GRADIENT ESTIMATES FOR HEAT-TYPE EQUATIONS ON MANIFOLDS EVOLVING BY THE RICCI FLOW

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Abstract: In this paper, certain localized and global gradient estimates for all positive solutions to the geometric heat equation coupled to the Ricci flow either forward or backward in time are proved. As a by product, we obtain various Li-Yau type differential Harnack estimates. We also discuss the case when the diffusion operator is perturbed with the curvature operator (precisely, when the Laplacian is replaced with " $\Delta - R(x, t)$ ", R being the scalar operator). This is well generalised to the case of an adjoint heat equation under the Ricci flow.

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1. Introduction

Let M be an n -dimensional compact (or noncompact without boundary) manifold on which a one parameter family of Riemannian metrics $g(t), t \in [0, T)$ is defined. We say $(M, g(t))$ is a solution to the Ricci flow if it is evolving by the following nonlinear weakly parabolic partial differential equation

$$\frac{\partial}{\partial t}g(x, t) = -2Rc(x, t), \quad (x, t) \in M \times [0, T] \quad (1.1)$$

with $g(x, 0) = g(0)$, where Rc is the Ricci curvature. By the positive solution to the heat equation on the manifold, we mean a smooth function at least C^2 in x and C^1 in t , $u \in C^{2,1}(M \times [0, T])$ which satisfies the following equation

$$\left(\Delta - \frac{\partial}{\partial t}\right)u(x, t) = 0, \quad (x, t) \in M \times [0, T], \quad (1.2)$$

where the symbol Δ is the Laplace-Beltrami operator acting on functions in space with respect to metric $g(t)$ in time. We couple the Ricci flow to the equation of the form (1.2) (either forward, backward, or perturbed with curvature operator), and obtain several gradient estimate on the logarithm of the positive solutions. We start with localized version for gradient estimate when Laplacian operator in (1.2) is replaced with $\Delta - R(x, t)$, where R is the scalar curvature of the Ricci flow $g(t)$. We show how the local results imply the the global ones. As an application of our gradient estimates, we derive several Harnack estimates.

In recent years, there have been increasing efforts towards the study of the Ricci flow coupled to heat-type equation. This actually arose from R. Hamilton's work [15], where he conceived the idea of investigating Ricci flow coupled to harmonic maps heat flow. He combined this with his previous results [13, 14] to study the formation of singularities in the Ricci flow. In [14], he proved a Harnack estimate on Riemannian manifolds with nonnegative positive curvature operator. Hamilton also proved Harnack estimates for surfaces [12] whose positive scalar curvature under the Ricci flow satisfies the heat equation with soliton potential. B. Chow [7] completed the proof of Harnack estimate for surfaces of positive scalar curvature in general. Thus, Harnack estimates for the Ricci flow on surfaces give a control on curvature growth, while in higher dimension, one uses the Harnack estimates to classify the ancient solutions of nonnegative curvature operators. The importance of gradient estimates as well as those of Harnack inequalities can not be overemphasised in the fields of Differential geometry and Analysis among their numerous applications. Differential Harnack inequalities are used to study the behaviours of solutions to the heat equation in space-time. Li and Yau's paper [17] can be said to mark the beginning of rigorous applications of these concepts, they derived gradient estimates for positive solutions to the heat operator defined on closed manifold with bounded Ricci curvature, from which they obtained Harnack inequalities. These inequalities were in turn used to establish various lower and upper bounds on the heat kernel. They also studied manifold satisfying Dirichlet and Neumann conditions. G. Perelman [19] recorded a breakthrough in the fields of

Topology and Geometry by the application of differential inequality, where he obtained gradient estimate for the fundamental solution to the conjugate heat equation on compact manifold evolving by the Ricci flow. Perelman's results are one of important ingredients in the proof of Poincaré conjecture. C. Guenther [11] also proved gradient estimates for positive solutions to the heat equation under the Ricci flow by adapting the methodology of Bakry and Qian [1] to time dependent metric case. As an application of her results, she got a Harnack-type inequality and obtain a lower bound for fundamental solutions. She also studied existence and basic properties of these solutions. S. Kuang and Q. Zhang [16] established a gradient estimate that holds for all positive solutions of the conjugate heat equation defined on a closed manifold whose metric is evolving by the Ricci flow. X. Cao [4] also used Perelman's approach to establish a differential inequality for all positive solutions to the conjugate heat equation also under the Ricci flow with nonnegative condition on the scalar curvature. Immediate consequences of their result in [16] and [5] are Harnack-type inequalities. Qi. Zhang [21] derived local gradient estimates for positive solutions to the heat equation coupled to the backward in time Ricci flow with assumption of lower bound on the Ricci curvature, his gradient estimate is used to prove a Gaussian bound for the conjugate heat equation. In [2], Băileşteanu, Cao, and Pulemotov applied Zhang's method to prove both space-only and space-time gradient estimates for heat equation coupled to forward in time Ricci flow. They also study manifolds with nonempty convex boundary evolving under the Ricci flow (see also [18] for a related result. Ecker, Knopf, Ni and Topping [10] have a local result on gradient estimate in relation to can value theorem and monotonicity for heat kernel.

