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L-JOIN MEET APPROXIMATION OPERATORS

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Abstract: In this paper, we define L-join meet and L-meet join approximation operators as a generalization of fuzzy rough set in complete residuated lattices. We show that L-join meet and L-meet join approximation operators are induced by various L-fuzzy relations. We investigate relations between their operations and Alexandrov L-topologies.

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1. Introduction

Pawlak [7,8] introduced rough set theory as a formal tool to deal with imprecision and uncertainty in data analysis. Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Radzikowska [9] developed fuzzy rough sets induced by various L-fuzzy relations in complete residuated lattice. Bělohlávek [1] investigated information systems and decision rules in complete residuated lattices. Zhang [5,6] introduced Alexandrov L-topologies induced by fuzzy rough sets. Algebraic structures of fuzzy rough sets are developed in many directions [3,4,10,11]

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We show that L-join meet and L-meet join approximation operators are induced by various L-fuzzy relations. We investigate relations between their operations and Alexandrov L-topologies.

2. Preliminaries

Definition 1. [1,2] An algebra $(L, \wedge, \vee, \odot, \to, \perp, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$;

(C2) $(L, \odot, \top)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

In this paper, we assume $(L, \wedge, \vee, \odot, \to,^* \perp, \top)$ is a complete residuated lattice with the law of double negation;i.e. $x^{**} = x$. For $\alpha \in L, A, \top_x \in L^X$, $(\alpha \to A)(x) = \alpha \to A(x)$, $(\alpha \odot A)(x) = \alpha \odot A(x)$ and $\top_x(x) = \top, \top_x(y) = \perp$, otherwise.

Lemma 2. [1,2] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $x \odot y \leq x \odot z$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$.

(2) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$.

(3) $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(4) $\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ and $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.

(5) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.

(6) $x \odot y = (x \to y^*)^*$.

(7) $x \odot (x \to y) \leq y$.

(8) $(x \to y) \odot (y \to z) \leq x \to z$.

Definition 3. [1,5] Let X be a set. A function $R : X \times X \to L$ is called:

(R1) *reflexive* if $R(x, x) = \top$ for all $x \in X$,

(R2) *symmetric* if $R(x, x) = \top$ for all $x \in X$,

(R3) *transitive* if $R(x, y) \odot R(y, z) \leq R(x, z)$, for all $x, y, z \in X$.

(R4) *Euclidean* if $R(x, z) \odot R(y, z) \leq R(x, y)$, for all $x, y, z \in X$.

If R satisfies (R1) and (R3), R is called an *L-fuzzy preorder.*

If R satisfies (R1), (R2) and (R3), R is called an *L-fuzzy equivalence relation.*

3. L-Join Meet Approximation Operators

Definition 4. A map $\mathcal{K} : L^X \to L^X$ is called an *L-join meet approximation operator* iff it satisfies the following conditions

(K1) $\mathcal{K}(A) \leq A^*$,

(K2) $\mathcal{K}(\alpha \odot A) = \alpha \to \mathcal{K}(A)$,

(K3) $\mathcal{K}(\bigvee_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{K}(A_i)$.

A map $\mathcal{M} : L^X \to L^X$ is called an *L-meet join approximation operator* iff it satisfies the following conditions

(M1) $A^* \leq \mathcal{M}(A)$,

(M2) $\mathcal{M}(\alpha \to A) = \alpha \odot \mathcal{M}(A)$,

(M3) $\mathcal{M}(\bigwedge_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{M}(A_i)$.

Lemma 5. (1) *If $\mathcal{K}$ is an L-join meet approximation operator, then $\alpha \odot \mathcal{K}(A) \leq \mathcal{K}(\alpha \to A)$.*

(2) *If $\mathcal{M}$ is an L-meet join approximation operator, then $\mathcal{M}(\alpha \odot A) \leq \alpha \to \mathcal{M}(A)$.*

Proof. (1) If $A \leq B$, by (K3), $\mathcal{K}(B) = \mathcal{K}(A \vee B) = \mathcal{K}(A) \wedge \mathcal{K}(B)$, that is, $\mathcal{K}(B) \leq \mathcal{K}(A)$. Since $\alpha \odot (\alpha \to A) \leq A$ from Lemma 2(7),

$$\mathcal{K}(A) \leq \mathcal{K}(\alpha \odot (\alpha \to A)) = \alpha \to \mathcal{K}(\alpha \to A).$$

Hence $\alpha \odot \mathcal{K}(A) \leq \mathcal{K}(\alpha \to A)$.

(2) Since $\mathcal{M}(A) \leq \mathcal{M}(B)$ for $B \leq A$ from (M3), for $A \leq \alpha \to \alpha \odot A$, we have

$$\mathcal{M}(\alpha \to \alpha \odot A) = \alpha \odot \mathcal{M}(\alpha \odot A) \leq \mathcal{M}(A).$$

Hence $\mathcal{M}(\alpha \odot A) \leq \alpha \to \mathcal{M}(A)$.

□

Theorem 6. (1) *A map $\mathcal{K} : L^X \to L^X$ is an L-join meet approximation operator iff there exists a reflexive L-fuzzy relation $R^* \in L^{X\times X}$ such that*

$$\mathcal{K}(A)(y) = \bigwedge_{x\in X} (A(x) \to R(x, y)).$$

(2) *A map $\mathcal{K} : L^X \to L^X$ is an L-join meet approximation operator with $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A)$ iff there exists an L-fuzzy preorder $R^* \in L^{X\times X}$ such that*

$$\mathcal{K}(A)(y) = \bigwedge_{x\in X} (A(x) \to R(x, y)).$$

(3) *A map* $\mathcal{K} : L^X \to L^X$ *is an L-join meet approximation operator with* $\mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A)$ *iff there exists a reflexive and Euclidean L-fuzzy relation* $R^* \in L^{X\times X}$ *such that*

$$\mathcal{K}(A)(y) = \bigwedge_{x\in X}(A(x) \to R(x,y)).$$

Proof. (1) ($\Rightarrow$) Define $R(x,y) = \mathcal{K}(\top_x)(y)$. By (K1), $R(x,x) = \mathcal{K}(\top_x)(x) \le \top_x^*(x)$. So, $R(x,x) = \bot$. Since $A = \bigvee_{x\in X}(A(x)\odot \top_x)$, we have

$$\begin{aligned}\mathcal{K}(A)(y) &= \mathcal{K}(\textstyle\bigvee_{x\in X}(A(x)\odot\top_x))(y)\\ &= \textstyle\bigwedge_{x\in X}(A(x)\to \mathcal{K}(\top_x)(y))\\ &= \textstyle\bigwedge_{x\in X}(A(x)\to R(x,y)).\end{aligned}$$

