NORMALITY AND SHARING VALUES

Gopal Datt
Department of Mathematics
University of Delhi
Delhi 110007, INDIA

Abstract: In this Paper we prove some normality criteria for a family of holomorphic functions, where a complex value is shared by every function from the family and its $k^{th}$ derivative. We use some results from Value Distribution Theory for proving the results.

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1. Introduction

We denote the complex plane by $\mathbb{C}$, and the unit disk by $\Delta$. Let $f$ be a meromorphic function in $\mathbb{C}$. We say that $f$ is a normal function if there exits a positive $M$ such that $f^#(z) \leq M$ for all $z \in \mathbb{C}$, where $f^# = \frac{|f'(z)|}{1+|f(z)|^2}$ denotes the spherical derivative of $f$. { [8], P. 171}.

A family $\mathcal{F}$ of analytic functions on a domain $\Omega \subseteq \mathbb{C}$ is normal in $\Omega$ if every sequence of functions $\{f_n\} \subseteq \mathcal{F}$ contains either a subsequence which converges to a limit function $f \neq \infty$ uniformly on each compact subset of $\Omega$, or a subsequence which converges uniformly to $\infty$ on each compact subset. { [10], P.33}. 

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In this paper, we use the following standard notations of value distribution theory,

\[ T(r, f); m(r, f); N(r, f); \overline{N}(r, f), \ldots \]

We denote \( S(r, f) \) any function satisfying

\[ S(r, f) = o\{T(r, f)\}, \text{ as } r \to +\infty, \]

possibly outside of a set with finite measure.

According to Bloch’s principle every condition which reduces a meromorphic function in the plane to a constant, makes the family of meromorphic functions in a domain \( D \) normal. Rubel gave four counter examples to Bloch principle.

Let \( f \) and \( g \) be meromorphic functions in a domain \( D \) and \( a \in \mathbb{C} \). Let zeros of \( f - a \) are zeros of \( g - a \) (ignoring multiplicity), we write \( f = a \Rightarrow g = a \). Hence \( f = a \iff g = a \) means that \( f - a \) and \( g - a \) have the same zeros (ignoring multiplicity). If \( f - a \iff g - a \), then we say that \( f \) and \( g \) share the value \( z = a \) IM. \{ [14], p. 108 \}.

Let us recall the following known results that establish connection between shared values and normality. Mues and Steinmetz proved the following result :

**Theorem 1.1.** [7] Let \( f \) be a non-constant meromorphic function in the plane. If \( f \) and \( f' \) share three distinct complex numbers \( a_1, a_2, a_3 \) then \( f \equiv f' \).

Wilhelm Schwick seems to have been the first to draw a connection between normality and shared values. He proved the following theorem :

**Theorem 1.2.** [11] Let \( F \) be a family of meromorphic functions on a domain \( G \) and \( a_1, a_2, a_3 \) be distinct complex numbers. If \( f \) and \( f' \) share \( a_1, a_2, a_3 \) for every \( f \in F \), then \( F \) is normal in \( G \).

In 2000, Chen and Hua proved the following result :

**Theorem 1.3.** [1] Let \( F \) be a family of holomorphic functions in a domain \( D \). Suppose that there exists a non zero \( a \in \mathbb{C} \) such that for each function \( f \in F \); \( f, f' \) and \( f'' \) share the value \( z = a \) IM in \( D \). Then the family \( F \) is normal in \( D \).

Fang and Xu improved theorem 1.1 and theorem 1.2 by proving the following theorems :

**Theorem 1.4.** [5] Let \( F \) be a family of holomorphic functions on a domain \( D \) and let \( a, b \) be two distinct finite complex numbers such that \( b \neq 0 \). If for any \( f \in F \), \( f \) and \( f' \) share \( z = a \) IM and \( f(z) = b \) whenever \( f'(z) = b \) then \( F \) is normal in \( D \).
Theorem 1.5. [5] Let $F$ be a family of holomorphic functions in a domain $D$, and let $a$ be a non zero finite complex number. If for any $f \in F$ $f$ and $f'$ share $z = a$ IM and $f^{(k)}(z) = a, f^{(k+1)}(z) = a$ whenever $f(z) = a$. Then $F$ is normal in $D$.

It is natural to consider: What can we say if $f'$ in Theorem 1.4 and Theorem 1.5 is replaced by the $k$-th derivative $f^{(k)}$? In this paper, we prove the following results.

