S^{\langle * \rangle_A}S^{n+1} \\ &= (TS)^{\langle * \rangle_A}(TS)^{n+1}.\end{aligned}$$
Hence, TS is in the class $[nQN]$.

4.
$$\begin{aligned}(T+S)^n(T+S)^{\langle * \rangle_A}(T+S) &= (T^n+S^n)(T^{\langle * \rangle_A}T+S^{\langle * \rangle_A}S) \\ &= T^nT^{\langle * \rangle_A}T+S^nS^{\langle * \rangle_A}S \\ &= T^{\langle * \rangle_A}T^{n+1}+S^{\langle * \rangle_A}S^{n+1} \\ &= (T+S)^{\langle * \rangle_A}(T+S)^{n+1}.\end{aligned}$$
Which implies that $T+S$ is in the class $[nQN]_A$.

Remark 2.2. It is clear that a $(A,2)$-power quasi-normal operator is a $(A,2k)$-power quasi-normal operator and a $(A,3)$-power quasi-normal operator is a $(A,3k)$-power quasi-normal operator.

Lemma 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ such that $AT = TA$. Then T is an (A,n)-power quasi-normal operator if and only if T is a n-power quasi-normal.*

Proof. Note first that the conditions imposed on A and on T imply that $\overline{\mathcal{R}(A)} = \mathcal{H}$ and that $T^* R(A) \subset R(A)$. So $T^{\langle * \rangle_A}$ exists. Moreover

$$T^{\langle * \rangle_A} = A^\dagger T^* A = A^\dagger A T^* = P_{\overline{R(A)}} T^* = T^*.$$

Thus, the assertion follows.

Proposition 2.1. *Let $T \in \mathcal{B}_A(\mathcal{H})$, $B = T^n + T^{\langle * \rangle_A} T$ and $C = T^n - T^{\langle * \rangle_A} T$. Then, we have*

1. *$T \in [nQN]_A$ if and only if B commutes with C.*
2. *if $T \in [nQN]_A$, then $M = T^n T^{\langle * \rangle_A} T$ commutes with B and C.*
3. *Assume that $\mathcal{N}(A)$ is an invariant subspace for T. Then we have*

$$T_\lambda := T - \lambda I \in [nQN]_A,\ \forall\ \lambda \in \mathbb{C} \iff T \text{ is } A-\text{normal}.$$

Proof.

1. We have

$$\begin{gathered} BC = CB \\ \Updownarrow \\ (T^n + T^{\langle * \rangle_A} T)(T^n - T^{\langle * \rangle_A} T) \\ = (T^n - T^{\langle * \rangle_A} T)(T^n + T^{\langle * \rangle_A} T) \\ \Updownarrow \\ T^{2n} - T^n T^{\langle * \rangle_A} T + T^{\langle * \rangle_A} T T^n - (T^{\langle * \rangle_A} T)^2 \\ = T^{2n} + T^n T^{\langle * \rangle_A} T - T^{\langle * \rangle_A} T T^n - (T^{\langle * \rangle_A} T)^2 \\ \Updownarrow \\ T^n T^{\langle * \rangle_A} T = T^{\langle * \rangle_A} T T^n. \end{gathered}$$

2. Assume that T is in the class $[nQN]$. Then

$$\begin{aligned} MB &= T^nT^{\langle*\rangle_A}T(T^n+T^{\langle*\rangle_A}T) \\ &= T^nT^{\langle*\rangle_A}TT^n+T^n(T^{\langle*\rangle_A}T)^2 \\ &= T^nT^nT^{\langle*\rangle_A}T+T^{\langle*\rangle_A}TT^nT^{\langle*\rangle_A}T \\ &= (T^n+T^{\langle*\rangle_A}T)T^nT^{\langle*\rangle_A}T \\ &= BM. \end{aligned}$$

The same steps can prove that M and C commute.

3. Assume that $(T-\lambda I)$ is in the class $[nQN]_A$ for every $\lambda \in \mathbb{C}$. Then for every $\lambda \in \mathbb{C}$, we have

$$(T-\lambda I)^n(T-\lambda I)^{\langle*\rangle_A}(T-\lambda I) = (T-\lambda I)^{\langle*\rangle_A}(T-\lambda)(T-\lambda I)^n.$$

Hence, if we put $a_k = (-1)^k\binom{n}{k}$, we obtain

$$\begin{aligned} &\sum_{k=0}^{n} a_k\lambda^kT^{n-k}(T^{\langle*\rangle_A}T-\lambda T^{\langle*\rangle_A}) \\ &= (T^{\langle*\rangle_A}T-\lambda T^{\langle*\rangle_A})\sum_{k=0}^{n} a_k\lambda^kT^{n-k}. \end{aligned}$$

