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NEW OSCILLATION AND NON-OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract: In this paper we have present new oscillation and non-oscillation criteria for nonlinear differential equations of second order with damping term

$$(r(t)x'(t))' + p(t)x'(t) + q(t)f(g(x(t))) = 0, \quad t \geq t_0 > 0, \tag{1}$$

where the coefficients are positive and continuous functions.

Also by using the generalized Riccati technique and positive function of Phillo we get a new oscillation and non-oscillation criteria for (3). The result is different from most known ones and improve some previous oscillation criteria and cover the cases which are not covered by known results.

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1. Introduction

In this paper we presented the conditions for what the solutions of the nonlinear differential equations of the second order of the type:

$$(r(t)x'(t))' + p(t)x'(t) + q(t)f(g(x(t))) = 0, \quad t \geq t_0 > 0 \tag{2}$$

are oscillatory or non-oscillatory where $r(t) > 0$ for $t \in I = [t_0, \infty)$, $f, g \in$

$C(\Re,\Re)$ and $r,p,q \in C([t_0,\infty),\Re)$.

We assume the following conditions:

A1) For all $t \in I$, $p(t)$, $q(t)$ are real valued and locally integrable functions over I, and for $t_0 > 0$: $\int_{t_0}^{\infty} \frac{1}{r(t)} dt = \infty$;

A2) $xg(x) > 0$ and $g'(x) \geq k > 0$;

A3)$xf(x) > 0$ and $f'(x) \geq k > 0$.

The solution of equation (3) or (4) is a function $x(t)$, $\mathrm{t} \in [\mathrm{t_x},\infty) \subset [t_0,\infty\}$ which is twice continuously differentiable and satisfies equation (3) or (4) on the given interval. The number depends on that particular solution x *(t)* under consideration.

We consider only non-trivial solutions. A solution *x(t)* of (3) or (4) is said to be oscillatory if there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of points in the interval $[t_0,\infty\}$, so that $\lim\limits_{n\to\infty} \lambda_n = \infty$ and $x(\lambda_n) = 0$, $n \in N$, otherwise it is said to be nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory, otherwise it is considered that is nonoscillatory equation.

We will investigate the oscillatory and non-oscillatory conditions for the differential equations of the form:

$$(r(t)x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0,$$

$$(r(t)x'(t))' + p(t)x'(t) + q(t)x(t) = 0.$$

The analogous probles have been studied by many authors (see for example [1], [2], [8], [9] etc., and more general equations of the form (3)).

2. Main Results

We consider the differential equations of the second order

$$(r(t)x'(t))' + p(t)x'(t) + q(t)f(g(x(t))) = 0, \quad t \geq t_0 > 0 \tag{3}$$

where the conditions $\mathrm{A}_1 - \mathrm{A}_3$ are valid. Our attention is to find the oscillatory or non-oscillatory solutions of (3) for $t \geq t_0 > 0$.

If we get the Ricati transform for (3) for

$$W(t) = \frac{r(t)x'(t)}{f(g(x(t))} \tag{4}$$

we have

$$W'(t) = -\frac{q(t)}{r(t)}\frac{r(t)x'(t)}{f(g(x(t))} - p(t) - \frac{W^2(t)f'(g(x(t))g'(x(t)}{r(t)},$$

$$W'(t) = -\frac{q(t)}{r(t)}W(t) - p(t) - \frac{W^2(t)f'(g(x(t))g'(x(t)}{r(t)}, \tag{5}$$

where we used A_2, and A_3. We obtain

$$W'(t) \leq -\frac{q(t)}{r(t)}W(t) - p(t) - \frac{W^2(t)k^2}{r(t)}. \tag{6}$$

So

$$W'(t) \leq -\frac{1}{r(t)}(kW(t) + \frac{q(t)}{2k})^2 - (p(t) - \frac{q^2(t)}{4r(t)k^2}). \tag{7}$$

Let us set

$$F(t) = e^{\int_{t_0}^{t} \frac{q(l)}{r(l)}dl}, \quad \text{for } t \geq t_0 > 0.(\textbf{??})$$

and

$$E(t) = \int_t^\infty (p(s) - \frac{q^2(s)}{4r(s)k^2})ds, \quad \text{for } t \geq t_0 > 0.$$

Therefore, we obtain the following result.