($\Leftarrow$) (K1) $\mathcal{K}(A)(y) = \bigwedge_{x\in X}(A(x)\to R(x,y)) \le A(y)\to R(y,y) = A^*(y)$.

(K2) $\mathcal{K}(\alpha\odot A)(y) = \bigwedge_{x\in X}(\alpha\odot A(x)\to R(x,y)) = \alpha \to \bigwedge_{x\in X}(A(x)\to R(x,y)) = \alpha\to\mathcal{K}(A)(y)$.

(2) ($\Rightarrow$) By (1), we define $R(x,y) = \mathcal{K}(\top_x)(y)$. Since $\mathcal{K}(\mathcal{K}^*(\top_x))(z) \ge \mathcal{K}(\top_x)(z)$ and $\mathcal{K}^*(\top_x) = \bigvee_{y\in X}(\mathcal{K}^*(\top_x)(y)\odot\top_y$, we have

$$\begin{aligned}\mathcal{K}(\top_x)(z) &\le \mathcal{K}(\mathcal{K}^*(\top_x))(z)\\ &= \mathcal{K}(\textstyle\bigvee_{y\in X}(\mathcal{K}^*(\top_x)(y)\odot\top_y))(z)\\ &= \textstyle\bigwedge_{y\in X}(\mathcal{K}^*(\top_x)(y)\to\mathcal{K}(\top_y)(z)\\ &= (\textstyle\bigvee_{y\in X}(\mathcal{K}^*(\top_x)(y)\odot\mathcal{K}^*(\top_y)(z)))^*\end{aligned}$$

Hence $(\mathcal{K}^*(\top_x)(y)\odot\mathcal{K}^*(\top_y)(z)))^* \le \mathcal{K}^*(\top_x)(z)$, that is, $\bigvee_{y\in X}(R^*(x,y)\odot R^*(y,z)) \le R^*(x,z)$.

($\Leftarrow$) Since R^* is an L-fuzzy preorder, $R^*(x,y)\odot R^*(y,z) \le R^*(x,z)$ iff $R^*(x,y) \le R^*(y,z)\to R^*(x,z) = R(x,z)\to R(y,z)$ iff $R(x,z)\le R^*(x,y)\to R(y,z)$, we have

$$\begin{aligned}\mathcal{K}(\mathcal{K}^*(A))(z) &= \textstyle\bigwedge_{y\in X}(\mathcal{K}^*(A)(y)\to R(y,z))\\ &= \textstyle\bigwedge_{y\in X}(\bigvee_{x\in X}(A(x)\odot R^*(x,y))\to R(y,z))\\ &= \textstyle\bigwedge_{y\in X}\bigwedge_{x\in X}(A(x)\to(R^*(x,y)\to R(y,z)))\\ &= \textstyle\bigwedge_{x\in X}(A(x)\to\bigwedge_{y\in X}(R^*(x,y)\to R(y,z)))\\ &\ge \textstyle\bigwedge_{x\in X}(A(x)\to R(x,z)) = \mathcal{K}(A)(z).\end{aligned}$$

(3) ($\Rightarrow$) By (1), we define $R(x,y) = \mathcal{K}(\top_x)(y)$. Since $\mathcal{K}(\mathcal{K}(\top_x))(z) \ge \mathcal{K}^*(\top_x)(z)$ and $\mathcal{K}(\top_x) = \bigvee_{y\in X}(\mathcal{K}(\top_x)(y)\odot\top_y$, we have

$$\begin{aligned}\mathcal{K}^*(\top_x)(z) &\le \mathcal{K}(\mathcal{K}(\top_x))(z)\\ &= \mathcal{K}(\textstyle\bigvee_{y\in X}(\mathcal{K}(\top_x)(y)\odot\top_y))(z)\\ &= \textstyle\bigwedge_{y\in X}(\mathcal{K}(\top_x)(y)\to\mathcal{K}(\top_y)(z)\\ &= \textstyle\bigwedge_{y\in X}(\mathcal{K}^*(\top_y)(z)\to\mathcal{K}^*(\top_x)(y)\end{aligned}$$

Hence $\bigvee_{z\in X}\mathcal{K}^*(\top_x)(z)\odot\mathcal{K}^*(\top_y)(z)\le\mathcal{K}^*(\top_x)(y)$, that is, $\bigvee_{z\in X}(R^*(x,z)\odot R^*(y,z))\le R^*(x,y)$. So, R^* is Euclidean.