Theorem 1.6. Let $F$ be a family of holomorphic functions on a domain $D$ such that all zeros of $f \in F$ are of multiplicity at least $k$, where $k$ is a positive integer. Let $a$, $b$ be two distinct finite complex numbers such that $b \neq 0$. Suppose for any $f \in F$ satisfies the following conditions

1. $f$ and $f^{(k)}$ share $z = a$ IM
2. $f(z) = b$ whenever $f^{(k)}(z) = b$

Then $F$ is normal in $D$.

Theorem 1.7. Let $F$ be a family of holomorphic functions in a domain $D$ such that all zeros of $f \in F$ are of multiplicity at least $k$, where $k$ is a positive integer and let $a$ be a non zero finite complex number. If for any $f \in F$ $f$ and $f^{(k)}$ share $z = a$ IM and $f^{(k+1)}(z) = a$ whenever $f(z) = a$. Then $F$ is normal in $D$.

Remark 1.8. The hypothesis $a \neq b$ can not be dropped in Theorem 1.6.

Example 1.9. Let $D = \Delta = \{z : |z| < 1\}$, let $k = 1$ and $a = b = 1$, and

$$F = \{e^{nz} - \frac{1}{n^k} + 1 : n = 1, 2, 3, \ldots \}.$$

Then for any $f \in F$, and

$$f = e^{nz} - \frac{1}{n^k} + 1, \quad f^{(k)} = n^k e^{nz}.$$ 

Clearly, all other conditions of Theorem 1.6 are satisfied. However, $F$ is not normal in $\Delta$. This example confirms that $b \neq 0$ is necessary in Theorem 1.6 as $f^{(k)}(z) \neq 0$.

Example 1.10. Let $D = \Delta = \{z : |z| < 1\}$, let $k = 1$, $a = ((-1)^{k+1} + 1)b$ and

$$F = \{(b \frac{(z - \frac{1}{n})^k}{k!} + (-1)^{k+1} \frac{1}{k! n (z - \frac{1}{n})} + a : n = 1, 2, 3, \ldots \}.$$
Then, for every \( f_n(z) \in \mathcal{F} \),
\[
f_n(z) = b \frac{(z - \frac{1}{n})^k}{k!} + \frac{(-1)^{k+1}}{k!n(z - \frac{1}{n})} + a, \quad f_n^{(k)}(z) = b - \frac{1}{n(z - \frac{1}{n})^{k+1}}.
\]

Clearly, \( f_n \) and \( f_n^{(k)} \) share \( a \) and \( f_n^{(k)}(z) \neq b \), so that \( f_n(z) = b \) whenever \( f_n^{(k)}(z) = b \). But \( \mathcal{F} \) is not normal in \( D \). This example shows that Theorem 1.6 is not valid for a family of meromorphic functions.

We will use the tools of Fang and Xu [5] which they used in their paper.

2. Some Lemmas

In order to prove our results we need the following Lemmas.

**Lemma 2.1.** [16] [9] (Zalcman’s lemma) Let \( \mathcal{F} \) be a family of holomorphic functions in the unit disk \( \Delta \), with the property that for every function \( f \in \mathcal{F} \), the zeros of \( f \) are of multiplicity at least \( k \). If \( \mathcal{F} \) is not normal at \( z_0 \) in \( \Delta \), then for \( 0 \leq \alpha < k \), there exist

1. a sequence of complex numbers \( z_n \to z_0 \), \( |z_n| < r < 1 \),
2. a sequence of functions \( f_n \in \mathcal{F} \),
3. a sequence of positive numbers \( \rho_n \to 0 \),

such that \( g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \) converges to a non-constant entire function \( g \) on \( \mathbb{C} \). Moreover \( g \) is of order at most one. If \( \mathcal{F} \) possesses the additional property that there exists \( M > 0 \) such that \( |f^{(k)}(z)| \leq M \) whenever \( f(z) = 0 \) for any \( f \in \mathcal{F} \), then we can take \( \alpha = k \).

