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A NOTE ON THE EXACT SOLUTION FOR THE FRACTIONAL BURGERS EQUATION

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Abstract: In this work, the fractional derivatives in the sense of the Jumaries modified Riemann-Lioville derivative of order α and the fractional complex transform are used to obtain the most general solution for the fractional Burgers equation.

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1. Introduction

The fractional partial differential equations have a wide range of applications in several branch of the pure and applied sciences. They appear for instance in various applications in physics, biology, engineering and many others (see for instance [1] and references therein). In this order of ideas, the problem of determining exact solutions for nonlinear fractional partial differential equations is a very important task in applied mathematics. One of the techniques more known is the use of the fractional complex transform (see [2]) for to reduce the partial fractional equation to an ordinary differential equation which can be solve using several powerful methods such as the tanh-coth method (see [3]), the Exp-function method (see [4]), the improved generalized tanh-coth method

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(see [5]) or any other classic methods known for this task. Our problem of interest in this work consists on determining the most general solution to the following fractional Burger equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \rho u \frac{\partial^\alpha u}{\partial x^\alpha} - \delta \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = 0, \quad t > 0, \quad 0 < \alpha \leq 1, \tag{1}$$

where ρ and δ are arbitrary real parameters. With this aim, we will use the Jumaries's modified Riemann-Lioville derivative of order α (see [6],[7]). This kind of fractional derivative is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} [f(\xi) - f(0)] d\xi, \quad 0 < \alpha < 1. \tag{2}$$

In the references (see [6],[7]) can be found several relations and important formulas respect to the fractional derivative, however, for sake of simplicity, we put here only the followings:

1. Integral respect to $(dt)^\alpha$:

$$\begin{cases} I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{-\alpha} f(\xi) d\xi, \\ I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha+1)} \int_0^t f(\xi)(d\xi)^\alpha, \quad 0 < \alpha \leq 1, \\ \int (dt)^\alpha = t^\alpha. \end{cases} \tag{3}$$

2. Some derivative formulas:

$$\begin{cases} D_t^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, \\ D_t^\alpha (u(t)v(t)) = [D_t^\alpha u(t)]v(t) + u(t)[D_t^\alpha v(t)], \\ D_t^\alpha (f(u(t))) = f'_u(u(t)) D_t^\alpha u(t) = D_t^\alpha f(u(t))(u'(t))^\alpha. \end{cases} \tag{4}$$

2. The General Solution to the Fractional Burgers Equation

Firstly, we will focus our attention in to reduce (1) to an ordinary differential equation. That is, we will consider the fractional complex transform (see [2]) given by

$$u(x,t) = v(\xi), \quad \xi = \frac{\lambda_1 x^\alpha}{\Gamma(1+\alpha)} + \frac{\lambda_2 t^\alpha}{\Gamma(1+\alpha)}, \tag{5}$$

where λ_1 and λ_2 are nonzero constants. Substituting (5) into (1) we obtain the following ordinary differential equation

$$\lambda_2 v'(\xi) + \rho\lambda_1 v(\xi)v'(\xi) - \delta\lambda_1^2 v''(\xi) = 0, \tag{6}$$

with $v'(\xi) = \frac{dv}{d\xi}$.

Integrating (6) once with respect to ξ we have

$$\lambda_2 v(\xi) + \frac{\rho}{2}\lambda_1 v^2(\xi) - \delta\lambda_1^2 v'(\xi) = k, \tag{7}$$

where k is a constant of integration. Now, the Eq. (7) is a general Riccati equation of the form

$$\phi'(\xi) = \alpha(t) + \beta(t)\phi(\xi) + \gamma(t)\phi(\xi)^2, \tag{8}$$

where $\alpha(t) = -\frac{k}{\delta\lambda_1^2}$, $\beta(t) = \frac{\lambda_2}{\delta\lambda_1^2}$, $\gamma(t) = \frac{\rho}{2\delta\lambda_1^2}$. The solution of (8) in the case that $(\beta^2(t) - 4\alpha(t)\gamma(t)) \neq 0$ is given by (see [8])

$$\phi(\xi) = -\frac{\sqrt{\beta^2(t) - 4\alpha(t)\gamma(t)}\tanh[\frac{1}{2}\sqrt{\beta^2(t) - 4\alpha(t)\gamma(t)}\xi + \xi_0] - \beta(t)}{2\gamma(t)}, \tag{9}$$

with ξ_0 an arbitrary constant. Therefore, in accordance with (5) the following is the general solution to the Eq. (1):

$$u(x,t) = -\frac{\delta\lambda_1^2}{\rho}[\sqrt{\frac{\lambda_2^2 + 2k\rho}{\delta^2\lambda_1^4}} \tanh(\frac{1}{2}\sqrt{\frac{\lambda_2^2 + 2k\rho}{\delta^2\lambda_1^4}}[\frac{\lambda_1 x^\alpha}{\Gamma(1+\alpha)} + \frac{\lambda_2 t^\alpha}{\Gamma(1+\alpha)}] + \xi_0) - \frac{\lambda_2}{\delta\lambda_1^2}]. \tag{10}$$

Here, λ_1, λ_2 are arbitrary constant.

3. Conclusions

The most general exact solution to the fractional (in the variables t and x) Burgers equation have been obtained using the fractional complex transformation and the solution of a general Riccati equation. The result obtained here can be compared with those obtained in (see [9]) using the Exp-function method to see that in this model the use of the mathematical software can be avoided.

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