($\Leftarrow$) Since R^* is Euclidean, $A(x)\odot R^*(x,z)\odot R^*(y,z))\le A(x)\odot R^*(x,y)$ iff $A(x)\odot R^*(x,z)\le R^*(y,z)\to A(x)\odot R^*(x,y)$, we have

$$\begin{aligned}\mathcal{K}(\mathcal{K}(A))(z) &= \textstyle\bigwedge_{y\in X}(\mathcal{K}(A)(y)\to R(y,z))\\ &= \textstyle\bigwedge_{y\in X}(\bigwedge_{x\in X}(A(x)\to R(x,y))\to R(y,z))\\ &= \textstyle\bigwedge_{y\in X}(R^*(y,z)\to(\bigwedge_{x\in X}(A(x)\to R(x,y)))^*)\\ &= \textstyle\bigwedge_{y\in X}(R^*(y,z)\to\bigvee_{x\in X}(A(x)\odot R^*(x,y)))\\ &\ge \textstyle\bigvee_{x\in X}(A(x)\odot R^*(x,z))=\mathcal{K}(A)(z).\end{aligned}$$

□

Theorem 7. (1) *A map $\mathcal{M}:L^X\to L^X$ is an L-meet join approximation operator iff there exists a reflexive L-fuzzy relation $R\in L^{X\times X}$ such that*

$$\mathcal{M}(A)(y)=\bigvee_{x\in X}(A^*(x)\odot R(x,y)).$$

(2) *A map $\mathcal{K}:L^X\to L^X$ is an L-join meet approximation operator with $\mathcal{M}(\mathcal{M}^*(A))=\mathcal{M}(A)$ iff there exists an L-fuzzy preorder $R\in L^{X\times X}$ such that*

$$\mathcal{M}(A)(y)=\bigvee_{x\in X}(A^*(x)\odot R(x,y)).$$

(3) *A map $\mathcal{M}:L^X\to L^X$ is an L-meet join approximation operator with $\mathcal{M}(\mathcal{M}(A))=\mathcal{M}^*(A)$ iff there exists reflexive and Euclidean L-fuzzy relation $R\in L^{X\times X}$ such that*

$$\mathcal{M}(A)(y)=\bigvee_{x\in X}(A^*(x)\odot R(x,y)).$$

Proof. $\Rightarrow$ Define $R(x,y)=\mathcal{M}(\top_x^*)(y)$. Since $A=\bigwedge_{x\in X}(A^*(x)\to\top_x^*)$, we have

$$\begin{aligned}\mathcal{M}(A)(y) &= \textstyle\mathcal{M}(\bigwedge_{x\in X}(A(x)^*\to\top_x^*)(y)\\ &= \textstyle\bigvee_{x\in X}(A(x)^*\odot\mathcal{M}(\top_x^*)(y))\\ &= \textstyle\bigvee_{x\in X}(A(x)^*\odot R(x,y)).\end{aligned}$$

(2) ($\Rightarrow$) By (1), we define $R(x,y)=\mathcal{M}(\top_x^*)(y)$. Since $\mathcal{M}(\mathcal{M}^*(\top_x^*))(z)\le\mathcal{M}(\top_x^*)(z)$ and $\mathcal{M}^*(\top_x^*)=\bigwedge_{y\in X}(\mathcal{M}(\top_x^*)(y)\to\top_y^*)$, we have

$$\begin{aligned}&\mathcal{M}(\mathcal{M}^*(\top_x^*))(z)\\ &= \textstyle\mathcal{M}(\bigwedge_{y\in X}(\mathcal{M}(\top_x^*)(y)\to\top_y^*))(z)\\ &= \textstyle\bigvee_{y\in X}(\mathcal{M}(\top_x^*)(y)\odot\mathcal{M}(\top_y^*)(z))\le\mathcal{M}(\top_x^*)(z)\\ &\Leftrightarrow \textstyle\bigvee_{z\in X}(R(x,y)\odot R(y,z))\le R(x,y)\end{aligned}$$

(⇐)

$$\begin{aligned}\mathcal{M}(\mathcal{M}^*(A))(z) &= \bigvee_{y\in X}(\mathcal{M}(A)(y)\odot R(y,z))\\ &= \bigvee_{y\in X}(\bigvee_{x\in X}(A^*(x)\odot R(x,y))\odot R(y,z))\\ &\le \bigwedge_{x\in X}(A^*(x)\odot R(x,z)) = \mathcal{M}(A)(z).\end{aligned}$$

(3) (⇒) By (1), we define $R(x,y)=\mathcal{M}(\top_x^*)(y)$. Since $\mathcal{M}(\mathcal{M}(\top_x^*))(z)\le \mathcal{M}^*(\top_x^*)(z)$ and $\mathcal{M}(\top_x^*)=\bigwedge_{y\in X}(\mathcal{M}^*(\top_x^*)(y)\to \top_y^*)$, we have

$$\begin{aligned}&\mathcal{M}(\mathcal{M}(\top_x^*))(z)\\ &= \mathcal{M}(\bigwedge_{y\in X}(\mathcal{M}^*(\top_x^*)(y)\to \top_y^*))(z)\\ &= \bigvee_{y\in X}(\mathcal{M}^*(\top_x^*)(y)\odot \mathcal{M}(\top_y^*)(z))\le \mathcal{M}^*(\top_x^*)(z)\\ &\Leftrightarrow \bigwedge_{y\in X}(\mathcal{M}(\top_y^*)(z)\to \mathcal{M}(\top_x^*)(y)\ge \mathcal{M}(\top_x^*)(z)\\ &\Leftrightarrow \bigvee_{z\in X}(\mathcal{M}(\top_x^*)(z)\odot \mathcal{M}(\top_y^*)(z))\le \mathcal{M}(\top_x^*)(y)\\ &\Leftrightarrow \bigvee_{z\in X}(R(x,z)\odot R(y,z))\le R(x,y)\end{aligned}$$

(⇐)

$$\begin{aligned}\mathcal{M}(\mathcal{M}(A))(z) &= \bigvee_{y\in X}(\mathcal{M}^*(A)(y)\odot R(y,z))\\ &= \bigvee_{y\in X}(\bigwedge_{x\in X}(R(x,y)\to A(x))\odot R(y,z))\\ &\le \bigwedge_{x\in X}(R(x,z)\to A(x)) = \mathcal{M}^*(A)(z).\end{aligned}$$

□

Definition 8. [4,5] A subset $\tau\subset L^X$ is called an *Alexandrov L-topology* if it satisfies:

(T1) $\bot_X,\top_X\in\tau$ where $\top_X(x)=\top$ and $\bot_X(x)=\bot$ for $x\in X$.

