International Journal of Pure and Applied Mathematics
Volume 93 No. 2 2014, 165-179
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v93i2.2

ijpam.eu

ALEXANDROV L-TOPOLOGIES

Yong Chan Kim

Department of Mathematics
Gangneung-Wonju University
Gangneung, Gangwondo 210-702, KOREA

Abstract: In this paper, we introduce upper and lower approximation operators as a generalization of fuzzy rough set in complete residuated lattices. We investigate relations between their operations and Alexandrov L-topologies.

AMS Subject Classification: 03E72, 03G10, 06A15, 06F07
Key Words: complete residuated lattices, upper and lower approximation operators, Alexandrov L-topologies

1. Introduction

Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Pawlak [6,7] introduced rough set theory as a formal tool to deal with imprecision and uncertainty in data analysis. Radzikowska [8] developed fuzzy rough sets in complete residuated lattice. Bělohlávek [1] investigated information systems and decision rules in complete residuated lattices. Zhang [11,12] introduced Alexandrov L-topologies induced by fuzzy rough sets. Algebraic structures of fuzzy rough sets are developed in many directions [3,9,10].

Received: October 24, 2013

In this paper, we introduce upper and lower approximation operators as a generalization of fuzzy rough set in complete residuated lattices. We investigate relations between their operations and Alexandrov L-topologies.

2. Preliminaries

Definition 1. [1,2] An algebra $(L, \wedge, \vee, \odot, \rightarrow, \perp, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$;

(C2) $(L, \odot, \top)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume $(L, \wedge, \vee, \odot, \rightarrow,^* \perp, \top)$ is a complete residuated lattice with the law of double negation;i.e. $x^{**} = x$. For $\alpha \in L, A, \top_x \in L^X$, $(\alpha \rightarrow A)(x) = \alpha \rightarrow A(x)$, $(\alpha \odot A)(x) = \alpha \odot A(x)$ and $\top_x(x) = \top, \top_x(x) = \perp$, otherwise.

Lemma 2. (see [1,2]) *For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

(1) *If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.*

(2) $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$.

(3) $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$.

(4) $\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ *and* $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.

(5) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

(6) $x \odot y = (x \rightarrow y^*)^*$.

(7) $x \odot (x \rightarrow y) \leq y$.

(8) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.

Definition 3. (see [1,4]) Let X be a set. A function $R : X \times X \rightarrow L$ is called:

(R1) *reflexive* if $R(x, x) = \top$ for all $x \in X$,

(R2) *transitive* if $R(x, y) \odot R(y, z) \leq R(x, z)$, for all $x, y, z \in X$.

If R satisfies (R1) and (R2), R is called a *preorder*.

3. Alexandrov L-Topologies

Definition 4. (1) A map $\mathcal{H}: L^X \to L^X$ is called an L-*upper approximation operator* iff it satisfies the following conditions

(H1) $A \le \mathcal{H}(A)$,

(H2) $\mathcal{H}(\alpha \odot A) = \alpha \odot \mathcal{H}(A)$ where $\alpha(x) = \alpha$ for all $x \in X$,

(H3) $\mathcal{H}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{H}(A_i)$.

(2) A map $\mathcal{J}: L^X \to L^X$ is called an L-*lower approximation operator* iff it satisfies the following conditions

(J1) $\mathcal{J}(A) \le A$,

(J2) $\mathcal{J}(\alpha \to A) = \alpha \to \mathcal{J}(A)$,

(J3) $\mathcal{J}(\bigwedge_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{J}(A_i)$.

Lemma 5. (1) *If $\mathcal{H}$ is an L-upper approximation operator, then $\mathcal{H}(\alpha \to A) \le \alpha \to \mathcal{H}(A)$.*

(2) *If $\mathcal{J}$ is an L-lower approximation operator, then $\alpha \odot \mathcal{J}(A) \le \mathcal{J}(\alpha \odot A)$.*

Proof. (1) By (H3), $\mathcal{H}$ is an increasing operator. Since

$$\alpha \odot \mathcal{H}(\alpha \to A) = \mathcal{H}(\alpha \odot (\alpha \to A)) \le \mathcal{H}(A).$$

Hence $\mathcal{H}(\alpha \to A) \le \alpha \to \mathcal{H}(A)$.

(2) By (J3), $\mathcal{J}$ is an increasing operator. Since

$$\mathcal{J}(A) \le \mathcal{J}(\alpha \to \alpha \odot A) = \alpha \to \mathcal{J}(\alpha \odot A).$$

Hence $\alpha \odot \mathcal{J}(A) \le \mathcal{J}(\alpha \odot A)$.

□

Theorem 6. *Let $\mathcal{H}: L^X \to L^X$ be an L-upper approximation operator iff there exists a reflexive relation $R \in L^{X\times X}$ such that*

$$\mathcal{H}(A)(y) = \bigvee_{x\in X} (A(x) \odot R(x,y)).$$

Proof. ($\Rightarrow$) Define $R(x,y) = \mathcal{H}(\top_x)(y)$. Since $\mathcal{H}(\top_x)(y) \ge \top_x(y)$, then $R(x,x) = \top$. Since $A = \bigvee_{x\in X}(A(x) \odot \top_x)$, we have

$$\begin{aligned}\mathcal{H}(A)(y) &= \mathcal{H}(\textstyle\bigvee_{x\in X} A(x) \odot \top_x)(y)\\ &= \textstyle\bigvee_{x\in X}(A(x) \odot \mathcal{H}(\top_x)(y))\\ &= \textstyle\bigvee_{x\in X}(A(x) \odot R(x,y)).\end{aligned}$$

(⇐) (H1) Since $\mathcal{H}(A)(x) \geq A(x) \odot R(x,x) = A(x)$. (H2) and (H3) are easily proved.

□

Theorem 7. *Let X and Y be two sets. $\mathcal{J}$ is an L-lower approximation operator iff there exists a reflexive relation $R \in L^{X\times X}$ such that*

$$\mathcal{J}(A)(y) = \bigwedge_{x\in X} (R(x,y) \to A(x)).$$

Proof. (⇒) Define $R(x,y) = \mathcal{J}(\top_x^*)^*(y)$. Since $A = \bigwedge_{x\in X}(A^*(x) \to \top_x^*)$, we have

$$\begin{aligned}\mathcal{J}(A)(y) &= \mathcal{J}(\textstyle\bigwedge_{x\in X}(A^*(x) \to \top_x^*))(y)\\ &= \textstyle\bigwedge_{x\in X}(A^*(x) \to \mathcal{J}(\top_x^*)(y))\\ &= \textstyle\bigwedge_{x\in X}(\mathcal{J}(\top_x^*)^*(y) \to A(x))\\ &= \textstyle\bigwedge_{x\in X}(R(x,y) \to A(x)).\end{aligned}$$

⇐ (J1) $\mathcal{J}(A)(y) = \bigwedge_{x\in X}(R(x,y) \to A(x)) \leq R(y,y) \to A(y) = A(y)$.