**Lemma 2.2.** [6] [15] Let \( f \) be a non-constant meromorphic function. Then for \( k \geq 1 \), \( b \neq 0, \infty \),
\[
T(r, f) \leq N(r, f) + N(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k+1)}}) - N(r, \frac{1}{f^{(k)}}) + S(r, f).
\]

3. Proof of Theorem 1.6

Since normality is a local property, we assume that \( D = \Delta = \{ z : |z| < 1 \} \). Suppose \( \mathcal{F} \) is not normal in \( D \). Without loss of generality we assume that \( \mathcal{F} \) is not normal at the point \( z_0 \) in \( \Delta \). Then by Lemma 2.1, there exist
1. a sequence of complex numbers \( z_n \to z_0, |z_n| < r < 1 \),

2. a sequence of functions \( f_n \in \mathcal{F} \) and

3. a sequence of positive numbers \( \rho_n \to 0 \),

such that \( g_n(\zeta) = \rho_n^{-k}[f_n(z_n + \rho_n \zeta) - a] \) converges locally uniformly to a non-constant entire function \( g \). Moreover \( g \) is of order at most one.

Now we claim that \( g = 0 \) if and only if \( g^{(k)} = a \) and \( g^{(k)} \neq b \). Suppose \( g(\zeta_0) = 0 \). Then by Hurwitz’s theorem there exist \( \zeta_n; \zeta_n \to \zeta_0 \) such that

\[
g_n(\zeta_n) = \rho_n^{-k}[f_n(z_n + \rho_n \zeta_n) - a] = 0.
\]

Thus \( f_n(z_n + \rho_n \zeta_n) = a \). Since \( f_n \) and \( f_n^{(k)} \) share \( z = a \) IM, we have

\[
g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n \zeta_n) = a.
\]

Hence

\[
g^{(k)}(\zeta) = \lim_{n \to \infty} g_n^{(k)}(\zeta_n) = a.
\]

Thus we have proved that \( g^{(k)} = a \) whenever \( g = 0 \).

On the other hand, if \( g^{(k)}(\zeta_0) = a \), then there exist \( \zeta_n; \zeta_n \to \zeta_0 \) such that

\[
g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n \zeta_n) = a, \quad n = 1, 2, \ldots \]

Hence \( f_n(z_n + \rho_n \zeta_n) = a \) and \( g_n(\zeta_n) = 0 \) for \( n = 1, 2, \ldots \). Thus

\[
g(\zeta_0) = \lim_{n \to \infty} g_n(\zeta_n) = 0.
\]

This shows that \( g = 0 \) whenever \( g^{(k)} = a \). Hence \( g = 0 \) if and only if \( g^k = a \).

Next, we prove \( g^{(k)}(\zeta) \neq b \). Suppose there exist \( \zeta_0 \) satisfying \( g^{(k)}(\zeta_0) = b \).

Then, by Hurwitz’s theorem, there exist a sequence \( \zeta_n \to \zeta_0 \) and \( g_n^{(k)}(\zeta_n) = b, \quad n = 1, 2, \ldots \). Since \( f_n(z) = b \) whenever \( f_n^{(k)}(z) = b \Rightarrow f_n(z_n + \rho_n \zeta_n) = b \) and,

\[
g_n(\zeta_n) = \rho_n^{-k}[f_n(z_n + \rho_n \zeta_n) - a] = \rho_n^{(k)}[b - a] \to \infty,
\]

this contradicts

\[
\lim_{n \to \infty} g_n(\zeta_n) = g(\zeta_0) \neq \infty
\]

. So \( g^{(k)}(\zeta) \neq b \). Hence we get,

\[
g^{(k)}(\zeta) = b + e^{A\zeta + B},
\]
where $A$ and $B$ are two constants. We claim that $A = 0$. Suppose that $A \neq 0$, then

$$g(\zeta) = \frac{b\zeta^k}{k!} + e^{A\zeta + B} + \frac{c_1\zeta^{k-1}}{(k-1)!} + \ldots + c_{k-1}\zeta + c_k,$$

(2)

where $c_1, c_2, \ldots, c_k$ are constants. Let $g^{(k)} = a$. Then by (1), (2) and $g(\zeta) = 0$ whenever $g^{(k)}(\zeta) = a$, we have

$$\frac{b\zeta^k}{k!} + \frac{c_1\zeta^{k-1}}{(k-1)!} + \ldots + c_k + \frac{b - a}{A^k} = 0.$$

This is a polynomial of degree $k$ in $\zeta$ this polynomial has $k$ solutions. Which contradicts the fact that $g^{(k)}$ has infinitely many solutions. Thus we have,

$$g^{(k)}(\zeta) = b + e^B$$

and

$$g(\zeta) = (b + e^B)\zeta^k$$

Since $g$ is non-constant, this contradicts $g(\zeta) = 0 \iff g^{(k)}(\zeta) = a$. Thus $F$ is normal in $D$. This completes the proof of theorem.