(T2) If $A_i\in\tau$ for $i\in\Gamma$, $\bigvee_{i\in\Gamma}A_i,\bigwedge_{i\in\Gamma}A_i\in\tau$.

(T3) $\alpha\odot A\in\tau$ for all $\alpha\in L$ and $A\in\tau$.

(T4) $\alpha\to A\in\tau$ for all $\alpha\in L$ and $A\in\tau$.

Theorem 9. (1) *τ is an Alexandrov L-topology on X iff $\tau_*=\{A^*\in L^X\mid A\in\tau\}$ is an Alexandrov L-topology on X.*

(2) *If $\mathcal{K}$ is an L-join meet approximation operator, then $\tau_{\mathcal{K}}=\{A\in L^X\mid \mathcal{K}(A)=A^*\}$ is an Alexandrov L-topology on X.*

(3) *If $\mathcal{K}$ is an L-join meet approximation operator with $\mathcal{K}(\mathcal{K}^*(A))=\mathcal{K}(A)$, then $\tau_{\mathcal{K}}=\{A\in L^X\mid \mathcal{K}(A)=A^*\}=\{\mathcal{K}^*(A)\mid A\in L^X\}$ is an Alexandrov L-topology on X.*

(4) *If $\mathcal{K}$ is an L-join meet approximation operator with $\mathcal{K}(\mathcal{K}(A))=\mathcal{K}^*(A)$, then $\tau_{\mathcal{K}}=\{\mathcal{K}(A)\mid A\in L^X\}=(\tau_{\mathcal{K}})_*$ is an Alexandrov L-topology on X.*

(5) *If $\mathcal{M}$ is an L-meet join operator, then $\tau_{\mathcal{M}}=\{A\in L^X\mid \mathcal{M}(A)=A^*\}$ is an Alexandrov L-topology on X.*

(6) *If $\mathcal{M}$ is an L-meet join approximation operator with $\mathcal{M}(\mathcal{M}^*(A)) = \mathcal{M}(A)$, then $\tau_{\mathcal{M}} = \{A \in L^X \mid \mathcal{M}(A) = A^*\} = \{\mathcal{M}^*(A) \mid A \in L^X\}$ is an Alexandrov L-topology on X.*

(7) *If $\mathcal{M}$ is an L-meet join approximation operator with $\mathcal{M}(\mathcal{M}(A)) = \mathcal{M}^*(A)$, then $\tau_{\mathcal{M}} = \{\mathcal{M}(A) \mid A \in L^X\} = (\tau_{\mathcal{M}})_*$ is an Alexandrov L-topology on X.*

Proof. (1) Let $A^* \in \tau_*$ for $A \in \tau$. Since $\alpha \odot A^* = (\alpha \to A)^*$ and $\alpha \to A^* = (\alpha \odot A)^*$, τ_* is an Alexandrov L-topology on X.

(2) (T1) Since $\mathcal{K}(\top_X) \le \top_X^* = \bot_X$ and $\mathcal{K}(\bot_X) = \mathcal{K}(\bot \odot A) = \bot \to \mathcal{K}(A) = \top_X$, $\bot_X = \mathcal{K}(\top_X)$ and $\top_X = \mathcal{K}(\bot_X)$. Then $\bot_X, \top_X \in \tau_{\mathcal{K}}$.

(T2) For $A_i \in \tau_{\mathcal{K}}$ for each $i \in \Gamma$, by (K3),

$$\mathcal{K}(\bigvee_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} \mathcal{K}(A_i) = \bigwedge_{i\in\Gamma} A_i^* = (\bigvee_{i\in\Gamma} A_i)^*$$

Thus $\bigvee_{i\in\Gamma} A_i \in \tau_{\mathcal{K}}$. Since $\mathcal{K}$ is a deceasing map for (K3), we have

$$\bigvee_{i\in\Gamma} A_i^* = \bigvee_{i\in\Gamma} \mathcal{K}(A_i) \le \mathcal{K}(\bigwedge_{i\in\Gamma} A_i) \le (\bigwedge_{i\in\Gamma} A_i)^*,$$

Thus, $\bigwedge_{i\in\Gamma} A_i \in \tau_{\mathcal{K}}$.

(T3) For $A \in \tau_{\mathcal{K}}$, by (K2), $\alpha \odot A \in \tau_{\mathcal{K}}$.

(T4) For $A \in \tau_{\mathcal{K}}$, by Lemma 5(1), $\mathcal{K}(\alpha \to A) \ge \alpha \odot \mathcal{K}(A) = \alpha \odot A^* = (\alpha \to A)^*$. Then $\alpha \to A \in \tau_{\mathcal{K}}$.

(3) Put $P = \{\mathcal{K}^*(A) \mid A \in L^X\}$. Let $\mathcal{K}^*(A) \in P$. Since $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A)$, then $\mathcal{K}^*(A) \in \tau_{\mathcal{K}}$. Let $A \in \tau_{\mathcal{K}}$. Then $A = \mathcal{K}^*(A) \in P$.

(4) Let $A \in \tau_{\mathcal{K}}$. Then $A^* = \mathcal{K}(A)$. Since $\mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A)$, then $\mathcal{K}(A^*) = A$. So $A^* \in (\tau_{\mathcal{K}})_*$. Let $A \in (\tau_{\mathcal{K}})_*$. Then $A = \mathcal{K}(A^*)$. Since $\mathcal{K}(\mathcal{K}(A^*)) = \mathcal{K}^*(A^*)$, then $\mathcal{K}(A) = A^*$. So $A \in \tau_{\mathcal{K}}$. Hence $\tau_{\mathcal{K}} = (\tau_{\mathcal{K}})_*$. Put $Q = \{\mathcal{K}(A) \mid A \in L^X\}$. Let $\mathcal{K}(A) \in Q$. Since $\mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A)$, then $\mathcal{K}(A) \in \tau_{\mathcal{K}}$. Let $A \in \tau_{\mathcal{K}}$. Then $A^* = \mathcal{K}(A) \in (\tau_{\mathcal{K}})_* = \tau_{\mathcal{K}}$. Hence $Q = \tau_{\mathcal{K}}$.