So,

$$\begin{aligned} &\sum_{k=1}^{n-1} a_k\lambda^k(T^{n-k}T^{\langle*\rangle_A}T-T^{\langle*\rangle_A}TT^{n-k}) \\ &-\sum_{k=1}^{n-1} a_k\lambda^{k+1}(T^{n-k}T^{\langle*\rangle_A}-T^{\langle*\rangle_A}T^{n-k}) = 0. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{k=1}^{n-1} a_k\lambda^k(T^{n-k}T^{\langle*\rangle_A}T-T^{\langle*\rangle_A}TT^{n-k}) \\ &-\sum_{k=1}^{n-2} a_k\lambda^{k+1}(T^{n-k}T^{\langle*\rangle_A}-T^{\langle*\rangle_A}T^{n-k}) \end{aligned}$$

$$-(-1)^n n\lambda^n (T^{\langle *\rangle_A}T - TT^{\langle *\rangle_A}) = 0.$$

Put $\lambda = re^{i\theta}, 0 \le \theta \le 2\pi, \;\; r > 0$, we get

$$\sum_{k=1}^{n-1} a_k (re^{i\theta})^k (T^{n-k}T^{\langle *\rangle_A}T - T^{\langle *\rangle_A}TT^{n-k})$$
$$-\sum_{k=1}^{n-2} a_k (re^{i\theta})^{k+1} (T^{n-k}T^{\langle *\rangle_A} - T^{\langle *\rangle_A}T^{n-k})$$
$$-(-1)^n n (re^{i\theta})^n (T^{\langle *\rangle_A}T - TT^{\langle *\rangle_A}) = 0.$$

So,

$$(T^{\langle *\rangle_A}T - TT^{\langle *\rangle_A})$$
$$= \frac{(-1)^n}{n(re^{i\theta})^n} \sum_{k=1}^{n-1} (-1)^k \tbinom{n}{k} (re^{i\theta})^k (T^{n-k}T^{\langle *\rangle_A}T - T^{\langle *\rangle_A}TT^{n-k})$$
$$- \frac{(-1)^n}{n(re^{i\theta})^n} \sum_{k=1}^{n-2} (-1)^k \tbinom{n}{k} (re^{i\theta})^{k+1} (T^{n-k}T^{\langle *\rangle_A} - T^{\langle *\rangle_A}T^{n-k}).$$

Letting $r \longrightarrow \infty$, we get $T^{\langle *\rangle_A}T - TT^{\langle *\rangle_A} = 0$. Hence, T is A-normal. Conversely it is known that an A- normal operators have translation invariant property i.e., if T is A- normal operator , then $(T - \lambda I)$ is A-normal for all $\lambda \in \mathbb{C}$ and hence $(T - \lambda I)$ is in the class $[nQN]_A$.

Proposition 2.2. *Let $T \in \mathcal{B}_A(\mathcal{H})$ with closed range. If $T \in [nQN]_A$ such that T is A-partial isometry, then*

$$T \in [(n+1)QN]_A.$$

Proof. Since T is A- partial isometry with closed range, then

$$TT^{\langle *\rangle_A}T = T \;\; (\text{ as in Proposition 1.4 }). \tag{2.2}$$

Multiplying (2.2) from left by $T^{\langle *\rangle_A}T^{n+1}$ and using the fact that T is in the class $[nNQ]_A$, we get

$$\begin{aligned} T^{\langle *\rangle_A}T^{n+2} &= T^{\langle *\rangle_A}T^{n+2}T^{\langle *\rangle_A}T \\ &= T^n T^{\langle *\rangle_A}T.TT^{\langle *\rangle_A}T \\ &= T^{n+1}T^{\langle *\rangle_A}T, \end{aligned}$$

which implies that T is in the class $[(n+1)QN]_A$.

Theorem 2.2. *Let $T \in \mathcal{B}_A(\mathcal{H})$.*

1. *If $T \in [2QN]_A \cap [3QN]_A$, then $T \in [nQN]_A$, $\forall\ n \geq 4$.*
2. *If T is in the class $[nQN]_A$ and in the class $[(n+1)QN]_A$, then it is in the class $[(n+2)QN]_A$, that is*
$$[nQN]_A \cap [(n+1)QN]_A \subset [(n+2)QN]_A.$$
3. *If $T \in [nQN]_A \cap [(n+1)QN]_A$ such that T is injective, then T is A-quasi-normal.*

Proof.

1. We shall prove the assertion by induction.
   The case $n = 4$ is a consequence of Theorem 2.1.

   Let us prove the assertion for $n = 5$.
   Since $T \in [2QN]_A$, then
$$T^2 T^{\langle * \rangle_A} T = T^{\langle * \rangle_A} T^3. \tag{2.3}$$
   So by multiplying (2.3) from left by T^3, we get
$$T^5 T^{\langle * \rangle_A} T = T^3 T^{\langle * \rangle_A} T^3.$$
   Thus, since $T \in [3QN]_A$, we have
$$\begin{aligned} T^5 T^{\langle * \rangle_A} T &= T^3 T^{\langle * \rangle_A} T^3 \\ &= T^{\langle * \rangle_A} T^4 T^2 \\ &= T^{\langle * \rangle_A} T^6. \end{aligned}$$
   Now, assume that the result is true for an integer $n \geq 5$, that is
$$T^n T^{\langle * \rangle_A} T = T^{\langle * \rangle_A} T T^n.$$
   Then,
$$\begin{aligned} T^{n+1} T^{\langle * \rangle_A} T &= T T^{\langle * \rangle_A} T^{n+1} \\ &= T T^{\langle * \rangle_A} T^3 T^{n-2} \\ &= T^3 T^{\langle * \rangle_A} T T^{n-2} \\ &= T^{\langle * \rangle_A} T^4 T^{(n-2)} \\ &= T^{\langle * \rangle_A} T^{n+2}. \end{aligned}$$
   Thus, T is in the class $[(n+1)QN]_A$.