1.1. Preliminaries and Notation

We remark that our manifold is endowed with Riemannian metric $ds^2 = g = g_{ij}dx^i dx^j$, where $\{x^i\}, 1 \leq i \leq n$ is a local coordinate system and n is the dimension of the manifold. The operator Δ is the Laplace-Beltrami operator on (M, g) which is defined by

$$\Delta_g = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j \right) \quad (1.3)$$

and $\nabla = g^{ij} \partial_i$ is the gradient operator, where $|g|$ = determinant of g and $g^{ij} = (g_{ij})^{-1}$, inverse metric. A natural function that will be defined on M is the distance function from a given point, namely, let $p \in M$ and define $d(x, p)$ for all $x \in M$ where $dist(\cdot, \cdot)$ is the geodesic distance. Note that $d(x, p)$ is only

Lipschitz continuous, i.e., everywhere continuous except on the cut locus of p and on the point where x and p coincide. It is then easy to see that

$$|\nabla d| = g^{ij} \partial_i d \partial_j d = 1 \quad \text{on} \quad M \setminus \{\{p\} \cup \text{cut}(p)\}.$$

Let $d(x, y, t)$ be the geodesic distance between x and y with respect to the metric $g(t)$, we define a smooth cut-off function $\varphi(x, t)$ with support in the geodesic cube

$$\mathcal{Q}_{2\rho, T} := \{(x, t) \in M \times (0, T] : d(x, y, t) \leq 2\rho\},$$

for any C^2 -function $\psi(s)$ on $[0, +\infty)$ with

$$\psi(s) = \begin{cases} 1, & s \in [0, 1], \\ 0, & s \in [2, +\infty) \end{cases}$$

and

$$\psi'(s) \leq 0, \quad \psi''(s) \geq -C_1 \quad \text{and} \quad \frac{|\psi'|^2}{\psi} \leq C_2,$$

where C_1, C_2 are absolute constants, such that

$$\varphi(x, t) = \psi\left(\frac{d(x, p, t)}{\rho}\right) \quad \text{and} \quad \varphi|_{\mathcal{Q}_{2\rho, T}} = 1.$$

We will apply maximum principle and invoke Calabi's trick to assume everywhere smoothness of $\varphi(x)$ since $\psi(s)$ is in general Lipschitz. We need Laplacian comparison theorem to do some calculation on $\varphi(x, t)$. Here is the statement of the theorem; Let M be a complete n -dimensional Riemannian manifold whose Ricci curvature is bounded from below by $Rc \geq (n-1)k$ for some constant $k \in \mathbb{R}$. Then the Laplacian of the distance function satisfies

$$\Delta d(x, p) \leq \begin{cases} (n-1)\sqrt{k} \cot(\sqrt{k}\rho), & k > 0, \\ (n-1)\rho^{-1}, & k = 0, \\ (n-1)\sqrt{|k|} \coth(\sqrt{|k|}\rho), & k < 0. \end{cases} \quad (1.4)$$

Throughout, we will impose boundedness condition on the Ricci curvature of the metric and note that when the metric evolves by the Ricci flow, boundedness and sign assumptions are preserved as long as the flow exists, so also the metrics

are uniformly equivalent, precisely, if $-K_1g \leq Rc \leq K_2g$, where $g(t), t \in [0, T]$ is a Ricci flow, then

$$e^{-k_1T}g(0) \leq g(t) \leq e^{k_2T}g(0). \quad (1.5)$$

See [8] and [9] for details on the theory of the Ricci flow. Most of our calculations are done in local coordinates where $\{x^i\}$ is fixed in a neighbourhood of every point $x \in M$. We switch between the Rc and R_{ij} for the Ricci curvature notation, and any time R_{ij} is used we assumed to be on local coordinates at x . We also define some classical identities on function $f \in C^\infty$ at point x in a local normal coordinate system, namely,