(J2) $\mathcal{J}(\alpha \to A)(y) = \bigwedge_{x\in X}(R(x,y) \to (\alpha \to A)(x)) = \alpha \to \bigwedge_{x\in X}(R(x,y) \to A(x)) = \alpha \to \mathcal{J}(A)(y)$.

(J3) It is easily proved.

□

Theorem 8. *Let $\mathcal{H} : L^X \to L^X$ be an L-upper approximation operators. Then the following properties hold.*

(1) Define $\mathcal{J}_1(A) = \bigvee\{B \mid \mathcal{H}(B) \leq A\}$. Then $\mathcal{J}_1 : L^X \to L^X$ is an L-lower approximation operator such that $\mathcal{H}(B) \leq A$ iff $B \leq \mathcal{J}_1(A)$ for all $A, B \in L^X$.

(2) Define $\mathcal{J}_2(A) = \mathcal{H}(A^*)^*$. Then $\mathcal{J}_2 : L^X \to L^X$ is an L-lower approximation operator

(3) If $\mathcal{H}(\top_y)(x) = \mathcal{H}(\top_x)(y)$ for all $x, y \in X$, then $\mathcal{J}_1 = \mathcal{J}_2$.

Proof. (1) (J1) Since $B \leq \mathcal{H}(B) \leq A$, $\mathcal{J}_1(A) \leq A$.

(J2) Since $\mathcal{H}(\mathcal{J}_1(A)) \leq A$ iff $\mathcal{J}_1(A) \leq \mathcal{J}_1(A)$, we have $\alpha \to \mathcal{J}_1(A) \leq \alpha \to \mathcal{J}_1(A)$ iff $\alpha \odot (\alpha \to \mathcal{J}_1(A)) \leq \mathcal{J}_1(A)$ iff $\mathcal{H}(\alpha \odot (\alpha \to \mathcal{J}_1(A))) = \alpha \odot \mathcal{H}(\alpha \to \mathcal{J}_1(A)) \leq A$ iff $\mathcal{H}(\alpha \to \mathcal{J}_1(A)) \leq \alpha \to A$ iff $\alpha \to \mathcal{J}_1(A) \leq \mathcal{J}_1(\alpha \to A)$. Hence $\mathcal{J}_1(\alpha \to A) \geq \alpha \to \mathcal{J}_1(A)$.

Suppose there exists $B \in L^X$ with $\mathcal{H}(B) \leq \alpha \to A$ such that

$$B \not\leq \alpha \to \mathcal{J}_1(A).$$

Since $\alpha \odot \mathcal{H}(B) = \mathcal{H}(\alpha \odot B) \leq A$, then $\mathcal{J}_1(A) \geq \alpha \odot B$. Thus,

$$\alpha \to \mathcal{J}_1(A) \geq \alpha \to (\alpha \odot B) \geq B.$$

It is a contradiction. Hence $\mathcal{J}_1(\alpha \to A) \le \alpha \to \mathcal{J}_1(A)$.

(J3) $\mathcal{J}_1(\bigwedge_{i\Gamma} A_i) = \bigwedge_{i\Gamma} \mathcal{J}_1(A_i)$. By the definition of $\mathcal{J}_1$, since $\bigwedge_{i\Gamma} A_i \le A_i$, then

$$\mathcal{J}_1(\bigwedge_{i\Gamma} A_i) \le \bigwedge_{i\Gamma} \mathcal{J}_1(A_i).$$

Since $\mathcal{H}(\bigwedge_{i\Gamma} \mathcal{J}_1(A_i)) \le \mathcal{H}(\mathcal{J}_1(A_i)) \le A_i$, then $\mathcal{H}(\bigwedge_{i\Gamma} \mathcal{J}_1(A_i)) \le \bigwedge_{i\Gamma} A_i$. Thus

$$\bigwedge_{i\Gamma} \mathcal{J}_1(A_i) \le \mathcal{J}_1(\bigwedge_{i\Gamma} A_i).$$

(2) (J1) Since $A^* \le \mathcal{H}(A^*)$, $\mathcal{J}_2(A) = \mathcal{H}(A^*)^* \le A$.

(J2)

$$\begin{aligned}\mathcal{J}_2(\alpha \to A) &= (\mathcal{H}(\alpha \to A)^*)^* = (\mathcal{H}(\alpha \odot A^*))^* \\ &= (\alpha \odot \mathcal{H}(A^*))^* = \alpha \to \mathcal{H}(A^*)^* \\ &= \alpha \to \mathcal{J}_2(A).\end{aligned}$$

(J3)

$$\begin{aligned}\mathcal{J}_2(\bigwedge_{i\in\Gamma} A_i) &= (\mathcal{H}(\bigwedge_{i\in\Gamma} A_i)^*)^* = (\mathcal{H}(\bigvee_{i\in\Gamma} A_i^*))^* \\ &= (\bigvee_{i\in\Gamma} \mathcal{H}(A_i^*))^* = \bigwedge_{i\in\Gamma}(\mathcal{H}(A_i^*))^* \\ &= \bigwedge_{i\in\Gamma} \mathcal{J}_2(A_i).\end{aligned}$$

(3)

$$\begin{aligned}\mathcal{J}_1(A)(x) &= \bigvee\{B(x) \mid \mathcal{H}(B)(y) \le A(y)\} \\ &= \bigvee\{B(x) \mid \mathcal{H}(\bigvee_{x\in X} B(x) \odot \top_x)(y) \le A(y)\} \\ &= \bigvee\{B(x) \mid \bigvee_{x\in X} B(x) \odot \mathcal{H}(\top_x)(y) \le A(y)\} \\ &= \bigvee\{B(x) \mid B(x) \le \mathcal{H}(\top_x)(y) \to A(y)\} \\ &= \bigwedge_{y\in X}(\mathcal{H}(\top_x)(y) \to A(y))\end{aligned}$$