2. Let $T \in [nQN]_A \cap [(n+1)QN]_A$. Since T is in the class $[nQN]_A$, then

$$T^n T^{\langle * \rangle_A} T = T^{\langle * \rangle_A} T T^n.$$

So,

$$T^{n+1} T^{\langle * \rangle_A} T T = T T^{\langle * \rangle_A} T T^{n+1}.$$

Since T is in the class $[(n+1)QN]_A$, then

$$T^{\langle * \rangle_A} T T^{n+2} = T^{n+2} T^{\langle * \rangle_A} T.$$

Hence, T is in the class $[(n+2)QN]_A$.

3. We have $T^n(T T^{\langle * \rangle_A} T - T^{\langle * \rangle_A} T^2) = 0$. Since T is injective, then $T T^{\langle * \rangle_A} T - T^{\langle * \rangle_A} T^2 = 0$. Hence, T is A- quasi-normal.

Proposition 2.3. *Let $T \in \mathcal{B}_A(\mathcal{H})$ such that $\mathcal{N}(A)$ is an invariant subspace under the action of T. Then, we have the following properties:*

1. *If T is in the class $[2QN]_A$ and T is an $(A,2)$-isometry, then T^2 is in the class $[nQN]_A$ for all positive integer $n \geq 2$.*
2. *If $T \in [2QN]_A$ such that T is an $(A,2)$-isometry, then T is an A-isometry.*
3. *If T is in the class $[2QN]_A \cap [3QN]_A$ and T is an (A,n)-isometry, then T is an A- isometry.*

Proof.

1. From Theorem 2.2, it suffices to prove that T^2 is in the class $[2QN]_A$ and T^2 is in the class $[3QN]_A$.
Since T is in the class $[2QN]_A$ and T is an $(A,2)$-isometry, we have

$$\begin{aligned}
T^4(T^{\langle * \rangle_A 2} T^2) &= T^4(2T^{\langle * \rangle_A} T - P) \\
&= 2T^4 T^{\langle * \rangle_A} T - T^4 P \\
&= 2T^2(T^{\langle * \rangle_A} T^3) - T^4 P \\
&= 2T^{\langle * \rangle_A} T^5 - P T^4 \\
&= (2T^{\langle * \rangle_A} T - P) T^4 \\
&= T^{\langle * \rangle_A 2} T^6.
\end{aligned}$$

Hence, T^2 is in the class $[2QN]_A$.

On the other hand,

$$\begin{aligned} T^6(T^{\langle * \rangle_A 2}T^2) &= T^6(2T^{\langle * \rangle_A}T - P) \\ &= 2T^6T^{\langle * \rangle_A}T - PT^6 \\ &= 2T^4(T^{\langle * \rangle_A}T^3) - PT^6 \\ &= 2T^{\langle * \rangle_A}T^7 - PT^6 \\ &= (2T^{\langle * \rangle_A}T - P)T^6 \\ &= (T^{\langle * \rangle_A 2}T^2)T^6. \end{aligned}$$

Thus, T^2 is in the class $[3QN]_A$.

2. By the definition of $(A, 2)$-isometry,

$$(T^{\langle * \rangle_A 2}T^2)(T^{\langle * \rangle_A}T) - 2(T^{\langle * \rangle_A}T)^2 + PT^{\langle * \rangle_A}T = 0.$$

Since T is in the class $[2QN]_A$, then

$$T^{\langle * \rangle_A 2}(T^{\langle * \rangle_A}T)T^2 - 2(T^{\langle * \rangle_A}T)^2 + PT^{\langle * \rangle_A}T = 0,$$

that is

$$T^{\langle * \rangle_A 3}T^3 - 2(T^{\langle * \rangle_A}T)^2 + T^{\langle * \rangle_A}PT = 0. \tag{2.4}$$

Also, we have

$$T^{\langle * \rangle_A}\left[T^{\langle * \rangle_A 2}T^2 - 2T^{\langle * \rangle_A}T + P\right]T = 0$$

i.e.

$$T^{\langle * \rangle_A 3}T^3 - 2T^{\langle * \rangle_A 2}T^2 + T^{\langle * \rangle_A}PT = 0. \tag{2.5}$$

From (2.4) and (2.5) $T^{\langle * \rangle_A 2}T^2 = (T^{\langle * \rangle_A}T)^2$ and so,

$$\begin{aligned} (T^{\langle * \rangle_A}T)^2 - 2(T^{\langle * \rangle_A}T) + P &= T^{\langle * \rangle_A 2}T^2 - 2T^{\langle * \rangle_A}T + P \\ &= \left(T^{\langle * \rangle_A}T - P\right)^2 \\ &= 0, \end{aligned}$$

or

$$T^{\langle * \rangle_A}T = P.$$

Hence, T is an A-isometry by Proposition 1.3.