(5) (T1) Since $\bot_X^* \le \mathcal{M}(\bot_X)$ and $\mathcal{M}(\top_X) = \mathcal{M}(\bot_X \to A) = \bot_X \odot \mathcal{M}(A) = \bot_X$, $\top_X = \mathcal{M}(\bot_X)$ and $\bot_X = \mathcal{M}(\top_X)$. Then $\bot_X, \top_X \in \tau_{\mathcal{M}}$.

(T2) For $A_i \in \tau_{\mathcal{M}}$ for each $i \in \Gamma$, by (M3),

$$\mathcal{M}(\bigwedge_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} \mathcal{M}(A_i) = \bigvee_{i\in\Gamma} A_i^* = (\bigwedge_{i\in\Gamma} A_i)^*.$$

Thus, $\bigwedge_{i\in\Gamma} A_i \in \tau_{\mathcal{M}}$. Since $\mathcal{M}$ is deceasing map for (M3), we have

$$(\bigvee_{i\in\Gamma} A_i)^* \le \mathcal{M}(\bigvee_{i\in\Gamma} A_i) \le \bigwedge_{i\in\Gamma} \mathcal{M}(A_i) = (\bigvee_{i\in\Gamma} A_i)^*,$$

Thus, $\bigvee_{i\in\Gamma} A_i \in \tau_{\mathcal{M}}$.

(T3) For $A \in \tau_{\mathcal{M}}$, by Lemma 5 (2), $(\alpha \odot A)^* \leq \mathcal{M}(\alpha \odot A) \leq \alpha \to \mathcal{M}(A) = \alpha \to A^*$. Then $\alpha \odot A \in \tau_{\mathcal{M}}$.

(T4) For $A \in \tau_{\mathcal{M}}$, by (M2), $\mathcal{M}(\alpha \to A) = \alpha \odot \mathcal{M}(A) = \alpha \odot A^* = (\alpha \to A)^*$. Then $\alpha \to A \in \tau_{\mathcal{M}}$.

(6) and (7) are similarly proved as (3) and (4), respectively.

□

Theorem 10. *Let $R \in L^{X\times X}$ be an L-fuzzy relation. Define operators as follows*

$$\begin{aligned}
\mathcal{K}_{R^*}(A)(y) &= \bigwedge_{x\in X}(A(x) \to R^*(x,y)),\\
\mathcal{K}_{R^{-1*}}(A)(y) &= \bigwedge_{x\in X}(A(x) \to R^{-1*}(x,y))\\
\mathcal{M}_R(A)(y) &= \bigvee_{x\in X}(A^*(x) \odot R(x,y)),\\
\mathcal{M}_{R^{-1}}(A)(y) &= \bigvee_{x\in X}(A^*(x) \odot R^{-1}(x,y)).
\end{aligned}$$

Then the following properties hold.

(1) *If R is reflexive, then $\tau_{\mathcal{K}_{\mathbf{R}^*}} = \tau_{(\mathcal{K}_{\mathbf{R}^{-1*}})_*} = \tau_{\mathcal{M}_{\mathbf{R}^{-1}}} = \tau_{(\mathcal{M}_{\mathbf{R}})_*}$.*

(2) *If R is an L-fuzzy preorder, then*

$$\begin{aligned}
\tau_{\mathcal{K}_{\mathbf{R}^*}} &= \{\bigvee_{x\in X}(A(x) \odot R(x,-)) \mid A \in L^X\}\\
\tau_{\mathcal{K}_{\mathbf{R}^{-1*}}} &= \{\bigvee_{x\in X}(A(x) \odot R(-,x)) \mid A \in L^X\}\\
\tau_{\mathcal{M}_{\mathbf{R}}} &= \{\bigwedge_{x\in X}(R(x,-) \to A(x)) \mid A \in L^X\}\\
\tau_{\mathcal{M}_{\mathbf{R}^{-1}}} &= \{\bigwedge_{x\in X}(R(-,x) \to A(x)) \mid A \in L^X\}.
\end{aligned}$$

(3) *If R is reflexive and Euclidean, then R is symmetric, R is an L-fuzzy preorder and $\tau_{\mathcal{K}_{\mathbf{R}^*}} = \tau_{\mathcal{K}_{\mathbf{R}^{-1*}}} = \tau_{(\mathcal{K}_{\mathbf{R}^{-1*}})_*} = \tau_{\mathcal{M}_{\mathbf{R}^{-1}}} = \tau_{\mathcal{M}_{\mathbf{R}}} = \tau_{(\mathcal{M}_{\mathbf{R}})_*}$ such that*

$$\begin{aligned}
\tau_{\mathcal{K}_{\mathbf{R}^*}} &= \{\bigwedge_{x\in X}(A(x) \to R^*(x,-)) \mid A \in L^X\}\\
\tau_{\mathcal{K}_{\mathbf{R}^{-1*}}} &= \{\bigwedge_{x\in X}(A(x) \to R^*(-,x)) \mid A \in L^X\}\\
\tau_{\mathcal{M}_{\mathbf{R}}} &= \{\bigvee_{x\in X}(R(x,-) \odot A^*(x)) \mid A \in L^X\}\\
\tau_{\mathcal{M}_{\mathbf{R}^{-1}}} &= \{\bigvee_{x\in X}(R(-,x) \odot A^*(x)) \mid A \in L^X\}.
\end{aligned}$$

Proof. (1) We only show that

$$\begin{aligned}
&\mathcal{K}_{R^*}(A) = A^* \text{ iff } \mathcal{K}_{R^{-1*}}(A^*) = A\\
&\text{ iff } \mathcal{M}_R(A^*) = A \text{ iff } \mathcal{M}_{R^{-1}}(A) = A^*.
\end{aligned}$$

First, $\mathcal{K}_{R^*}(A) = A^*$ iff $\mathcal{K}_{R^{-1*}}(A^*) = A$.

