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NORMALITY AND SHARING VALUES

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Abstract: In this Paper we prove some normality criteria for a family of holomorphic functions, where a complex value is shared by every function from the family and its k^{th} derivative. We use some results from Value Distribution Theory for proving the results.

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1. Introduction

We denote the complex plane by $\mathbb{C}$, and the unit disk by Δ. Let f be a meromorphic function in $\mathbb{C}$. We say that f is a normal function if there exits a positive M such that$f^{\#}(z) \leq M$ for all $z \in \mathbb{C}$, where $f^{\#} = \frac{|f'(z)|}{1+|f(z)|^2}$ denotes the spherical derivative of f. { [8], P. 171}.

A family $\mathcal{F}$ of analytic functions on a domain $\Omega \subseteq \mathbb{C}$ is normal in Ω if every sequence of functions $\{f_n\} \subseteq \mathcal{F}$ contains either a subsequence which converges to a limit function $f \not\equiv \infty$ uniformly on each compact subset of Ω, or a subsequence which converges uniformly to ∞ on each compact subset. { [10], P.33}.

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In this paper, we use the following standard notations of value distribution theory,

$$T(r, f); m(r, f); N(r, f); \overline{N}(r, f), \ldots.$$

We denote $S(r, f)$ any function satisfying

$$S(r, f) = o\{T(r, f)\}, \text{ as } r \to +\infty,$$

possibly outside of a set with finite measure.

According to Bloch's principle every condition which reduces a meromorphic function in the plane to a constant, makes the family of meromorphic functions in a domain $\mathcal{D}$ normal. Rubel gave four counter examples to Bloch principle.

Let f and g be meromorphic functions in a domain D and $a \in \mathbb{C}$. Let zeros of $f - a$ are zeros of $g - a$ (ignoring multiplicity), we write $f = a \Rightarrow g = a$. Hence $f = a \iff g = a$ means that $f - a$ and $g - a$ have the same zeros (ignoring multiplicity). If $f - a \iff g - a$, then we say that f and g share the value $z = a$ IM. { [14], p. 108}.

Let us recall the following known results that establish connection between shared values and normality. Mues and Steinmetz proved the following result :

Theorem 1.1. *[7] Let f be a non-constant meromorphic function in the plane. If f and f' share three distinct complex numbers a_1, a_2, a_3 then $f \equiv f'$.*

Wilhelm Schwick seems to have been the first to draw a connection between normality and shared values. He proved the following theorem :

Theorem 1.2. *[11] Let $\mathcal{F}$ be a family of meromorphic functions on a domain G and a_1, a_2, a_3 be distinct complex numbers . If f and f' share a_1, a_2, a_3 for every $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in G.*

In 2000, Chen and Hua proved the following result :

Theorem 1.3. *[1] Let $\mathcal{F}$ be a family of holomorphic functions in a domain D. Suppose that there exists a non zero $a \in \mathbb{C}$ such that for each function $f \in F$; f, f' and f'' share the value $z = a$ IM in D. Then the family $\mathcal{F}$ is normal in D.*

Fang and Xu improved theorem 1.1 and theorem 1.2 by proving the following theorems :

Theorem 1.4. *[5] Let $\mathcal{F}$ be a family of holomorphic functions on a domain D and let a, b be two distinct finite complex numbers such that $b \neq 0$. If for any $f \in \mathcal{F}$, f and f' share $z = a$ IM and $f(z) = b$ whenever $f'(z) = b$ then $\mathcal{F}$ is normal in D.*

Theorem 1.5. *[5] Let $\mathcal{F}$ be a family of holomorphic functions in a domain D, and let a be a non zero finite complex number. If for any $f \in \mathcal{F}$ f and f' share $z = a$ IM and $f^{(k)}(z) = a, f^{(k+1)}(z) = a$ whenever $f(z) = a$. Then $\mathcal{F}$ is normal in D.*

It is natural to consider : *What can we say if f' in Theorem 1.4 and Theorem 1.5 is replaced by the k-th derivative $f^{(k)}$?* In this paper, we prove the following results.