Theorem 1. *If $t \geq t_0 > 0$, $p(t) > 0$ and*

$$\int_{t_0}^\infty \frac{1}{F(s)r(s)}ds = \infty, \tag{8}$$

$$E(t_0) = \infty. \tag{9}$$

Then the equation (3) is oscillatory.

Proof. From (8) we have $F'(t) = \frac{q(t)}{r(t)}F(t)$, where $F(t) > 0$.
Thus, we may re-write the differential equation

$$(F(t)r(t)x'(t))' + F(t)p(t)f(g(x(t))) = 0 \tag{10}$$

to the form

$$F(t)[(r(t)x'(t))' + q(t)x'(t) + p(t)f(g(x(t)))] = 0.$$

Here we can se that the differential equations (3) is oscillatory if and only if the equation (**??**) is oscillatory. Assume that differential equation (3) is non-oscillatory. Let $x(t)$ is a solution of (3). So we may assume that $x(t) > 0$ for

$t \geq t_0$. For the case where $x(t) < 0$, put $y(t) = -x(t)$. In following we show that $x'(t) > 0$ for $t \geq t_0$.

We can se from (10) and A_2, A_3 that

$$(F(t)r(t)x'(t))' = -F(t)p(t)f(g(x(t))) \leq 0.$$

The function $F(t)r(t)x'(t)$ is not increasing for $t \geq t_0$. Assume that for some $t_1 > t_0$ we get $F(t_1)r(t_1)x'(t_1) < 0$. Put $F(t_1)r(t_1)x'(t_1) = B < 0$ then for $t \geq t_1$ we have

$$F(t)r(t)x'(t) \leq B.$$

Dividing the both sides of the last inequality by $F(t)r(t)$ we get

$$x'(t) \leq \frac{B}{F(t)r(t)}$$

integrate from t_1 to $t > t_1$ we obtain

$$x(t) - x(t_1) \leq B \int_{t_1}^{t} \frac{1}{F(s)r(s)} ds.$$

Thus from (9) it follows that $x(t) < 0$ for sufficiently large t which is a contradiction. So we have $x'(t) > 0$ for $t \geq t_1$. Now we can si that for above given condition and$x'(t) > 0$ we have $W(t) > 0$. To use Ricatti transform for (3) we get the inequality (7). Integrate that from t_1 to $t > t_1$ we have

$$W(t) - W(t_1) \leq -\int_{t_1}^{t} \frac{1}{r(s)}(kW(s) + \frac{q(s)}{2k})^2 ds - \int_{t_1}^{t} (p(s) - \frac{q^2(s)}{4r(s)k^2}) ds,$$

$$W(t) \leq W(t_1) - \int_{t_1}^{t} (p(s) - \frac{q^2(s)}{4r(s)k^2}) ds.$$

To consider (9) there exists a $t_2 \geq t_1$ such that for $t \geq t_2$

$$W(t) \leq 0$$

which is contradiction because we have $W(t) > 0$ for $t > t_1$. Thus the proof is complete.

We note here that the result of theorem 1 is more general form than those presented in [2], [5] etc.

To use inequality (5) we immediately obtain the following.