For $A^*(x) = \bigvee_{y\in Y}(A^*(y) \odot \top_y(x))$,

$$\begin{aligned}\mathcal{J}_2(A)(x) &= (\mathcal{H}(A^*))^*(x) = (\mathcal{H}(\bigvee_{y\in X}(A^*(y) \odot \top_y)))^*(x) \\ &= (\bigvee_{y\in X}(A^*(y) \odot \mathcal{H}(\top_y)))^*(x) \\ &= \bigwedge_{y\in X}(\mathcal{H}(\top_y)(x) \to A(y)).\end{aligned}$$

□

Remark 9. Define $\mathcal{H}_R : L^X \to L^X$ as $\mathcal{H}_R(A)(y) = \bigvee_{x\in X}(A(x) \odot R(x,y))$. By Theorem 8(1), we obtain $\mathcal{J}_1(A)(x) = \bigvee\{B \mid \mathcal{H}_R(B) \le A\} = \bigwedge_{y\in Y}(R(x,y) \to A(y))$. If R is reflexive and $R(x,y) = R(y,x)$ for $x, y \in X$, then $\mathcal{H}_R(\top_y)(x) = R(x,y) = R(y,x) = \mathcal{H}_R(\top_x)(y)$. Hence $\mathcal{J}_1(A) = \bigvee\{B \mid \mathcal{H}_R(B) \le A\} = \mathcal{H}_R(A^*)^*$.

Theorem 10. *Let $\mathcal{J}: L^X \to L^X$ be an L-lower approximation operators. Then the following properties hold.*

(1) Define $\mathcal{H}_1(B) = \bigwedge\{A \mid B \leq \mathcal{J}(A)\}$. Then $\mathcal{H}_1 : L^X \to L^X$ is an L-upper approximation operator such that $(\mathcal{H}_1, \mathcal{J})$ is a residuated connection.

(2) Define $\mathcal{H}_2(A) = \mathcal{J}(A^*)^*$. Then $\mathcal{H}_2 : L^X \to L^X$ is an L-upper approximation operator

(3) If $\mathcal{J}(\top_y^*)^*(x) = \mathcal{J}(\top_x^*)^*(y)$ for all $x, y \in X$, then $\mathcal{H}_1 = \mathcal{H}_2$.

Proof. (1) Since $B \leq \mathcal{J}(A) \leq A$, $B \leq \mathcal{H}_1(B)$.

(H2) $\mathcal{H}_1(\alpha \odot B) = \alpha \odot \mathcal{H}_1(B)$.

Let $\mathcal{H}_1(B) \leq \bigwedge A$ iff $B \leq \mathcal{J}(\bigwedge A)$. Then

$$\alpha \odot B \leq \alpha \odot \mathcal{J}(\bigwedge A) \leq \mathcal{J}(\alpha \odot \bigwedge A)$$

Hence $\mathcal{H}_1(\alpha \odot B) \leq \alpha \odot \bigwedge A$. So, $\mathcal{H}_1(\alpha \odot B) \leq \alpha \odot \mathcal{H}_1(B)$.

Suppose $\mathcal{H}_1(\alpha \odot B) \not\geq \alpha \odot \mathcal{H}_1(B)$. Suppose there exists $A \in L^X$ with $\mathcal{J}(A) \geq \alpha \odot B$ such that

$$A \not\geq \alpha \odot \mathcal{H}_1(B).$$

Since $\mathcal{J}(\alpha \to A) = \alpha \to \mathcal{J}(A) \geq \alpha \to \alpha \odot B \geq B$, then $\mathcal{H}_1(B) \geq \alpha \to B$. Thus,

$$\alpha \odot \mathcal{H}_1(B) \leq \alpha \odot (\alpha \to A) \leq A.$$

It is a contradiction. Hence $\mathcal{H}_1(\alpha \odot B) \geq \alpha \odot \mathcal{H}_1(B)$.

(H3) We show that $\mathcal{H}_1(\bigvee_{i\Gamma} A_i) = \bigvee_{i\Gamma} \mathcal{H}_1(A_i)$. By the definition of $\mathcal{J}$, since $A_i \leq \bigvee_{i\Gamma} A_i$, then

$$\mathcal{H}_1(\bigvee_{i\Gamma} A_i) \geq \bigvee_{i\Gamma} \mathcal{H}_1(A_i).$$

Since $\mathcal{J}(\bigvee_{i\Gamma} \mathcal{H}_1(A_i)) \geq \mathcal{J}(\mathcal{H}_1(A_i)) \geq A_i$, then $\mathcal{J}(\bigvee_{i\Gamma} \mathcal{H}_1(A_i)) \geq \bigvee_{i\Gamma} A_i$. Thus

$$\mathcal{H}_1(\bigvee_{i\Gamma} A_i) \leq \bigvee_{i\Gamma} \mathcal{H}_1(A_i).$$

(2) (H1) Since $A^* \geq \mathcal{J}(A^*)$, $\mathcal{H}_2(A) = \mathcal{J}(A^*)^* \geq A$.

(H2)

$$\begin{aligned}\mathcal{H}_2(\alpha \odot A) &= (\mathcal{J}(\alpha \odot A)^*)^* = (\mathcal{J}(\alpha \to A^*))^* \\ &= (\alpha \to \mathcal{J}(A^*))^* = \alpha \odot \mathcal{J}(A^*)^* \\ &= \alpha \odot \mathcal{H}_2(A).\end{aligned}$$

(H3)

$$\begin{aligned}\mathcal{H}_2(\textstyle\bigvee_{i\in\Gamma} A_i) &= (\mathcal{J}(\textstyle\bigvee_{i\in\Gamma} A_i)^*)^* = (\mathcal{J}(\textstyle\bigwedge_{i\in\Gamma} A_i^*))^* \\ &= (\textstyle\bigwedge_{i\in\Gamma} \mathcal{J}(A_i^*))^* = \textstyle\bigvee_{i\in\Gamma}(\mathcal{J}(A_i^*))^* \\ &= \textstyle\bigvee_{i\in\Gamma} \mathcal{H}_2(A_i).\end{aligned}$$