Since $\mathcal{K}_{R^*}(A)(y) = \bigwedge_{x\in X}(A(x) \to R^*(x,y)) \geq A^*(y)$, then

$$A(x) \leq \bigwedge_{y\in X} A^*(y) \to R^*(x,y),$$

iff $A(x) \leq \mathcal{K}_{R^{-1*}}(A^*)(x)$. Hence $\mathcal{K}_{R^{-1*}}(A^*) = A$. Conversely, it is similarly proved.

Second, $\mathcal{K}_{R^*}(A) = A^*$ iff $\mathcal{M}_R(A^*) = A$.

Since $\mathcal{K}_{R^*}(A)(y) = \bigwedge_{x\in X}(A(x) \to R^*(x,y) \geq A^*(y)$, then $(\bigwedge_{x\in X}(A(x) \to R^*(x,y))^* = \bigvee_{x\in X}(A(x) \odot R(x,y)) = \mathcal{M}_R(A^*)(y) \leq A(y)$. Thus $\mathcal{M}_R(A^*) = A$. Conversely, it is similarly proved.

Finally, by a second method, we have $\mathcal{K}_{R^{-1*}}(A^*) = A$ iff $\mathcal{M}_{R^{-1}}(A) = A^*$.

(2) Since R and R^{-1} are L-fuzzy preorders, by Theorems 6(2) and 7(2), then

$$\mathcal{K}_{R^*}(\mathcal{K}^*_{R^*}(A)) = \mathcal{K}_{R^*}(A),\ \mathcal{K}_{R^{-1*}}(\mathcal{K}^*_{R^{-1*}}(A)) = \mathcal{K}_{R^{-1*}}(A)$$

$$\mathcal{M}_R(\mathcal{M}^*_R(A)) = \mathcal{M}_R(A),\ \mathcal{M}_{R^{-1}}(\mathcal{M}^*_{R^{-1}}(A)) = \mathcal{M}_{R^{-1}}(A),$$

$$\begin{array}{ll} \tau_{\mathcal{K}_{\mathbf{R}^*}} & = \{\mathcal{K}^*_{R^*}(A) = \bigvee_{x\in X}(A(x) \odot R(x,-)) \mid A \in L^X\} \\ \tau_{\mathcal{K}_{\mathbf{R}^{-1*}}} & = \{\mathcal{K}^*_{R^{-1*}}(A) = \bigvee_{x\in X}(A(x) \odot R(-,x)) \mid A \in L^X\} \\ \tau_{\mathcal{M}_{\mathbf{R}}} & = \{\mathcal{M}^*_R(A) = \bigwedge_{x\in X}(R(x,-) \to A(x)) \mid A \in L^X\} \\ \tau_{\mathcal{M}_{\mathbf{R}^{-1}}} & = \{\mathcal{M}^*_{R-1}(A) = \bigwedge_{x\in X}(R(-,x) \to A(x)) \mid A \in L^X\}. \end{array}$$

(2) Since R and R^{-1} are reflexive and Euclidean L-fuzzy preorder, by Theorems 6(3) and 7(3),

$$\mathcal{K}_{R^*}(\mathcal{K}_{R^*}(A)) = \mathcal{K}^*_{R^*}(A),\ \mathcal{K}_{R^{-1*}}(\mathcal{K}_{R^{-1*}}(A)) = \mathcal{K}^*_{R^{-1*}}(A)$$

$$\mathcal{M}_R(\mathcal{M}_R(A)) = \mathcal{M}^*_R(A),\ \mathcal{M}_{R^{-1}}(\mathcal{M}_{R^{-1}}(A)) = \mathcal{M}^*_{R^{-1}}(A),$$

$$\begin{array}{ll} \tau_{\mathcal{K}_{\mathbf{R}^*}} & = \{\mathcal{K}_{R^*}(A) = \bigwedge_{x\in X}(A(x) \to R^*(x,-)) \mid A \in L^X\} \\ \tau_{\mathcal{K}_{\mathbf{R}^{-1*}}} & = \{\mathcal{K}_{R^{-1*}}(A) = \bigwedge_{x\in X}(A(x) \to R^*(-,x)) \mid A \in L^X\} \\ \tau_{\mathcal{M}_{\mathbf{R}}} & = \{\mathcal{M}_R(A) = \bigvee_{x\in X}(R(x,-) \odot A^*(x)) \mid A \in L^X\} \\ \tau_{\mathcal{M}_{\mathbf{R}^{-1}}} & = \{\mathcal{M}_{R-1}(A) = \bigvee_{x\in X}(R(-,x) \odot A^*(x)) \mid A \in L^X\}. \end{array}$$

□

Example 11. Let $(L = [0,1], \odot, \to, ^*)$ be a complete residuated lattice with the law of double negation defined by

$$x \odot y = (x + y - 1) \vee 0,\ \ x \to y = (1 - x + y) \wedge 1,\ x^* = 1 - x.$$

Let $X = \{a, b, c\}$ and we define $R \in L^{X\times X}$ as follows

$$R = \begin{pmatrix} 1 & 0.7 & 0.3 \\ 0.8 & 1 & 0.4 \\ 0.9 & 0.6 & 1 \end{pmatrix}.$$

(1) Since R is a reflexive L-fuzzy relation, by Theorem 6(1), $\mathcal{K}_{R^*}$ is an L-join meet approximation operator. Moreover, by Theorem 7(1), $\mathcal{M}_R$ is an L-join meet approximation operator.