Lema 1. *Assume that* $p(t) \geq 0$, $q(t) \geq 0$ *for* $t \geq t_0$ *and (7) are valid.If the equation* (3) *has a positive solution* $x(t)$ *we have*

$$\lim_{t\to\infty} \frac{r(t)x'(t)}{f(g(x(t)))} = 0. \tag{11}$$

Proof. Let the solution of the (3) is $x(t) > 0$.From the proof of teorem1. for $t \geq t_0$, $p(t) \geq 0$ since (8) is valid, exists $t_1 \geq t_0$ such that $x'(t) > 0$ for $t \geq t_1$.
Since

$$W(t) = \frac{r(t)x'(t)}{f(g(x(t))} > 0.$$

If dividing inequality (6) with $W^2(t)$ we have

$$\frac{W'(t)}{W^2(t)} \leq -\frac{q(t)}{r(t)} - \frac{p(t)}{W(t)} - \frac{k^2}{r(t)} \leq -\frac{k^2}{r(t)}$$

$$-\frac{W'(t)}{W^2(t)} \geq \frac{k^2}{r(t)}.$$

Integrating the above inequality over $[t_1, t)$ we have

$$\frac{1}{W(t)}/_{t_1}^{t} \geq k^2 \int_{t_1}^{t} \frac{ds}{r(s)},$$

where from A$_1$) we get $\lim\limits_{t\to\infty} \frac{1}{W(t)} = \infty$, for $W(t) > 0$ we have $\lim\limits_{t\to\infty} W(t) = 0$.
The proof is complete. □

Theorem 2. *Assume that* $A_1 - A_3$ *and* (8) *are valid. Then if exists* $t_1 \geq t_0$ *and a continuously differentiable function W(t) satisfying*

$$W(t_1) > \int_{t_1}^{t} \frac{1}{r(s)} (\sqrt{f'(g(x(t)))g'((x(t))}W(s) + \frac{q(s)}{2\sqrt{f'(g(x(t)))g'((x(t))}})^2 ds,$$

$$+ \int_{t_1}^{t} (p(s) - \frac{q^2(s)}{4r(s)f'(g(x(t)))g'(x(t))})ds. \tag{12}$$

Then equation (3) *has a positive solution for* $t > t_1$.

Proof. Since (8) is valid we get that $x'(t) > 0$.
If in the equation (5) integrating from t_1 to t ($>t_1$) we have

$$W(t) - W(t_1) = -\int_{t_1}^{t} \frac{1}{r(s)} (\sqrt{f'(g(x(t)))g'((x(t))}W(s)$$

$$+\frac{q(s)}{2\sqrt{f'(g(x(t)))g'((x(t))}})^2ds-\int_{t_1}^{t}(p(s)-\frac{q^2(s)}{4r(s)f'(g(x(t)))g'(x(t))})ds.$$

Therefore $W(t)>0$ for $t>t_1$. Then we conclude that $x(t)>0$ for $t>t_1$. The proof is complete.

We say that a function $H=H(t,s)$ belongs to a function class X, denote by $H\in X$,if $H\in C(D,R)$, where $D=\{(t,s),-\infty<s\le t<\infty\}$, which completes $H(t,s)>0$, for $t>s$, $H(t,t)=0$ and has continuous partial derivatives on D such that $\frac{\partial H(t,s)}{\partial s}=-h_2(t,s)\sqrt{H(t,s)}$, $\frac{\partial H(t,s)}{\partial t}=h_1(t,s)\sqrt{H(t,s)}$. □

Teorema 3. *Let assumptions A1) – A3) hold and $H\in X$. If there exist $(a,b)\subseteq[t_0,\infty), c\in(a,b)$, such that*

$$\frac{1}{H(c,a)}\int_a^c H(s,a)p(s)ds+\frac{1}{H(b,c)}\int_c^b H(b,s)p(s)ds\le$$

$$\frac{1}{H(c,a)}\int_a^c\frac{r(s)}{4k^2}(\frac{q(s)}{r(s)}-h_1(c,a)\sqrt{H(c,a)})^2ds+$$

$$\frac{1}{H(c,b)}\int_c^b\frac{r(s)}{4k^2}(\frac{q(s)}{r(s)}+h_2(b,c)\sqrt{H(b,c)})^2ds \tag{13}$$

Then every solution of eq. (3) *is osillatory.*

Proof. Suppose to the contrary, that *x(t)* be a non-oscillatory solution of (3), say $x(t)\neq 0$ on $[t_0,\infty)$.