(3)

$$\begin{aligned}\mathcal{H}_1(B)(x) &= \bigwedge\{A(x) \mid B \le \mathcal{J}(A)\}\\ &= \bigwedge\{A(x) \mid B(y) \le \bigwedge_{x\in X}(A^*(x) \to \mathcal{J}(\top_x^*)(y))\}\\ &= \bigwedge\{A(x) \mid A^*(x) \le \bigwedge_{y\in X} B(x) \to \mathcal{F}(\top_x^*)(y)\}\\ &= \bigwedge\{A(x) \mid A(x) \ge \bigvee_{y\in X}(B(y) \odot \mathcal{J}(\top_x^*)^*(y)\}\\ &= \bigvee_{y\in X}(B(y) \odot \mathcal{J}(\top_x^*)^*(y)\end{aligned}$$

For $A^* = \bigwedge_{y\in X}(A(y) \to \top_y^*)$,

$$\begin{aligned}\mathcal{H}_2(A)(x) &= (\mathcal{J}(A^*))^*(x) = (\mathcal{J}(\bigwedge_{y\in X}(A(y) \to \top_y^*)))^*(x)\\ &= (\bigwedge_{y\in X}(A(y) \to \mathcal{J}(\top_y^*)))^*(x)\\ &= \bigvee_{y\in X}(A(y) \odot \mathcal{J}(\top_y^*)^*(x)).\end{aligned}$$

□

Remark 11. Define $\mathcal{J}_R : L^X \to L^X$ as $\mathcal{J}_R(A)(y) = \bigwedge_{x\in X}(R(x,y) \to A(x))$. By Theorem 10, $\mathcal{H}_1(A)(x) = \bigwedge\{B \mid \mathcal{J}_R(B) \le A\} = \bigvee_{y\in Y}(R(x,y) \odot A(y))$. If R is reflexive and $R(x,y) = R(y,x)$ for $x,y \in X$, then $\mathcal{J}_R(\top_y^*)^*(x) = R(y,x) = R(x,y) = \mathcal{J}_R(\top_x^*)^*(y)$. Hence $\mathcal{H}_1(A) = \bigwedge\{B \mid \mathcal{J}_R(B) \le A\} = \mathcal{J}_R(A^*)^*$.

Definition 12. [4] A subset $\tau \subset L^X$ is called an *Alexandrov L-topology* if it satisfies:

(T1) $\bot_X, \top_X \in \tau$ where $\top_X(x) = \top$ and $\bot_X(x) = \bot$ for $x \in X$.

(T2) If $A_i \in \tau$ for $i \in \Gamma$, $\bigvee_{i\in\Gamma} A_i, \bigwedge_{i\in\Gamma} A_i \in \tau$.

(T3) $\alpha \odot A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

(T4) $\alpha \to A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

Theorem 13. (1) *τ is an Alexandorv L-topology on X iff $\tau_* = \{A^* \in L^X \mid A \in \tau\}$ is an Alexandorv L-topology on X.*

(2) *If $\mathcal{H}$ is an L-upper approximation operator, then $\tau_{\mathcal{H}} = \{A \in L^X \mid \mathcal{H}(A) = A\}$ is an Alexandorv L-topology on X.*

(3) *If $\mathcal{J}$ is an L-lower approximation operator, then $\tau_{\mathcal{J}} = \{A \in L^X \mid \mathcal{J}(A) = A\}$ is an Alexandorv L-topology on X.*

Proof. (1) Let $A^* \in \tau_*$ for $A \in \tau$. Since $\alpha \odot A^* = (\alpha \to A)^*$ and $\alpha \to A^* = (\alpha \odot A)^*$, τ_* is an Alexandorv topology on X.

(2) (T1) Since $\top_X \le \mathcal{H}(\top_X)$ and $\mathcal{H}(\bot_X) = \mathcal{H}(\bot_X \odot A) = \bot_X \odot \mathcal{H}(A)$, $\top_X = \mathcal{H}(\top_X)$ and $\top_X = \mathcal{H}(\top_X)$. Then $\bot_X, \top_X \in \tau_{\mathcal{H}}$.

(T2) For $A_i \in \tau_{\mathcal{H}}$ for each $i \in \Gamma$, by (H3), $\bigvee_{i\in\Gamma} A_i \in \tau_{\mathcal{H}}$. Since $\bigwedge_{i\in\Gamma} A_i \le \mathcal{H}(\bigwedge_{i\in\Gamma} A_i) \le \bigwedge_{i\in\Gamma} \mathcal{H}(A_i) = \bigwedge_{i\in\Gamma} A_i$, Thus, $\bigwedge_{i\in\Gamma} A_i \in \tau_{\mathcal{H}}$.

(T3) For $A \in \tau_{\mathcal{H}}$, by (H2), $\alpha \odot A \in \tau_{\mathcal{H}}$.

(T4) For $A \in \tau_{\mathcal{H}}$, by Lemma 5 (1), $\mathcal{H}(\alpha \to A) \leq \alpha \to \mathcal{H}(A) = \alpha \to A$. Then $\alpha \to A \in \tau_{\mathcal{H}}$.

(3) (T1) Since $\mathcal{J}(\perp_X) \leq \perp_X$ and $\mathcal{J}(\top_X) = \mathcal{J}(\perp \to A) = \perp \to \mathcal{J}(A) = \top_X$, $\perp_X = \mathcal{J}(\perp_X)$ and $\top_X = \mathcal{J}(\top_X)$. Then $\perp_X, \top_X \in \tau_{\mathcal{J}}$.

(T2) For $A_i \in \tau_{\mathcal{J}}$ for each $i \in \Gamma$, by (J3), $\bigwedge_{i\in\Gamma} A_i \in \tau_{\mathcal{J}}$. Since $\bigvee_{i\in\Gamma} A_i \leq \bigvee_{i\in\Gamma} \mathcal{J}(A_i) \leq \bigvee_{i\in\Gamma} \mathcal{J}(A_i) = \bigvee_{i\in\Gamma} A_i$, Thus, $\bigvee_{i\in\Gamma} A_i \in \tau_{\mathcal{J}}$.

(T3) For $A \in \tau_{\mathcal{J}}$, by Lemma 5(2), $\alpha \odot \mathcal{J}(A) \leq \mathcal{J}(\alpha \odot A) \leq \alpha \odot A$. Then $\alpha \odot A \in \tau_{\mathcal{J}}$.

(T4) For $A \in \tau_{\mathcal{J}}$, by (J2), $\alpha \to A \in \tau_{\mathcal{J}}$.