Theorem 1.6. *Let $\mathcal{F}$ be a family of holomorphic functions on a domain D such that all zeros of $f \in \mathcal{F}$ are of multiplicity at least k, where k is a positive integer. Let a, b be two distinct finite complex numbers such that $b \neq 0$. Suppose for any $f \in \mathcal{F}$ satisfies the following conditions*

1. *f and $f^{(k)}$ share $z = a$ IM*
2. *$f(z) = b$ whenever $f^{(k)}(z) = b$*

Then $\mathcal{F}$ is normal in D.

Theorem 1.7. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D such that all zeros of $f \in \mathcal{F}$ are of multiplicity at least k, where k is a positive integer and let a be a non zero finite complex number. If for any $f \in \mathcal{F}$ f and $f^{(k)}$ share $z = a$ IM and $f^{(k+1)}(z) = a$ whenever $f(z) = a$. Then $\mathcal{F}$ is normal in D.*

Remark 1.8. The hypothesis $a \neq b$ can not be dropped in Theorem 1.6.

Example 1.9. Let $D = \Delta = \{z : |z| < 1\}$, let $k = 1$ and $a = b = 1$, and

$$\mathcal{F} = \{e^{nz} - \tfrac{1}{n^k} + 1 : n = 1, 2, 3, \ldots\}.$$

Then for any $f \in \mathcal{F}$, and

$$f = e^{nz} - \tfrac{1}{n^k} + 1, \quad f^{(k)} = n^k e^{nz}.$$

Clearly, all other conditions of Theorem 1.6 are satisfied. However, $\mathcal{F}$ is not normal in Δ. This example confirms that $b \neq 0$ is necessary in Theorem 1.6 as $f^{(k)}(z) \neq 0$.

Example 1.10. Let $D = \Delta = \{z : |z| < 1\}$, let $k = 1$, $a = ((-1)^{k+1} + 1)b$ and

$$\mathcal{F} = \{b\frac{(z - \frac{1}{n})^k}{k!} + \frac{(-1)^{k+1}}{k!n(z - \frac{1}{n})} + a : n = 1, 2, 3, \ldots\}.$$

Then, for every $f_n(z) \in \mathcal{F}$,

$$f_n(z) = b\frac{(z-\frac{1}{n})^k}{k!} + \frac{(-1)^{k+1}}{k!n(z-\frac{1}{n})} + a, \qquad f_n^{(k)}(z) = b - \frac{1}{n(z-\frac{1}{n})^{k+1}}.$$

Clearly, f_n and $f_n^{(k)}$ share a and $f_n^{(k)}(z) \neq b$, so that $f_n(z) = b$ whenever $f_n^{(k)}(z) = b$. But $\mathcal{F}$ is not normal in D. This example shows that Theorem 1.6 is not valid for a family of meromorphic functions.

We will use the tools of Fang and Xu [5] which they used in their paper.

2. Some Lemmas

In order to prove our results we need the following Lemmas.

Lemma 2.1. *[16] [9](Zalcman's lemma) Let $\mathcal{F}$ be a family of holomorphic functions in the unit disk Δ, with the property that for every function $f \in \mathcal{F}$, the zeros of f are of multiplicity at least k. If $\mathcal{F}$ is not normal at z_0 in Δ, then for $0 \le \alpha < k$, there exist*

1. *a sequence of complex numbers $z_n \to z_0$, $|z_n| < r < 1$,*
2. *a sequence of functions $f_n \in \mathcal{F}$,*
3. *a sequence of positive numbers $\rho_n \to 0$,*

such that $g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n\zeta)$ converges to a non-constant entire function g on $\mathbb{C}$. Moreover g is of order at most one . If $\mathcal{F}$ possesses the additional property that there exists $M > 0$ such that $|f^{(k)}(z)| \le M$ whenever $f(z) = 0$ for any $f \in \mathcal{F}$, then we can take $\alpha = k$.