(2) Since R is an L-fuzzy preorder on X, by Theorem 10(1), we have

$$\tau_{\mathcal{K}_{\mathbf{R}^*}} = \tau_{(\mathcal{K}_{\mathbf{R}^{-1*}})_*} = \tau_{\mathcal{M}_{\mathbf{R}^{-1}}} = \tau_{(\mathcal{M}_{\mathbf{R}^{-1}})_*}.$$

(3) For $A(a) = 0.7, A(b) = 0.5, A(c) = 0.5$, since R is an L-fuzzy preorder on X, by Theorem 7(2),

$$\mathcal{K}_{R^*}(A) = (0.3, 0.5, 0.5) = \mathcal{K}_{R^*}(\mathcal{K}^*_{R^*}(A)).$$

$$\mathcal{K}^*_{R^*}(A) = (0.7, 0.5, 0.5) \in \tau_{\mathcal{K}_{\mathbf{R}^*}}.$$

Since $0.9 = \bigvee_{x\in X} R(a,x) \odot R(c,x) \not\leq R(a,c) = 0.3$, R is not Euclidean. By Theorem 6(3), we have

$$\mathcal{K}^*_{R^*}(A) = (0.7, 0.5, 0.5) \neq \mathcal{K}_{R^*}(\mathcal{K}_{R^*}(A)) = (0.6, 0.5, 0.5).$$

$$\mathcal{K}_{R^*}(A) = (0.3, 0.5, 0.5) \not\in \tau_{\mathcal{K}_{\mathbf{R}^*}}.$$

For $B(a) = 0.4, B(b) = 0.5, B(c) = 0.6$, since R is an L-fuzzy preorder on X, by Theorem 7(2),

$$\mathcal{M}_R(B) = (0.6, 0.5, 0.4) = \mathcal{M}_R(\mathcal{M}^*_R(B)).$$

$$\mathcal{M}^*_R(B) = (0.4, 0.5, 0.6) \in \tau_{\mathcal{M}_{\mathbf{R}}}.$$

Since $0.9 = \bigvee_{x\in X} R(a,x) \odot R(c,x) \not\leq R(a,c) = 0.3$, R is not Euclidean. By Theorem 6(3),

$$\mathcal{M}^*_R(B) = (0.4, 0.5, 0.6) \neq \mathcal{M}_R(\mathcal{M}_R(B)) = (0.5, 0.5, 0.6).$$

$$\mathcal{M}_R(B) = (0.6, 0.5, 0.4) \not\in \tau_{\mathcal{M}_{\mathbf{R}}}.$$

(4) Put $R^{(2)}(y,z) = \bigvee_{x\in X}(R(y,x) \odot R(z,x))$, we obtain an L-fuzzy relation $R^{(2)}$ as

$$R^{(2)} = \begin{pmatrix} 1 & 0.8 & 0.9 \\ 0.8 & 1 & 0.7 \\ 0.9 & 0.7 & 1 \end{pmatrix}.$$

Since $R^{(2)}$ is Euclidean and $R^{(2)}(x,y) = R^{(2)}(y,x)$ for all $x, y \in X$, then $R^{(2)}$ is an L-fuzzy preorder. By Theorems 6 and 7, for $C \in L^X$, we obtain

$$\mathcal{K}_{R^{(2)*}}(C) = \mathcal{K}_{R^{(2)}}(\mathcal{K}^*_{R^{(2)*}}(C)),$$

$$\mathcal{K}^*_{R^{(2)*}}(C) = \mathcal{K}_{R^{(2)*}}(\mathcal{K}_{R^{(2)*}}(C)),$$
$$\mathcal{M}^*_{R^{(2)}}(C) = \mathcal{M}_{R^{(2)}}(\mathcal{M}_{R^{(2)}}(C)),$$
$$\mathcal{M}_{R^{(2)}}(C) = \mathcal{M}_{R^{(2)}}(\mathcal{M}^*_{R^{(2)}}(C)).$$

By Theorem 10(3), we have

$$\tau_{\mathcal{K}_{\mathbf{R}^{(2)*}}} = \tau_{(\mathcal{K}_{\mathbf{R}^{(2)-1*}})_*} = \tau_{(\mathcal{K}_{\mathbf{R}^{(2)*}})_*} = \tau_{\mathcal{M}_{\mathbf{R}^{(2)}}} = \tau_{\mathcal{M}_{\mathbf{R}^{(2)-1}}} = \tau_{(\mathcal{M}_{\mathbf{R}^{(2)}})_*}.$$

(5) Since $0.9 = \bigvee_{x\in X} R(x,a)\odot R(x,c) \not\leq R(a,c) = 0.3$, for $A(a) = 0.5, A(b) = 0.6, A(c) = 0.7$, we have

$$\mathcal{K}^*_{R^{-1*}}(A) = (0.5, 0.6, 0.7) \neq \mathcal{K}_{R^{-1*}}(\mathcal{K}_{R^{-1*}}(A)) = (0.5, 0.6, 0.6).$$

Moreover, since R^{-1} is an L-fuzzy preorder, by Theorem 9,

$$\mathcal{K}^*_{R^{-1*}}(A) = (0.5, 0.6, 0.7) \in \tau_{\mathcal{K}_{\mathbf{R}^{-1*}}}, \mathcal{K}_{R^{-1*}}(A) = (0.5, 0.4, 0.3) \notin \tau_{\mathcal{K}_{\mathbf{R}^{-1*}}}.$$

(6) For $A(a) = 0.5, A(b) = 0.1, A(c) = 0.2$, we have

$$\mathcal{M}^*_{R^{-1}}(A) = (0.4, 0.1, 0.2) \neq \mathcal{M}_{R^{-1}}(\mathcal{M}_{R^{-1}}(A)) = (0.4, 0.2, 0.3).$$