□

Theorem 14. (1) *If τ is an Alexandorv L-topology on X, then there exists a preorder R_τ such that $\tau = \tau_{\mathcal{H}_{R_\tau}}$. Moreover, we define $h_\tau : L^X \to L^X$ as $h_\tau(A) = \bigwedge\{A_i \mid A \leq A_i,\ A_i \in \tau\}$. Then $h_\tau = \mathcal{H}_{R_\tau}$ is an L-upper approximation operator on X.*

(2) *If τ is an Alexandorv L-topology on X, then there exists a preorder R_τ^{-1} such that $\tau_* = \tau_{\mathcal{H}_{R_\tau^{-1}}}$. Moreover, we define $h_{\tau_*} : L^X \to L^X$ as $h_{\tau_*}(A) = \bigwedge\{A_i \mid A \leq A_i,\ A_i \in \tau_*\}$. Then $h_{\tau_*} = \mathcal{H}_{R_\tau^{-1}}$ is an L-upper approximation operator on X.*

(3) *If τ is an Alexandorv L-topology on X, then there exists a preorder R_τ^{-1} such that $\tau = \tau_{\mathcal{J}_{R_\tau^{-1}}}$. Moreover, we define $j_\tau : L^X \to L^X$ as $j_\tau(A) = \bigvee\{A_i \mid A_i \leq A,\ A_i \in \tau\}$. Then $j_\tau = \mathcal{J}_{R_\tau^{-1}}$ is an L-lower approximation operator on X.*

(4) *If τ is an Alexandorv L-topology on X, then there exists a preorder R_τ such that $\tau_* = \tau_{\mathcal{J}_{R_\tau}}$. Moreover, we define $j_{\tau_*} : L^X \to L^X$ as $j_{\tau_*}(A) = \bigvee\{A_i \mid A_i \leq A,\ A_i \in \tau_*\}$. Then $j_{\tau_*} = \mathcal{J}_{R_\tau}$ is an L-lower approximation operator on X.*

Proof. (1) Define $R_\tau(x,y) = \bigwedge_{B\in\tau}(B(x) \to B(y))$. Then R_τ is a preorder. Since $A \in \tau$, $R_\tau(x,y) \odot A(x) = \bigwedge_{B\in\tau}(B(x) \to B(y)) \odot A(x) \leq (A(x) \to A(y)) \odot A(x) \leq A(y)$. Hence $\mathcal{H}_{R_\tau}(A) = A \in \tau_{\mathcal{H}_{R_\tau}}$.

Let $\mathcal{H}_{R_\tau}(A) = A \in \tau_{\mathcal{H}_{R_\tau}}$. Then

$$\begin{aligned} A(x) &= \mathcal{H}_{R_\tau}(A)(x) = \bigvee_{y\in X}(A(y) \odot R_\tau(y,x)) \\ &= \bigvee_{y\in X}(A(y) \odot \bigwedge_{B\in\tau}(B(y) \to B(x))) \end{aligned}$$

Since $\bigwedge_{B\in\tau}(B(y) \to B) \in \tau$ and $\bigvee_{y\in X}(A(y) \odot \bigwedge_{B\in\tau}(B(y) \to B)) \in \tau$, we have $A \in \tau$. Hence $\tau = \tau_{\mathcal{H}_{R_\tau}}$.

Since $A_i \in \tau = \tau_{\mathcal{H}_{R_\tau}}$ with $A \le A_i,\ A_i \in \tau$, then

$$\bigwedge_i A_i \le \mathcal{H}_{R_\tau}(\bigwedge_i A_i) \le A_i = \mathcal{H}_{R_\tau}(A_i).$$

So, $\mathcal{H}_{R_\tau}(\bigwedge_i A_i) = \bigwedge_i A_i$. Thus

$$\mathcal{H}_{R_\tau}(A) \le \mathcal{H}_{R_\tau}(\bigwedge_i A_i) = \bigwedge_i A_i = h_\tau(A).$$

Since R_τ is reflexive and transitive, we have $\bigvee_{y\in X} R_\tau(z,y)\odot R_\tau(y,x) = R_\tau(z,x)$. Therefore,

$$\begin{aligned}\mathcal{H}_{R_\tau}(\mathcal{H}_{R_\tau}(A))(x) &= \bigvee_{y\in X}(\mathcal{H}_{R_\tau}(A)(y)\odot R_\tau(y,x))\\ &= \bigvee_{y\in X}(\bigvee_{z\in X}(A(z)\odot R_\tau(z,y))\odot R_\tau(y,x))\\ &= \bigvee_{z\in X}(A(z)\odot \bigvee_{y\in X}(R_\tau(z,y)\odot R_\tau(y,x)))\\ &= \bigvee_{z\in X}(A(z)\odot R_\tau(z,x))\\ &= \mathcal{H}_{R_\tau}(A)(x).\end{aligned}$$

Thus, $\mathcal{H}_{R_\tau}(\mathcal{H}_{R_\tau}(A)) = \mathcal{H}_{R_\tau}(A) \ge A$ and $\mathcal{H}_{R_\tau}(A) \in \tau_{\mathcal{H}_{R_\tau}} = \tau$, $h_\tau \le \mathcal{H}_{R_\tau}(A)$. Hence $h_\tau = \mathcal{H}_{R_\tau}$.

(2) By (1), since $R_\tau^{-1}(x,y) = \bigwedge_{B\in\tau}(B(y)\to B(x))$, $R_\tau^{-1}(x,x) = \top$ and

$$\begin{aligned}R_\tau^{-1}(x,y)\odot R_\tau^{-1}(y,z) &= \bigwedge_{B\in\tau}(B(y)\to B(x))\odot \bigwedge_{B\in\tau}(B(z)\to B(x)))\\ &\le \bigwedge_{B\in\tau}((B(y)\to B(x))\odot(B(z)\to B(y)))\\ &\le \bigwedge_{B\in\tau}(B(z)\to B(x)).\end{aligned}$$

Thus R_τ^{-1} is a preorder. Since $A\in\tau$, $R_\tau^{-1}(x,y)\odot A^*(x) = \bigwedge_{B\in\tau}(B^*(x)\to B^*(y))\odot A^*(x) \le (A^*(x)\to A^*(y))\odot A^*(x) \le A^*(y)$. Hence $\mathcal{H}_{R_\tau^{-1}}(A^*) = A^* \in \tau_{\mathcal{H}_{R_\tau^{-1}}}$.