4. Proof of Theorem 1.7

Suppose $F$ is not normal in $\Delta$; without loss of generality we assume that $F$ is not normal at the point $z = 0$. Then by Lemma 2.1, there exist

1. a sequence of complex numbers $z_n \to 0$, $|z_n| < r < 1$,

2. a sequence of functions $f_n \in F$ and

3. a sequence of positive numbers $\rho_n \to 0$,

such that $g_n(\zeta) = \rho_n^{-k}[f_n(z_n + \rho_n\zeta) - a] \text{ converges locally uniformly to a non-constant entire function } g$. Moreover $g$ is of order at most one. Now we claim that $g = 0 \iff g^{(k)} = a \text{ and } g^{(k+1)} = 0 \text{ whenever } g = 0$. Let $g(\zeta_0) = 0$. Then by Hurwitz’s theorem there exist $\zeta_n; \zeta_n \to \zeta_0$ such that

$$g_n(\zeta_n) = \rho_n^{-k}[f_n(z_n + \rho_n\zeta_n) - a] = 0.$$
Thus \( f_n(z_n + \rho_n \zeta_n) = a \), since \( f_n \) and \( f_n^{(k)} \) share \( z = a \) IM, we have
\[
g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n \zeta_n) = a
\]
and
\[
g_n^{(k+1)}(\zeta_n) = \rho_n f_n^{(k+1)}(z_n + \rho_n \zeta_n).
\]
Which implies that
\[
g^{(k)}(\zeta_0) = \lim_{n \to \infty} g_n^{(k)}(\zeta_n) = a
\]
and
\[
g^{(k+1)}(\zeta_0) = \lim_{n \to \infty} g_n^{(k+1)}(\zeta_n) = 0.
\]
Thus we get, \( g^{(k)} = a \) whenever \( g = 0 \) and \( g^{(k+1)} = 0 \) whenever \( g = 0 \).

On the other hand, if \( g^{(k)}(\zeta_0) = a \) then there exist \( \zeta_n \to \zeta_0 \) such that \( g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n \zeta_n) = a, \ n = 1, 2, \ldots \). Hence \( f_n(z_n + \rho_n \zeta_n) = a \) and \( g_n(\zeta_n) = 0 \) for \( n = 1, 2, \ldots \). Thus
\[
g(\zeta_0) = \lim_{n \to \infty} g_n(\zeta_n) = 0.
\]
This shows that \( g = 0 \) whenever \( g^{(k)} = a \). Hence \( g = 0 \) if and only if \( g^k = a \) and \( g^{(k+1)} = 0 \) whenever \( g = 0 \).

Now using Lemma 2.2 and Nevanlinna’s first fundamental theorem, we have
\[
T(r, g) \leq \overline{N}(r, g) + N(r, \frac{1}{g}) + N(r, \frac{1}{g^{(k)} - a}) - N(r, \frac{1}{g^{(k+1)}}) + S(r, g)
\]
\[
= N(r, \frac{1}{g}) + N(r, \frac{1}{g^{(k)} - a}) - N(r, \frac{1}{g^{(k+1)}}) + S(r, g)
\]
\[
\leq N(r, \frac{1}{g^{(k)} - a}) - \overline{N}(r, \frac{1}{g^{(k+1)}}) + S(r, g)
\]
\[
\leq T(r, \frac{1}{g^{(k)} - a}) - \overline{N}(r, \frac{1}{g^{(k+1)}}) + S(r, g)
\]
\[
\leq T(r, g^{(k)} - a) - \overline{N}(r, \frac{1}{g^{(k+1)}}) + S(r, g)
\]
\[
\leq T(r, g) - \overline{N}(r, \frac{1}{g^{(k+1)}}) + S(r, g).
\]
Thus we get
\[
\overline{N}(r, \frac{1}{g^{(k+1)}}) = S(r, g),
\]
by (3), (4) and the claim \( g = 0 \) if and only if \( g^{(k)} = a, g^{(k+1)} = 0 \) whenever \( g = 0 \) we get a contradiction: \( T(r, g) = S(r, g) \). It proves the theorem.
References