3. By the definition of (A,n)-isometry,

$$\begin{aligned} T^{\langle *\rangle_A n}T^nT^{\langle *\rangle_A}T &- \binom{n}{1}T^{\langle *\rangle_A n-1}T^{n-1}T^{\langle *\rangle_A}T+... \\ &+ (-1)^{n-2}\binom{n}{n-2}T^{\langle *\rangle_A 2}T^2T^{\langle *\rangle_A}T \\ &+ (-1)^{n-1}\binom{n}{n-1}T^{\langle *\rangle_A}TT^{\langle *\rangle_A}T \\ &+ (-1)^nPT^{\langle *\rangle_A}T=0. \end{aligned} \tag{2.6}$$

Since T is in $[2QN]_A\cap[3QN]_A$, we have by Theorem 2.2

$$\begin{aligned} T^{\langle *\rangle_A n+1}T^{n+1} &- \binom{n}{1}T^{\langle *\rangle_A n}T^n+... \\ &+ (-1)^{n-2}\binom{n}{n-2}T^{\langle *\rangle_A 3}T^3 \\ &+ (-1)^n\binom{n}{n-1}(T^{\langle *\rangle_A}T)^2 \\ &+ (-1)^nT^{\langle *\rangle_A}PT=0. \end{aligned} \tag{2.7}$$

Also, we have

$$\begin{aligned} T^{\langle *\rangle_A}\Big[T^{\langle *\rangle_A n}T^n &- \binom{n}{1}T^{\langle *\rangle_A n-1}T^{n-1}+... \\ &+ (-1)^{n-1}\binom{n}{n-1}TT^{\langle *\rangle_A}+(-1)^nP\Big]T=0, \end{aligned}$$

that is

$$\begin{aligned} T^{\langle *\rangle_A n+1}T^{n+1} &- \binom{n}{1}T^{\langle *\rangle_A n}T^n+... \\ &+ (-1)^{n-1}\binom{n}{n-1}T^{\langle *\rangle_A 2}T^2 \\ &+ (-1)^nT^{\langle *\rangle_A}PT=0. \end{aligned} \tag{2.8}$$

From (2.6) and (2.7), we get $T^{2\langle *\rangle_A}T^2=(T^{\langle *\rangle_A}T)^2$.
Consequently

$$(T^{\langle *\rangle_A})^kT^k=(T^{\langle *\rangle_A}T)^k,\ \ \forall\ \ k\in\mathbb{N}.$$

Then,

$$\begin{aligned} (T^{\langle *\rangle_A}T)^n &- \binom{n}{1}(T^{\langle *\rangle_A}T)^{n-1}+.... \\ &+ (-1)^{n-1}\binom{n}{n-1}(T^{\langle *\rangle_A}T) \\ &+ (-1)^nP \\ &= 0 \\ &= (P-T^{\langle *\rangle_A}T)^n. \end{aligned}$$

This completes the proof.

Theorem 2.3. *Let $T \in \mathcal{B}_A(\mathcal{H})$ such that $\mathcal{N}(T^*A) \subset \mathcal{N}(T)$ and $\mathcal{R}(AT) \subset \mathcal{R}(T)$. Then the following properties hold:*

1. *If $T \in [nQN]_A$, then $T^{\langle * \rangle_A} T^n = T^n T^{\langle * \rangle_A}$ and T^n is $A-$ normal.*
2. *If $T \in [2QN]_A \cap [3QN]_A$, then T is A-normal.*

Proof.

1. Since $T \in [nQN]_A$, we have $T^n T^{\langle * \rangle_A} T = T^{\langle * \rangle_A} T T^n$.
   We deduce that
   $$\big(T^n T^{\langle * \rangle_A} - T^{\langle * \rangle_A} T^n\big) T = 0.$$
   It follows that
   $$T^n T^{\langle * \rangle_A} - T^{\langle * \rangle_A} T^n = 0 \text{ on } \mathcal{R}(T).$$
   By hypothesis, we obtain
   $$T^n T^{\langle * \rangle_A} - T^{\langle * \rangle_A} T^n = 0 \text{ on } \overline{\mathcal{R}(AT)}.$$
   On the other hand, since $\mathcal{N}(T^*A) \subset \mathcal{N}(T)$, we have
   $$T^n T^{\langle * \rangle_A} - T^{\langle * \rangle_A} T^n = 0 \text{ on } \mathcal{N}(T^*A) = \mathcal{N}\big((AT)^*\big).$$
   Then, the result follows from the identity
   $$\mathcal{H} = \mathcal{N}\big((AT)^*\big) \oplus \mathcal{N}\big((AT)^*\big)^{\perp}.$$
2. Since $T \in [2QN]_A \cap [3QN]_A$, then
   $$T^2 T^{\langle * \rangle_A} T = T^{\langle * \rangle_A} T T^2 \text{ and } T^3 T^{\langle * \rangle_A} T = T^{\langle * \rangle_A} T T^3,$$
   so
   $$\big(T T^{\langle * \rangle_A} - T^{\langle * \rangle_A} T\big) T^3 = 0.$$
   It follows that $\big(T T^{\langle * \rangle_A} - T^{\langle * \rangle_A} T\big) T^2 = 0$ on $\mathcal{R}(T)$ and hence $\big(T T^{\langle * \rangle_A} - T^{\langle * \rangle_A} T\big) T^2 = 0$ on $\mathcal{R}(AT)$. Since $\mathcal{N}(T^*A) \subset \mathcal{N}(T)$, we get $\big(T T^{\langle * \rangle_A} - T^{\langle * \rangle_A} T\big) T^2 = 0$ on $\mathcal{N}(T^*A)$. Hence,
   $$\big(T T^{\langle * \rangle_A} - T^{\langle * \rangle_A} T\big) T^2 = 0 \text{ on } \mathcal{H}.$$
   By repeating this process, we obtain
   $$T T^{\langle * \rangle_A} - T^{\langle * \rangle_A} T = 0.$$