Let $\mathcal{H}_{R_\tau^{-1}}(A) = A \in \tau_{\mathcal{H}_{R_\tau^{-1}}}$. Then

$$\begin{aligned}A(x) &= \mathcal{H}_{R_\tau^{-1}}(A)(x) = \bigvee_{y\in X}(A(y)\odot R_\tau^{-1}(y,x))\\ &= \bigvee_{y\in X}(A(y)\odot \bigwedge_{B\in\tau}(B^*(y)\to B^*(x)))\end{aligned}$$

Since $\bigwedge_{B\in\tau}(B^*(y)\to B^*)\in\tau_*$ and $\bigvee_{y\in X}(A(y)\odot\bigwedge_{B\in\tau}(B^*(y)\to B^*))\in\tau_*$, we have $A\in\tau_*$. Hence $\tau_* = \tau_{\mathcal{H}_{R_\tau^{-1}}}$.

(3) By (2), $R_\tau^{-1}(x,y) = \bigwedge_{B\in\tau}(B(y)\to B(x))$ is a preorder. Since $A\in\tau$,

$$\begin{aligned}A(y)\odot R_\tau^{-1}(x,y) &= A(y)\odot\bigwedge_{B\in\tau}(B(y)\to B(x))\\ &\le A(y)\odot(A(y)\to A(x)) \le A(x).\end{aligned}$$

Thus $A(y) \leq R_\tau^{-1}(x,y) \to A(x)$. Hence $\mathcal{J}_{R_\tau^{-1}}(A) = A \in \tau_{\mathcal{J}_{R_\tau^{-1}}}$.

Let $\mathcal{J}_{R_\tau^{-1}}(A) = A \in \tau_{\mathcal{J}_{R_\tau^{-1}}}$. Then $\bigwedge_y (R_\tau^{-1}(x,y) \to A(x)) = A(y)$. So, $A(y) \leq R_\tau^{-1}(x,y) \to A(x)$ iff $A(y) \odot R_\tau^{-1}(x,y) \leq A(x)$ iff $A(y) \odot R_\tau(y,x) \leq A(x)$ .. Thus $\mathcal{H}_{R_\tau}(A)(x) \leq A(x)$. Hence $A = \mathcal{H}_{R_\tau}(A)$.

$$\begin{array}{ll} A(x) & = \mathcal{H}_{R_\tau}(A)(x) = \bigvee_{y\in X}(A(y) \odot R_\tau(y,x)) \\ & = \bigvee_{y\in X}(A(y) \odot \bigwedge_{B\in\tau}(B(y) \to B(x))) \end{array}$$

Since $\bigwedge_{B\in\tau}(B(y) \to B) \in \tau$ and $\bigvee_{y\in X}(A(y) \odot \bigwedge_{B\in\tau}(B(y) \to B)) \in \tau$, we have $A \in \tau$. Hence $\tau = \tau_{\mathcal{J}_{R_\tau^{-1}}}$.

Since $A_i \in \tau = \tau_{\mathcal{J}_{R_\tau^{-1}}}$, then $A_i = \mathcal{J}_{R_\tau^{-1}}(A_i) \leq \mathcal{J}_{R_\tau^{-1}}(\bigvee_i A_i) \leq \bigvee_i A_i$, then $\mathcal{J}_{R_\tau^{-1}}(\bigvee_i A_i) = \bigvee_i A_i$. Thus $j_\tau(A) \leq \mathcal{J}_{R_\tau^{-1}}(A)$. Since R_τ^{-1} is a preorder, we have $\bigvee_{y\in X} R_\tau^{-1}(x,y) \odot R_\tau^{-1}(y,z) = R_\tau^{-1}(y,z)$. Therefore,

$$\begin{array}{ll} \mathcal{J}_{R_\tau^{-1}}(\mathcal{J}_{R_\tau^{-1}}(A))(x) & = \bigwedge_{y\in X}(R_\tau^{-1}(x,y) \to \mathcal{J}_{R_\tau^{-1}}(A)(y)) \\ & = \bigwedge_{y\in X}(R_\tau^{-1}(x,y) \to \bigwedge_{z\in X}(R_\tau^{-1}(y,z) \to A(z))) \\ & = \bigwedge_{y\in X}\bigwedge_{z\in X}(R_\tau^{-1}(x,y) \odot R_\tau^{-1}(y,z) \to A(z)) \\ & = \bigwedge_{z\in X}(\bigvee_{y\in X} R_\tau^{-1}(x,y) \odot R_\tau^{-1}(y,z) \to A(z)) \\ & = \bigwedge_{z\in X}(R_\tau^{-1}(x,z) \to A(z)) \\ & = \mathcal{J}_{R_\tau^{-1}}(A)(x). \end{array}$$

Thus, $\mathcal{J}_{R_\tau^{-1}}(\mathcal{J}_{R_\tau^{-1}}(A)) = \mathcal{J}_{R_\tau^{-1}}(A) \leq A$ and $\mathcal{J}_{R_\tau^{-1}}(A) \in \tau_{\mathcal{J}_{R_\tau^{-1}}} = \tau$, $\mathcal{J}_{R_\tau^{-1}}(A) \leq j_\tau(A)$. Hence $j_\tau = \mathcal{J}_{R_\tau^{-1}}$.

(4) It is proved in a similar way as (3). □

Theorem 15. (1) *If R is a preorder on X, then $R = R_{\tau_{\mathcal{H}_R}}$.*

(2) *If R is a preorder on X, then $R = R_{\tau_{\mathcal{J}_{R^{-1}}}}$.*

Proof. (1) Since R is transitive, $R(x,y) \odot B(z) \odot R(z,x) \leq B(z) \odot R(z,y)$ iff $R(x,y) \leq B(z) \odot R(z,x) \to B(z) \odot R(z,y)$. Thus

$$\begin{array}{ll} R_{\tau_{\mathcal{H}_R}}(x,y) & = \bigwedge_{B\in\tau_{\mathcal{H}_R}}(B(x) \to B(y)) \\ & = \bigwedge_{B\in\tau_{\mathcal{H}_R}}(\mathcal{H}_R(B)(x) \to \mathcal{H}_R(B)(y)) \\ & \geq \bigwedge_{B\in\tau_{\mathcal{H}_R}}\bigvee_{z\in X}(B(z) \odot R(z,x) \to B(z) \odot R(z,y)) \\ & \geq R(x,y). \\ R_{\tau_{\mathcal{H}_R}}(x,y) & = \bigwedge_{B\in\tau_{\mathcal{H}_R}}(B(x) \to B(y)) \\ & \leq \bigwedge_{z\in X}(\mathcal{H}_R(\top_z)(x) \to \mathcal{H}_R(\top_z)(y)) \\ & = \bigwedge_{z\in X}(R(z,x) \to R(z,y)) \leq R(x,x) \to R(x,y) \\ & = R(x,y). \end{array}$$