Lemma 2.2. *[6] [15] Let f be a non-constant meromorphic function. Then for $k \ge 1$, $b \neq 0, \infty$,*

$$T(r,f) \le \overline{N}(r,f) + N(r,\tfrac{1}{f}) + N(r,\tfrac{1}{f^{(k)}-b}) - N(r,\tfrac{1}{f^{(k+1)}}) + S(r,f).$$

3. Proof of Theorem 1.6

Since normality is a local property, we assume that $D = \Delta = \{z : |z| < 1\}$. Suppose $\mathcal{F}$ is not normal in D. Without loss of generality we assume that $\mathcal{F}$ is not normal at the point z_0 in Δ. Then by Lemma 2.1, there exist

1. a sequence of complex numbers $z_n \to z_0$, $|z_n| < r < 1$,
2. a sequence of functions $f_n \in \mathcal{F}$ and
3. a sequence of positive numbers $\rho_n \to 0$,

such that $g_n(\zeta) = \rho_n^{-k}[f_n(z_n + \rho_n\zeta) - a]$ converges locally uniformly to a non-constant entire function g. Moreover g is of order at most one.

Now we claim that $g = 0$ if and only if $g^{(k)} = a$ and $g^{(k)} \neq b$. Suppose $g(\zeta_0) = 0$. Then by Hurwitz's theorem there exist ζ_n; $\zeta_n \to \zeta_0$ such that

$$g_n(\zeta_n) = \rho_n^{-k}[f_n(z_n + \rho_n\zeta_n) - a] = 0.$$

Thus $f_n(z_n + \rho_n\zeta_n) = a$. Since f_n and $f_n^{(k)}$ share $z = a$ IM, we have

$$g_n^k(\zeta_n) = f_n^{(k)}(z_n + \rho_n\zeta_n) = a.$$

Hence

$$g^{(k)}(\zeta_0) = \lim_{n\to\infty} g_n^{(k)}(\zeta_n) = a.$$

Thus we have proved that $g^{(k)} = a$ whenever $g = 0$.

On the other hand, if $g^{(k)}(\zeta_0) = a$, then there exist ζ_n; $\zeta_n \to \zeta_0$ such that $g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n\zeta_n) = a$, $n = 1, 2, \ldots$. Hence $f_n(z_n + \rho_n\zeta_n) = a$ and $g_n(\zeta_n) = 0$ for $n = 1, 2, \ldots$. Thus

$$g(\zeta_0) = \lim_{n\to\infty} g_n(\zeta_n) = 0.$$

This shows that $g = 0$ whenever $g^{(k)} = a$. Hence $g = 0$ if and only if $g^k = a$.

Next, we prove $g^{(k)}(\zeta) \neq b$. Suppose there exist ζ_0 satisfying $g^{(k)}(\zeta_0) = b$. Then, by Hurwitz's theorem, there exist a sequence $\zeta_n \to \zeta_0$ and $g_n^{(k)}(\zeta_n) = b$, $n = 1, 2, \ldots$. Since $f_n(z) = b$ whenever $f_n^{(k)}(z) = b \Rightarrow f_n(z_n + \rho_n\zeta_n) = b$ and,

$$g_n(\zeta_n) = \rho_n^{-k}[f_n(z_n + \rho_n\zeta_n) - a] = \rho_n^{(k)}[b - a] \to \infty,$$

this contradicts

$$\lim_{n\to\infty} g_n(\zeta_n) = g(\zeta_0) \neq \infty$$

. So $g^{(k)}(\zeta) \neq b$. Hence we get,

$$g^{(k)}(\zeta) = b + e^{A\zeta + B}, \tag{1}$$

where A and B are two constants. We claim that $A = 0$. Suppose that $A \neq 0$, then

$$g(\zeta) = \frac{b\zeta^k}{k!} + \frac{e^{A\zeta+B}}{A^k} + \frac{c_1\zeta^{k-1}}{(k-1)!} + \ldots + c_{k-1}\zeta + c_k, \tag{2}$$

where $c_1, c_2, \ldots, c_k$ are constants. Let $g^{(k)} = a$. Then by (1), (2) and $g(\zeta) = 0$ whenever$g^{(k)}(\zeta) = a,$ we have

$$\frac{b\zeta^k}{k!} + \frac{c_1\zeta^{k-1}}{(k-1)!} + \ldots + c_k + \frac{b-a}{A^k} = 0.$$

This is a polynomial of degree k in ζ this polynomial has k solutions. Which contradicts the fact that $g^{(k)}$ has infinitely many solutions. Thus we have,

$$g^{(k)}(\zeta) = b + e^B$$

and

$$g(\zeta) = (b + e^B)\frac{\zeta^k}{k!} + \frac{c_1\zeta^{k-1}}{(k-1)!} + \ldots + c_k.$$

Since g is non-constant, this contradicts $g(\zeta) = 0 \Leftrightarrow g^{(k)}(\zeta) = a$. Thus $\mathcal{F}$ is normal in D. This completes the proof of theorem.