Proposition 2.4. *Let $T \in \mathcal{B}_A(\mathcal{H})$ such that $\mathcal{N}(A)$ is an invariant subspace under the action of T,*

$$\mathcal{R}(AT) = \mathcal{R}(T) \ \text{ and } \ \mathcal{N}(A(T-kI)) = \mathcal{N}(T-kI) \ \text{ for } k = 0, 1.$$

So,

if T and $T-I$ are in the class class $[2QN]_A$, then T is A-normal.

Proof. It is easy to see that the condition on $T-I$ implies that

$$T^2(T^{\langle * \rangle_A}T) - T^2T^{\langle * \rangle_A} - 2T(T^{\langle * \rangle_A}T) + 2TT^{\langle * \rangle_A}$$

$$= (T^{\langle * \rangle_A}T)T^2 - T^{\langle * \rangle_A}T^2 - 2(T^{\langle * \rangle_A}T)T + 2T^{\langle * \rangle_A}T.$$

Since T is in the class $[2QN]_A$, we have

$$-T^2T^{\langle * \rangle_A} - 2T(T^{\langle * \rangle_A}T) + 2TT^{\langle * \rangle_A} = -T^{\langle * \rangle_A}T^2 - 2(T^{\langle * \rangle_A}T)T + 2T^{\langle * \rangle_A}T.$$

As $TT^{\langle * \rangle_A}$, $T^{\langle * \rangle_A}T$ are A-self-adjoint operators, $\mathcal{R}(TT^{\langle * \rangle_A}) \subset \overline{\mathcal{R}(A)}$ and $\mathcal{R}(T^{\langle * \rangle_A}T) \subset \overline{\mathcal{R}(A)}$, we have

$$\left(TT^{\langle * \rangle_A}\right)^{\langle * \rangle_A} = TT^{\langle * \rangle_A} \ \text{ and } \ \left(T^{\langle * \rangle_A}T\right)^{\langle * \rangle_A} = T^{\langle * \rangle_A}T.$$

It follows that

$$\begin{aligned} - \quad & TT^{\langle * \rangle_A 2} - 2(T^{\langle * \rangle_A}T)T^{\langle * \rangle_A} + 2TT^{\langle * \rangle_A} \\ & = -T^{\langle * \rangle_A 2}T - 2T^{\langle * \rangle_A}(T^{\langle * \rangle_A}T) + 2T^{\langle * \rangle_A}T \end{aligned} \tag{2.9}$$

Let us now show that (2.9) implies

$$\mathcal{N}(T^{\langle * \rangle_A}) \subset \mathcal{N}(T). \tag{2.10}$$

We suppose that $T^{\langle * \rangle_A}\xi = 0$. Then from (2.9), we get

$$-3T^{\langle * \rangle_A 2}T\xi + 2T^{\langle * \rangle_A}T\xi = 0. \tag{2.11}$$

So,

$$-3T^{\langle * \rangle_A 3}T\xi + 2T^{\langle * \rangle_A 2}T\xi = 0.$$

Therefore, as T is in the class $[2QN]_A$, then

$$-3T^{\langle * \rangle_A}TT^{\langle * \rangle_A 2}\xi + 2T^{\langle * \rangle_A 2}T\xi = 0$$

and hence,

$$2T^{\langle * \rangle_A 2}T\xi = 0.$$

Consequently, (2.11) gives $2T^{\langle * \rangle_A}T\xi = 0$ or $AT\xi = 0$, i.e., $\mathcal{N}(T^{\langle * \rangle_A}) \subset \mathcal{N}(AT) \subset \mathcal{N}(T)$. This proves (2.10). Theorem 2.3 implies that T^2 is A- normal. This together with (2.9) gives

$$-T(T^{\langle * \rangle_A}T) + TT^{\langle * \rangle_A} = -(T^{\langle * \rangle_A}T)T + T^{\langle * \rangle_A}T$$

or

$$T^{\langle * \rangle_A}(T^{\langle * \rangle_A}T - TT^{\langle * \rangle_A}) = T^{\langle * \rangle_A}T - TT^{\langle * \rangle_A}. \tag{2.12}$$

If $\mathcal{N}(T^{\langle * \rangle_A} - I) = \{0\}$, then (2.12) implies that T is A-normal.