(2) Since R is transitive, $R(x,y)\odot(R(x,z)\to A(z))\odot R(y,z)\le A(z)$ iff $R(x,y)\le(R(x,z)\to A(z))\to(R(y,z)\to A(z))$ iff $R(x,y)\le(R^{-1}(z,x)\to A(z))\to(R^{-1}(z,y)\to A(z))$. Thus

$$\begin{aligned}
R_{\tau_{\mathcal{J}_{R^{-1}}}}(x,y) &= \bigwedge_{B\in\tau_{\mathcal{J}_{R^{-1}}}}(B(x)\to B(y))\\
&= \bigwedge_{B\in\tau_{\mathcal{J}_{R^{-1}}}}(\mathcal{J}_{R^{-1}}(B)(x)\to\mathcal{J}_{R^{-1}}(B)(y))\\
&\ge \bigwedge_{B\in\tau_{\mathcal{J}_{R^{-1}}}}\bigwedge_{z\in X}((R^{-1}(z,x)\to B(z))\to(R^{-1}(z,y)\to B(z)))\\
&\ge R(x,y).\\
R_{\tau_{\mathcal{J}_{R^{-1}}}}(x,y) &= \bigwedge_{B\in\tau_{\mathcal{J}_{R^{-1}}}}(B(x)\to B(y))\\
&\le \bigwedge_{z\in X}(\mathcal{J}_{R^{-1}}(\top_z^*)(x)\to\mathcal{J}_{R^{-1}}(\top_z^*)(y))\\
&= \bigwedge_{z\in X}(R^*(x,z)\to R^*(y,z))=\bigwedge_{z\in X}(R(y,z)\to R(x,z))\\
&\le R(y,y)\to R(x,y)=R(x,y).
\end{aligned}$$

□

Example 16. Let $(L=[0,1],\odot,\to,^*)$ be a complete residuated lattice with the law of double negation defined by

$$x\odot y=(x+y-1)\vee 0,\;\; x\to y=(1-x+y)\wedge 1,\;\; x^*=1-x.$$

Let $X=\{a,b,c\}$ and $A\in L^X$ as follows:

$$A(a)=0.1, A(b)=0.9, A(c)=0.8.$$

Define $R\in L^{X\times X}$ as follows

$$R=\begin{pmatrix} 1 & 0.7 & 0.3\\ 0.8 & 1 & 1\\ 0.9 & 0.5 & 1\end{pmatrix}.$$

Since $0.9=R(b,c)\odot R(c,a)\not\le R(b,a)=0.8$, R is not transitive. Thus, we have

$$\mathcal{H}_R(A)=(0.7,0.9,0.9)\ne\mathcal{H}_R(\mathcal{H}_R(A))=(0.8,0.9,0.9).$$

Moreover, by Theorem 14,

$$\mathcal{H}_R\le h_{\tau_{\mathcal{H}_R}},\; h_{\tau_{\mathcal{H}_R}}(A)\not\le\mathcal{H}_R(A).$$

We obtain a transitive relation R^2 as

$$R^2=\begin{pmatrix} 1 & 0.7 & 0.7\\ 0.9 & 1 & 1\\ 0.9 & 0.6 & 1\end{pmatrix}.$$

Then we have

$$\mathcal{H}_{R^2}(A) = \mathcal{H}_{R^2}(\mathcal{H}_{R^2}(A)) = (0.8, 0.9, 0.9).$$

Moreover, since R^2 is a preorder, by Theorem 14, we have

$$\mathcal{H}_{R^2} = h_{\tau_{\mathcal{H}_{R^2}}}, \quad R^2 = R_{\tau_{\mathcal{H}_{R^2}}}.$$

Example 17. Let $(L = [0,1], \odot)$ be a complete residuated latticestsc as Example 16. Let $X = \{a, b, c\}$ and $A \in L^X$ as follows:

$$A(a) = 0.5, A(b) = 0.8, A(c) = 0.3.$$

(1) Define $R(x, y) = A(x) \to A(y)$. Then

$$R = \begin{pmatrix} 1 & 1 & 0.8 \\ 0.7 & 1 & 0.5 \\ 1 & 1 & 1 \end{pmatrix}.$$

Hence $\mathcal{H}_R(B)(y) = \bigvee_{x \in X}(B(x) \odot R(x, y))$ is an L-upper approximation operator. Moreover, $\tau_{\mathcal{H}_R} = \{\alpha \odot A, \alpha \to A\}$ is an Alexandorv L-topology as $A \in \tau_{\mathcal{H}_R}$ from as follows:

$$\begin{array}{l} \mathcal{H}_R(A)(y) = A(x) \odot (A(x) \to A(y)) \leq A(y) \\ \text{iff } \mathcal{H}_R(A) = A. \end{array}$$

Moreover, $R_{\tau_{\mathcal{H}_R}}(x, y) = \bigwedge_{B \in \tau_{\mathcal{K}_R}} B(x) \to B(y) = A(x) \to A(y) = R(x, y)$ from

$$\begin{array}{rl} A(x) \to A(y) & \leq (\alpha \odot A)(x) \to (\alpha \odot A)(y) \\ A(x) \to A(y) & \leq (\alpha \to A)(x) \to (\alpha \to A)(y) \\ A_i(x) \to A_i(y) & \leq (\bigwedge A_i(x) \to \bigwedge A_i(y)) \\ A_i(x) \to A_i(y) & \leq (\bigvee A_i(x) \to \bigvee A_i(y)). \end{array}$$

From the above theorem, $h_{\tau_{\mathcal{H}_R}} = \mathcal{H}_R$.

(2) Define $R^{-1}(x, y) = A(y) \to A(x) = A^*(x) \to A^*(y)$. Then

$$R^{-1} = \begin{pmatrix} 1 & 0.7 & 1 \\ 1 & 1 & 1 \\ 0.8 & 0.5 & 1 \end{pmatrix}.$$

Hence $\mathcal{J}_{R^{-1}}(B)(y) = \bigwedge_{x \in X}(R^{-1}(x, y) \to B(x))$ is an L-lower approximation operator. We obtain $\tau_{\mathcal{J}_{R^{-1}}} = \{\alpha \odot A, \alpha \to A\}$ is an Alexandorv L-topology from:

$$\begin{array}{l} A(y) \leq R^{-1}(x, y) \to A(x) \text{ iff } A(y) \odot R^{-1}(x, y) \leq A(x) \\ \text{iff } A(y) \odot (A(y) \to A(x)) \leq A(x). \end{array}$$

Moreover, $R_{\tau_{\mathcal{J}_{R^{-1}}}}(x,y) = \bigwedge_{B\in\tau_{\mathcal{J}_{R^{-1}}}} B(x) \to B(y) = A(x) \to A(y) = R(x,y)$. Similarly, $\tau_{\mathcal{J}_R} = \{\alpha \odot A^*, \alpha \to A^*\} = \{B^* \mid B \in \tau_{\mathcal{J}_{R^{-1}}}\}$ is an Alexandorv L-topology.