Moreover, since R^{-1} is an L-fuzzy preorder, by Theorem 9, we have

$$\mathcal{M}^*_{R^{-1}}(A) = (0.4, 0.1, 0.2) \in \tau_{\mathcal{M}_{\mathbf{R}^{-1}}}, \mathcal{M}_{R^{-1}}(A) = (0.6, 0.9, 0.8) \notin \tau_{\mathcal{M}_{\mathbf{R}^{-1}}}.$$

(7) Put $R^{[2]}(y,z) = \bigvee_{x\in X} R(x,y)\odot R(x,z)$, we obtain an L-fuzzy relation $R^{[2]}, R^{[3]}$ as

$$R^{[2]} = \begin{pmatrix} 1 & 0.8 & 0.9 \\ 0.8 & 1 & 0.6 \\ 0.9 & 0.6 & 1 \end{pmatrix} \quad R^{[3]} = \begin{pmatrix} 1 & 0.8 & 0.9 \\ 0.8 & 1 & 0.7 \\ 0.9 & 0.7 & 1 \end{pmatrix}.$$

Since $\mathcal{M}_{(R^{[2]})^{-1}}(\mathcal{M}_{(R^{[2]})^{-1}}(\top^*_x))(z) \leq \mathcal{M}^*_{(R^{[2]})^{-1}}(\top^*_x)(z)$ and $\mathcal{M}_{(R^{[2]})^{-1}}(\top^*_x) = \bigwedge_{y\in X}(\mathcal{M}^*_{(R^{[2]})^{-1}}(\top^*_x)(y) \to \top^*_y)$, we have $\mathcal{M}_{(R^{[2]})^{-1}}(\top^*_c) \notin \tau_{\mathcal{M}_{(\mathbf{R}^{[2]})^{-1}}}$ from:

$$\begin{aligned}
&0.7 = \textstyle\bigvee_{x\in X} R^{[2]}(x,b)\odot R^{[2]}(x,c) \not\leq R^{[2]}(b,c) = 0.6\\
&\Leftrightarrow \textstyle\bigvee_{x\in X}(\mathcal{M}_{(R^{[2]})^{-1}}(\top^*_b)(x)\odot \mathcal{M}_{(R^{[2]})^{-1}}(\top^*_c)(x)) \not\leq \mathcal{M}_{(R^{[2]})^{-1}}(\top^*_c)(b)\\
&\Leftrightarrow \textstyle\bigwedge_{y\in X}(\mathcal{M}_{(R^{[2]})^{-1}}(\top^*_b)(x) \to \mathcal{M}_{(R^{[2]})^{-1}}(\top^*_c)(b) \not\geq \mathcal{M}_{(R^{[2]})^{-1}}(\top^*_c)(x)\\
&\Leftrightarrow \textstyle\bigvee_{b\in X}(\mathcal{M}_{(R^{[2]})^{-1}}(\top^*_b)(x)\odot \mathcal{M}^*_{(R^{[2]})^{-1}}(\top^*_c)(b) \not\leq \mathcal{M}^*_{(R^{[2]})^{-1}}(\top^*_c)(x)\\
&\Leftrightarrow \mathcal{M}_{(R^{[2]})^{-1}}(\textstyle\bigwedge_{b\in X}\mathcal{M}^*_{(R^{[2]})^{-1}}(\top^*_c)(b) \to \top^*_b)(x) \not\leq \mathcal{M}^*_{(R^{[2]})^{-1}}(\top^*_c)(x)\\
&\Leftrightarrow \mathcal{M}_{(R^{[2]})^{-1}}(\mathcal{M}_{(R^{[2]})^{-1}}(\top^*_c))(x) \not\leq \mathcal{M}^*_{(R^{[2]})^{-1}}(\top^*_c)(x)
\end{aligned}$$

Since

$$\mathcal{M}_{(R^{[2]})^{-1}}(\mathcal{M}^*_{(R^{[2]})^{-1}}(1^*_c))(b) = \bigvee_{a\in X} R^{[2]}(c,a)\odot R^{[2]}(a,b)$$
$$\not\leq R^{[2]}(c,b) = 0.6 = \mathcal{M}_{(R^{[2]})^{-1}}(1^*_c))(b),$$

$$\mathcal{M}^*_{(R^{[2]})^{-1}}(1^*_c) \notin \tau_{\mathcal{M}_{(\mathbf{R}^{[2]})^{-1}}}$$

(8) Since $R^{[3]}$ is Euclidean and $R^{[3]}(x,y) = R^{[3]}(y,x)$ for all $x,y\in X$, then $R^{[3]}$ is an L-fuzzy preorder. By Theorems 6 and 7, for $C\in L^X$, we obtain

$$\mathcal{K}_{R^{[3]*}}(C) = \mathcal{K}_{R^{[3]}}(\mathcal{K}^*_{R^{[3]*}}(C)),$$

$$\mathcal{K}^*_{R^{[3]*}}(C) = \mathcal{K}_{R^{[3]*}}(\mathcal{K}_{R^{[3]*}}(C)),$$
$$\mathcal{M}^*_{R^{[3]}}(C) = \mathcal{M}_{R^{[3]}}(\mathcal{M}_{R^{[3]}}(C)),$$
$$\mathcal{M}_{R^{[3]}}(C) = \mathcal{M}_{R^{[3]}}(\mathcal{M}^*_{R^{[3]}}(C)).$$

By Theorem 10(3), we have

$$\tau_{\mathcal{K}_{\mathbf{R}^{[3]*}}} = \tau_{(\mathcal{K}_{\mathbf{R}^{[3]-1*}})_*} = \tau_{(\mathcal{K}_{\mathbf{R}^{[3]*}})_*} = \tau_{\mathcal{M}_{\mathbf{R}^{[3]}}} = \tau_{\mathcal{M}_{\mathbf{R}^{[3]-1}}} = \tau_{(\mathcal{M}_{\mathbf{R}^{[3]}})_*}.$$

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