Now, assume that $\mathcal{N}(T^{\langle * \rangle_A} - I)$ is non trivial. Let

$$T^{\langle * \rangle_A 2}T\xi - T^{\langle * \rangle_A}T\xi = T^{\langle * \rangle_A}T\xi - T\xi.$$

Since $T^{\langle * \rangle_A 2}T = TT^{\langle * \rangle_A 2}$, we get

$$T^{\langle * \rangle_A}T\xi = T\xi.$$

Therefore,

$$||T\xi||_A^2 = < T^{\langle * \rangle_A}T\xi \mid \xi >_A = < T\xi \mid \xi >_A = < \xi \mid T^{\langle * \rangle_A}\xi >_A = ||\xi||_A^2.$$

Hence,

$$\begin{aligned} ||T\xi - \xi||_A^2 &= ||T\xi||_A^2 + ||\xi||_A^2 - 2Re < T\xi \mid \xi >_A \\ &= ||T\xi||_A^2 - ||\xi||_A^2 \\ &= 0. \end{aligned}$$

But $AT\xi = A\xi$. Then $\mathcal{N}(T^{\langle * \rangle_A} - I) \subset \mathcal{N}(AT - A) \subset \mathcal{N}(T - I)$.
This together with (2.11), yields to

$$T(T^{\langle * \rangle_A}T - TT^{\langle * \rangle_A}) = T^{\langle * \rangle_A} * T - TT^{\langle * \rangle_A}$$

and so,

$$T(T^{\langle * \rangle_A}T - TT^{\langle * \rangle_A})T = (T^{\langle * \rangle_A}T - TT^{\langle * \rangle_A})T$$

or

$$TT^{\langle * \rangle_A}T^2 - T^2T^{\langle * \rangle_A}T = T^{\langle * \rangle_A}T^2 - TT^{\langle * \rangle_A}T.$$

Since $T^2T^{\langle * \rangle_A} = T^{\langle * \rangle_A}T^2$ and $T^3T^{\langle * \rangle_A} = T^{\langle * \rangle_A}T$, we deduce that $T^{\langle * \rangle_A}T^2 = TT^{\langle * \rangle_A}T$.
Thus T is A- quasi-normal i.e., $T \in [QN]_A$. From Theorem 2.3, the A-normality of T follows.

In attempt to extend the above result for operators in the class $[nQN]_A$, we state the following result.

Theorem 2.4. *Let $T \in \mathcal{B}_A(\mathcal{H})$ such that $\mathcal{N}(A)$ is an invariant subspace under the action of T,*

$$\mathcal{R}(A(T)) = \mathcal{R}(T) \text{ and } \mathcal{N}(A(T-kI)) = \mathcal{N}(T-kI) \text{ for } k = 0,1.$$

If T is in the class $[2QN]_A \cap [3QN]_A$ such that $T - I$ is in the class $[nQN]_A$, then T is A- normal.

Proof. Since $T - I$ is in the class $[nQN]_A$, we have

$$\sum_{k=1}^{n} a_k T^k T^{\langle * \rangle_A} T - \sum_{k=1}^{n} a_k T^k T^{\langle * \rangle_A}$$
$$= T^{\langle * \rangle_A} T \sum_{k=1}^{n} a_k T^k - T^{\langle * \rangle_A} \sum_{k=1}^{n} a_k T^k,\ a_k = (-1)^{n-k}\binom{n}{k}.$$

Under the condition on T, we have by Theorem 2.3

$$a_1 T(T^{\langle * \rangle_A} T) - \Big(\sum_{k=1}^{n} a_k T^k\Big) T^{\langle * \rangle_A} = a_1 (T^{\langle * \rangle_A} T) T - T^{\langle * \rangle_A} \Big(\sum_{k=1}^{n} a_k T^k\Big)$$

or

$$a_1 (T^{\langle * \rangle_A} T) T^{\langle * \rangle_A} - T \sum_{k=1}^{n} a_k T^{\langle * \rangle_A k} =$$
$$a_1 T^{\langle * \rangle_A} (T^{\langle * \rangle_A} T) - (\sum_{k=1}^{n} a_k T^{\langle * \rangle_A k}) T, \tag{2.13}$$

which implies that $\mathcal{N}(T^{\langle * \rangle_A}) \subset \mathcal{N}(T)$.
In fact, let $T^{\langle * \rangle_A} \xi = 0$. From (2.13), we have

$$a_1 T^{\langle * \rangle_A 2} T \xi - \Big(\sum_{k=1}^{n} a_k T^{*k}\Big) T \xi = 0.$$

Since T is in $[2QN]_A \cap [3QN]_A$, we deduce that

$$a_1 T^{\langle * \rangle_A 2} T \xi - a_1 T^{\langle * \rangle_A} T \xi - a_2 T^{\langle * \rangle_A 2} T \xi = 0. \tag{2.14}$$

Then

$$a_1 T^{\langle * \rangle_A 3} T \xi - a_1 T^{\langle * \rangle_A 2} T - a_2 T^{\langle * \rangle_A 3} T \xi = 0.$$

Hence,

$$a_1 T^{\langle * \rangle_A 2} T \xi = 0.$$

Consequently (2.14) gives $T^{\langle*\rangle_A}T\xi=0$, which implies that

$$AT\xi=0.$$

It follows by Theorem 2.3 that T^k is a A-normal for $k=2,3,...,n$ and hence,

$$T(T^{\langle*\rangle_A}T)-TT^{\langle*\rangle_A}=(T^{\langle*\rangle_A}T)T-T^{\langle*\rangle_A}T$$

or,

$$T^{\langle*\rangle_A}(TT^{\langle*\rangle_A}-T^{\langle*\rangle_A}T)=TT^{\langle*\rangle_A}-T^{\langle*\rangle_A}T.$$

Hence,

$$(T^{\langle*\rangle_A}-I)(TT^{\langle*\rangle_A}-T^{\langle*\rangle_A}T)=0.$$

A similar argument as in the proof of Proposition 2.4 gives the desired result.