(3) From Remark 9, we obtain

$$\mathcal{J}_1(A)(x) = \bigvee\{B \mid \mathcal{H}_R(B) \le A\} = \bigwedge_{y\in X}(R(x,y) \to A(y))$$
$$= \bigwedge_{y\in X}(R^{-1}(y,x) \to A(y)) = \mathcal{J}_{R^{-1}}(A)(x)$$

such that

$$\tau_{\mathcal{J}_1} = \tau_{\mathcal{H}_R} = \{\alpha \odot A, \alpha \to A\}.$$

Moreover, $\mathcal{J}_2(A)(y) = \mathcal{H}_R(A^*)^*(y) = \bigwedge_{x\in X}(R(x,y) \to A(x)) = \mathcal{J}_R(A)(y)$ such that

$$\tau_{\mathcal{J}_2} = \{\alpha \odot A^*, \alpha \to A^*\}.$$

(4) From Remark 11, we obtain

$$\mathcal{H}_1(A)(x) = \bigwedge\{B \mid A \le \mathcal{J}_R(B)\} = \bigvee_{y\in X}(R(x,y) \odot A(y))$$

such that

$$\tau_{\mathcal{H}_1} = \tau_{\mathcal{H}_{R^{-1}}} = \{\alpha \odot A^*, \alpha \to A^*\}.$$

Moreover, $\mathcal{H}_2(A)(y) = \mathcal{J}_R(A^*)^*(y) = \bigvee_{x\in X}(R(y,x) \odot A(y)) = \mathcal{H}_R(A)(y)$ such that

$$\tau_{\mathcal{H}_2} = \{\alpha \odot A, \alpha \to A\}.$$

(5) Define $S(x,y) = (A(x) \to A(y)) \wedge (A(y) \to A(x))$. Then

$$S = \begin{pmatrix} 1 & 0.7 & 0.8 \\ 0.7 & 1 & 0.5 \\ 0.8 & 0.5 & 1 \end{pmatrix}.$$

Hence $\mathcal{H}_S(B)(y) = \bigvee_{x\in X}(B(x) \odot S(x,y))$ is an L-upper approximation operator. Moreover, since $B \le \mathcal{H}_S(B) \le \mathcal{H}_R(B)$ and $B \le \mathcal{H}_S(B) \le \mathcal{H}_{R^{-1}}(B)$, $\tau_{\mathcal{H}_R} \subset \tau_{\mathcal{H}_S}$ and $\tau_{\mathcal{H}_{R^{-1}}} \subset \tau_{\mathcal{H}_S}$ having

$$(0.4, 0.5, 0.6)^t \in \tau_{\mathcal{H}_S},\ (0.4, 0.5, 0.6)^t \notin \tau_{\mathcal{H}_R}$$

because $\mathcal{H}_R((0.4, 0.5, 0.6)^t) = (0.5, 0.5, 0.6)^t$.

From Remark 9, we obtain

$$\begin{aligned}\mathcal{J}_1(A)(x) &= \bigvee\{B \mid \mathcal{H}_S(B) \leq A\} = \bigwedge_{y\in X}(S(x,y) \to A(y)) \\ &= \bigwedge_{y\in X}(S(y,x) \to A(y)) = \mathcal{J}_S(A)(x).\end{aligned}$$

Since $\mathcal{H}_S(\top_x)(y) = S(x,y) = S(y,x) = \mathcal{H}_S(1_y)(x)$, we have

$$\mathcal{J}_1(B) = \mathcal{J}_S(B) = \mathcal{H}_S(B^*)^*.$$

From Remark 11, we obtain

$$\begin{aligned}\mathcal{H}_1(A)(x) &= \bigwedge\{B \mid A \leq \mathcal{J}_S(B)\} \\ &= \bigvee_{y\in X}(S(x,y) \odot A(y)) = \bigvee_{y\in X}(S(y,x) \odot A(y)) = \mathcal{H}_S(A).\end{aligned}$$

Moreover, since $\mathcal{J}_S(\top_x^*)(y) = S^*(x,y) = S^*(y,x) = \mathcal{J}_S(\top_y^*)(x)$, we have

$$\mathcal{H}_1(B) = \mathcal{H}_S(B) = \mathcal{J}_S(B^*)^*.$$

References

[1] R. Bělohlávek,*Fuzzy Relational Systems*, Kluwer Academic Publishers, New York , (2002).

[2] P. Hájek, *Metamathematices of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht (1998).

[3] Y.C. Kim, L-approximatins and join preserving maps, *J. Math. Comput. Sci.,* **3** (5)(2013), 1193-1210.

[4] H. Lai, D. Zhang, Fuzzy preorder and fuzzy topology, *Fuzzy Sets and Systems,* **157** (2006), 1865-1885.

[5] H. Lai, D. Zhang, Concept lattices of fuzzy contexts: Formal concept analysis vs. rough set theory, *Int. J. Approx. Reasoning,* **50** (2009), 695-707.

[6] Z. Pawlak, Rough sets, *Int. J. Comput. Inf. Sci.*, **11** (1982), 341-356.

[7] Z. Pawlak, Rough probability, *Bull. Pol. Acad. Sci. Math.*, **32**(1984), 607-615.

[8] A. M. Radzikowska, E.E. Kerre, A comparative study of fuzy rough sets, *Fuzzy Sets and Systems*, **126**(2002), 137-155.

[9] Y.H. She, G.J. Wang, An axiomatic approach of fuzzy rough sets based on residuated lattices, *Computers and Mathematics with Applications*, **58** (2009), 189-201.

[10] Zhen Ming Ma, Bao Qing Hu, Topological and lattice structures of L-fuzzy rough set determined by lower and upper sets, *Information Sciences*, **218**(2013), 194-204.