From Teorem 1, if inequation (6) multiplying with $H(t,s)$ and integrate from c to t where $t\in(c,b), s\in(c,t)$ we have

$$\int_c^t H(t,s)p(s)ds\le H(t,c)W(c)+\int_c^t(\frac{W(s)}{\sqrt{r(s)}}+(\frac{q(s)}{r(s)}+h_2(t,s)\sqrt{H(t,s)})\frac{r(s)}{4k^2})ds$$

$$+\int_c^t(\frac{q(r)}{r(s)}+h_2(t,s)\sqrt{H(t,s)})^2\frac{r(s)}{4k^2})ds \tag{14}$$

Let $t\to b_-$ in (14) and dividing it by *H(b,c)* we get

$$\frac{1}{H(b,c)}\int_c^b H(b,s)p(s)ds\le W(c)$$

$$+\frac{1}{H(b,c)}\int_c^b(\frac{q(r)}{r(s)}+h_2(b,s)\sqrt{H(b,s)})^2\frac{r(s)}{4k^2})ds \tag{15}$$

If (6) multiplying with $H(s,t)$ and integrate over (t,c) where $t \in (a,c), s \in (t,c)$

$$\int_t^c H(s,t)p(s)ds \le H(c,t)W(c) + \int_t^c (\frac{W(s)}{\sqrt{r(s)}} + (\frac{q(s)}{r(s)} - h_1(s,t)\sqrt{H(s,t)})\frac{r(s)}{4k^2})ds$$

$$+ \int_c^t (\frac{q(r)}{r(s)} + h_2(s,t)\sqrt{H(s,t)})^2 \frac{r(s)}{4k^2})ds$$

Let $t \to a^+$ in (12) and dividing it by *H(c,a)* we obtain

$$\frac{1}{H(c,a)} \int_a^c H(s,a)p(s)ds \le W(c) + \frac{1}{H(c,a)} \int_a^c (\frac{q(r)}{r(s)} + h_2(c,a)\sqrt{H(c,a)})^2 \frac{r(s)}{4k^2})ds \quad (16)$$

Adding (15) and (16) we have the following inequality

$$\frac{1}{H(c,a)} \int_a^c H(s,a)p(s)ds + \frac{1}{H(b,c)} \int_c^b H(b,s)p(s)ds$$

$$\le \frac{1}{H(c,a)} \int_a^c (\frac{q(r)}{r(s)} + h_2(c,a)\sqrt{H(c,a)})^2 \frac{r(s)}{4k^2})ds + \frac{1}{H(b,c)} \int_c^b (\frac{q(s)}{r(s)} + h_2(b,s)\sqrt{H(b,s)})^2 \frac{r(s)}{4k^2})ds.$$

Which contradict to the condition (13), therefore every solution of equation (3) be oscillatory. The proof is complete. □

Corollary 1 *Let assumptions A1) – A3) hold and* $H \in X$. *If there exist* $(a,b) \subseteq [t_0, \infty), c \in (a,b)$, *such that*

$$\frac{1}{H(c,a)} \int_a^c (\frac{r(s)}{4k^2}(\frac{q(s)}{r(s)} - h_1(c,a)\sqrt{H(c,a)})^2 - H(s,a)p(s)ds \ge 0$$

$$\frac{1}{H(b,c)} \int_c^b (\frac{q(s)}{r(s)} + h_2(b,s)\sqrt{H(b,s)})^2 \frac{r(s)}{4k^2}) - H(b,s)p(s)ds \ge 0.$$

Then the equation (3) *is oscilatory.*

The proof follow from Theorem 3.

Some results given above are studies in [1], [2], [3], [9] for *g(x)=x* and *f(x)=x.*

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