Theorem 2.5. *Let $T\in\mathcal{B}_A(\mathcal{H})$ such that $T\in\big([2QN]_A\cap[3QN]_A\big)$, $T^2\in[2QN]_A$ and $\mathcal{N}(T^2)\subset\mathcal{N}(T^{\langle*\rangle_A 2})$. If A is injective, then T^2 is A-quasi-normal.*

Proof. The condition T^2 is in the class $[2QN]_A$ gives

$$T^{\langle*\rangle_A 4}(T^{\langle*\rangle_A 2}T^2)=(T^{\langle*\rangle_A 2}T^2)T^{\langle*\rangle_A 4},$$

which implies that

$$T^{\langle*\rangle_A 5}(T^{\langle*\rangle_A}T)T=(T^{\langle*\rangle_A 2}T^2)T^{\langle*\rangle_A 4}.$$

But T is in the class $[3QN]_A$, so

$$T^{\langle*\rangle_A 2}(T^{\langle*\rangle_A}T)T^{\langle*\rangle_A 3}T=(T^{\langle*\rangle_A 2}T\langle*\rangle_A 2)T^{\langle*\rangle_A 4}.$$

Hence,

$$T^{\langle*\rangle_A 2}(T^{\langle*\rangle_A}T)^2T^{\langle*\rangle_A 2}=(T^{\langle*\rangle_A 2}T^2)T^{\langle*\rangle_A 4}\quad\big(T\in\ [2QN]_A\big).$$

Then the facts that $T\in[2QN]_A$ and $(T^{\langle*\rangle_A})^{\langle*\rangle_A}=T$ give

$$(T^{\langle*\rangle_A}T)^2T^{\langle*\rangle_A 4}=(T^{\langle*\rangle_A 2}T^2)T^{\langle*\rangle_A 4}$$

and

$$T^4((T^{\langle*\rangle_A}T)^2-T^{\langle*\rangle_A 2}T^2)=0.$$

Since $\mathcal{N}(T^2)\subset\mathcal{N}(T^{\langle*\rangle_A 2})$, then

$$T^{\langle*\rangle_A 2}T^2((T^{\langle*\rangle_A}T)^2-T^{\langle*\rangle_A 2}T^2)=0$$

or

$$T^2[(T^{\langle *\rangle_A}T)^2 - T^{\langle *\rangle_A 2}T^2)] = 0. \tag{2.15}$$

Hence,

$$T^{\langle *\rangle_A 2}[((T^{\langle *\rangle_A}T)^2 - T^{\langle *\rangle_A 2}T^2)] = 0, \ \ \big(\mathcal{N}(T^2) \subset \mathcal{N}(T^{\langle *\rangle_A 2})\big).$$

Or

$$[((T^{\langle *\rangle_A}T)^2 - T^{\langle *\rangle_A 2}T^2)]T^2 = 0. \tag{2.16}$$

Since T is in the class $[2QN]_A$, T^2 commutes with $(T^{\langle *\rangle_A}T)^2$, then the desired conclusion follows from (2.15) and (2.16). Hence,

$$[((T^{\langle *\rangle_A}T)^2 - T^{\langle *\rangle_A 2}T^2)]T^2 = 0. \tag{2.17}$$

Proposition 2.5. *Take an arbitrary nonnegative integer n. Let $T \in \mathcal{L}_A(\mathcal{H})$ such that T is in the class $(A,1)$- power quasi-normal. Then*

(1) $(T^{\langle *\rangle_A}T)^nT = T(T^{\langle *\rangle_A}T)^n$.

(2) $(T^{\langle *\rangle_A}T)^n = T^{\langle *\rangle_A n}T^n$.

(3) $(T^{\langle *\rangle_A n}T^n)T = T(T^{\langle *\rangle_A n}T^n)$.

Proof. (1) Since $TT^{\langle *\rangle_A}T = T^{\langle *\rangle_A}T^2$, then $T^n(T^{\langle *\rangle_A}T)^n = (T^{\langle *\rangle_A}T)^nT^n$.

(2) $(T^{\langle *\rangle_A}T)^n = \underbrace{T^{\langle *\rangle_A}TT^{\langle *\rangle_A}T...T^{\langle *\rangle_A}T}_{k-times} = (T^{\langle *\rangle_A})^nT^n$.