4. Proof of Theorem 1.7

Suppose $\mathcal{F}$ is not normal in Δ; without loss of generality we assume that $\mathcal{F}$ is not normal at the point $z = 0$. Then by Lemma 2.1, there exist

1. a sequence of complex numbers $z_n \to 0$, $|z_n| < r < 1$,
2. a sequence of functions $f_n \in \mathcal{F}$ and
3. a sequence of positive numbers $\rho_n \to 0$,

such that $g_n(\zeta) = \rho_n^{-k}[f_n(z_n + \rho_n\zeta) - a]$ converges locally uniformly to a non-constant entire function g. Moreover g is of order at most one.

Now we claim that $g = 0$ iff $g^{(k)} = a$ and $g^{(k+1)} = 0$ whenever $g = 0$. Let $g(\zeta_0) = 0$. Then by Hurwitz's theorem there exist ζ_n; $\zeta_n \to \zeta_0$ such that

$$g_n(\zeta_n) = \rho_n^{-k}[f_n(z_n + \rho_n\zeta_n) - a] = 0.$$

Thus $f_n(z_n+\rho_n\zeta_n)=a$, since f_n and $f_n^{(k)}$ share $z=a$ IM, we have

$$g_n^k(\zeta_n)=f_n^{(k)}(z_n+\rho_n\zeta_n)=a$$

and

$$g_n^{(k+1)}(\zeta_n)=\rho_n f_n^{(k+1)}(z_n+\rho_n\zeta_n).$$

Which implies that

$$g^{(k)}(\zeta_0)=\lim_{n\to\infty} g_n^{(k)}(\zeta_n)=a$$

and

$$g^{(k+1)}(\zeta_0)=\lim_{n\to\infty} g_n^{(k+1)}(\zeta_n)=0.$$

Thus we get, $g^{(k)}=a$ whenever $g=0$ and $g^{(k+1)}=0$ whenever $g=0$.

On then other hand, if $g^{(k)}(\zeta_0)=a$ then there exist $\zeta_n\to\zeta_0$ such that $g_n^{(k)}(\zeta_n)=f_n^{(k)}(z_n+\rho_n\zeta_n)=a$, $n=1,2,\ldots$. Hence $f_n(z_n+\rho_n\zeta_n)=a$ and $g_n(\zeta_n)=0$ for $n=1,2,\ldots$. Thus

$$g(\zeta_0)=\lim_{n\to\infty} g_n(\zeta_n)=0.$$

This shows that $g=0$ whenever $g^{(k)}=a$. Hence $g=0$ if and only if $g^k=a$ and $g^{(k+1)}=0$ whenever $g=0$.

Now using Lemma 2.2 and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned}
T(r,g) &\le \overline{N}(r,g)+N(r,\frac{1}{g})+N(r,\frac{1}{g^{(k)}-a})-N(r,\frac{1}{g^{(k+1)}})+S(r,g)\\
&= N(r,\frac{1}{g})+N(r,\frac{1}{g^{(k)}-a})-N(r,\frac{1}{g^{(k+1)}})+S(r,g)\\
&\le N(r,\frac{1}{g^{(k)}-a})-\overline{N}(r,\frac{1}{g^{(k+1)}})+S(r,g)\\
&\le T(r,\frac{1}{g^{(k)}-a})-\overline{N}(r,\frac{1}{g^{(k+1)}})+S(r,g)\\
&\le T(r,g^{(k)}-a)-\overline{N}(r,\frac{1}{g^{(k+1)}})+S(r,g)\\
&\le T(r,g)-\overline{N}(r,\frac{1}{g^{(k+1)}})+S(r,g). \qquad (3)
\end{aligned}$$

Thus we get

$$\overline{N}(r,\frac{1}{g^{(k+1)}})=S(r,g), \qquad (4)$$

by (3), (4) and the claim($g=0$ if and only if $g^{(k)}=a, g^{(k+1)}=0$ whenever $g=0$) we get a contradiction: $T(r,g)=S(r,g)$. It proves the theorem.

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