(3)

$$\begin{aligned}
(T^{\langle *\rangle_A}T)^nT &= (T^{\langle *\rangle_A})^{n-1}(T^{\langle *\rangle_A}T)T^n \\
&= (T^{\langle *\rangle_A})^{n-1}T^n(T^{\langle *\rangle_A}T) = (T^{\langle *\rangle_A})^{n-2}T^{n-1}(T^{\langle *\rangle_A})^2T^2 \\
&= (T^{\langle *\rangle_A})^{n-3}T^{n-2}(T^{\langle *\rangle_A})^3T^3 \\
&= ..... \\
&= T(T^{\langle *\rangle_A})^nT^n.
\end{aligned}$$

3. Tensor Product of (A, n)-Power Quasi-Normal Operators

Let $\mathcal{H}\overline{\otimes}\mathcal{H}$ denote the completion, endowed with a reasonable uniform crose-norm, of the algebraic tensor product $\mathcal{H}\otimes\mathcal{H}$ of $\mathcal{H}$ with itself. Given non-zero operators $T,S\in\mathcal{B}(\mathcal{H})$, let $T\otimes S\in\mathcal{B}(\mathcal{H}\overline{\otimes}\mathcal{H})$ denote the tensor product defined on the Hilbert space $\mathcal{H}\overline{\otimes}\mathcal{H}$ as follows

$$\langle T\otimes S(\xi_1\otimes\eta_1)|\,(\xi_2\otimes\eta_2)\rangle=\langle T\xi_1|\,\xi_2\rangle\langle S\eta_1|\,\eta_2\rangle.$$

The operation of taking tensor products $T\otimes S$ preserves many properties of $T,S\in\mathcal{B}(\mathcal{H})$, but by no means all of them. Whereas $T\otimes S$ is normal if and only if T and S are normal, there exist paranormal operators T and S such that $T\otimes S$ is not paranormal. It is proved that for non-zero $T,S\in\mathcal{B}(\mathcal{H}),T\otimes S$ is p-hyponormal if and only if T and S are p-hyponormal. These results are extended to p-quasihyponormal operators for more details see [9], [10], [11], [12], [18] and the references therein.

In the following proposition we will prove the stability of the class of (A,n)-power quasinormal operators under the direct sum and tensor product.
Recall that $(T\otimes S)^*(T\otimes S)=T^*T\otimes S^*S$ and by the uniqueness of positive square roots, $|T\otimes S|^r=|T|^r\otimes|S|^r$ for any positive rational number r. From the density of the rational set in the real set, we obtain $|T\otimes S|^p=|T|^p\otimes|S|^p$ for every positive real number p. Observe also that

$$A\otimes B=(A\otimes I)(I\otimes B)=(I\otimes B)(A\otimes I).$$

Proposition 3.1. *Let $A_i\in\mathcal{L}_+(\mathcal{H})$ and $T_i\in\mathcal{L}_{A_i}(\mathcal{H})$ such that T_i is in the class $[nQN]_{A_i}$ for $i=1,2,...,p$, then we have the following properties*

1. *$T_1\oplus T_2\oplus...\oplus T_p$ is in the class $[nQN]_{\left(A_1\oplus A_2\oplus...\oplus A_p\right)}$.*
2. *$T_1\otimes T_2\otimes...\otimes T_p$ is in the class $[nQN]_{\left(A_1\otimes A_2\otimes..\otimes A_p\right)}$.*

Proof.

1.

$$\begin{aligned}&(T_1\oplus...\oplus T_p)^n(T_1\oplus...\oplus T_p)^{\langle*\rangle_{\left(A_1\oplus...\oplus A_p\right)}}(T_1\oplus...\oplus T_p)\\=\;&T_1^nT_1^{\langle*\rangle_{A_1}}T_1\oplus T_2^nT_2^{\langle*\rangle_{A_2}}T_2\oplus...\oplus T_p^nT_p^{\langle*\rangle_{A_p}}T_p\\=\;&T_1^{\langle*\rangle_{A_1}}T_1T_1^n\oplus T_2^{\langle*\rangle_{A_2}}T_2T_2^n\oplus...\oplus T_p^{\langle*\rangle_{A_p}}T_pT_p^n\\=\;&(T_1\oplus...\oplus T_p)^{\langle*\rangle_{\left(A_1\oplus...\oplus A_p\right)}}(T_1\oplus...\oplus T_p)(T_1\oplus...\oplus T_p)^n.\end{aligned}$$

Hence, $T_1\oplus...\oplus T_p$ is in the class $[nQN]_{\left(A_1\oplus A_2\oplus...\oplus A_p\right)}$.

2. Let $x_1, x_2, ..., x_p \in \mathcal{H}$. Then

$$(T_1 \otimes ... \otimes T_p)^n (T_1 \otimes ... \otimes T_p)^{\langle * \rangle \left(A_1 \otimes ... \otimes A_p\right)} (T_1 \otimes ... \otimes T_p)(x_1 \otimes .... \otimes x_p)$$

$$\begin{aligned} &= T_1^n T_1^{\langle * \rangle A_1} T_1 x_1 \otimes .... \otimes T_p^n T_p^{\langle * \rangle A_p} T_p x_p \\ &= T_1^{\langle * \rangle A_1} T_1 T_1^n x_1 \otimes .... \otimes T_p^{\langle * \rangle A_p} T_p T_p^n x_p \end{aligned}$$

$$= (T_1 \otimes ... \otimes T_p)^{\langle * \rangle \left(A_1 \otimes .... \otimes A_p\right)} (T_1 \otimes ... \otimes T_p)^{n+1} (x_1 \otimes .... \otimes x_p).$$

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