Bochner-Weitzenböck identity

$$\Delta|\nabla f|^2 = 2 \sum_{ij} |f_{ij}|^2 + 2 \sum_{ij} f_j f_{jji} + 2 \sum_{ij} R_{ij} f_i f_j \quad (1.6)$$

and Ricci identity

$$\sum_{ij} f_j f_{jii} - \sum_{ij} f_j f_{jji} = R_{ij} f_i f_j. \quad (1.7)$$

We make use of the above identities and we try as much as possible to be explicit at any point where they are used. The rest of this paper is planned as follows; In section two we discuss gradient estimate on the operator of the form $(\partial_t - \Delta + R)$ under both backward and forward in time Ricci flow, where we only require boundedness of scalar curvature. In section three, we discuss local space-time gradient estimates for heat equation on manifold evolving by the Ricci flow, while as an application of these results we obtain various Li-Yau type Harnack inequality in section four.

2. The Conjugate Heat Equation under Ricci Flow

In this section, we discuss the localized version of gradient estimate on the heat equation perturbed with curvature operator under both forward and backward Ricci flow. The estimate under backward action of Ricci flow is related to the local monotonicity for heat kernel and mean value theorem of Ecker, Knopf, Ni and Topping in [10, Section 3]. Here, they worked in general geometric flow, we follow their approach. We also remark that our result can be easily generalized to Perelman's conjugate heat equation and entropy monotonicity [19] and we can use it to prove a global Gaussian estimate as done in [21, Section 5].

We consider the conjugate heat equation coupled to the backward and forward Ricci flow respectively as follows

$$\begin{cases} (\partial_t - \Delta + R)u(x, t) = 0 \\ \frac{\partial}{\partial t}g(x, t) = 2Ric(x, t) \end{cases} \quad (2.1)$$

and

$$\begin{cases} (\partial_t - \Delta + R)u(x, t) = 0 \\ \frac{\partial}{\partial t}g(x, t) = -2Ric(x, t). \end{cases} \quad (2.2)$$

Suppose $u = u(x, t)$ solves the conjugate heat equation and satisfies $0 < u \leq A$ in the geodesic cube $\mathcal{Q}_{2\rho} \subset M$ as defined by

$$\mathcal{Q}_{2\rho, T} := \{(x, t) \in M \times (0, T] : d(x, y, t) \leq 2\rho\}, \quad (2.3)$$

then we have

Theorem 2.1. *Let $(M, g(t)), t \in [0, T]$ be a complete solution to the backward Ricci flow (2.1b) with $R \geq -k_1, Ric \geq -k_2$ and $|\nabla R| \leq k_3$, for some constants $k_1, k_2, k_3 \geq 0$. Let $u = u(x, t)$ be any positive solution to the heat equation (2.1a) defined in $\mathcal{Q}_{2\rho, T} \subset (M \times [0, T])$, satisfying $0 < u \leq A$. There exist absolute constants C_1, C_2 depending on n such that*

$$\frac{|\nabla u|^2}{u^2} \leq \left(1 + \log\left(\frac{A}{u}\right)\right)^2 \left(\frac{1}{t} + C_2 k_1 + 4k_2 + 2k_3 + \frac{1}{\rho^2}(\rho C_1 \sqrt{k_2} + C_2)\right). \quad (2.4)$$

The boundedness assumption may be weakened in the case of the forward Ricci flow (2.2) as we can state our result as follows;

Theorem 2.2. *Let $(M, g(t)), t \in [0, T]$ be a complete solution to the forward Ricci flow (2.2b) with $R \geq -k_1$ and $|\nabla R| \leq k_3$, for some constants $k_1, k_2, k_3 \geq 0$. Let $u = u(x, t)$ be any positive solution to the heat equation (2.2a) defined in $\mathcal{Q}_{2\rho, T} \subset (M \times [0, T])$, satisfying $0 < u \leq A$. There exist absolute constants C_1, C_2 depending on n such that*

$$\frac{|\nabla u|^2}{u^2} \leq \left(1 + \log\left(\frac{A}{u}\right)\right)^2 \left(\frac{1}{t} + C_2 k_1 + 2k_3 + \frac{C_2}{\rho^2}\right). \quad (2.5)$$

Proof. (of Theorem 2.1)

Let $f = \log u$ and $w = |\nabla \ln(1 - f)|^2 = \frac{|\nabla f|^2}{(1-f)^2}$. By standard calculation, we have $\partial_t f = \Delta f + |\nabla f|^2 - R$ and

$$\partial_t(|\nabla f|^2) = (\partial_t g^{ij})\partial_i f \partial_j f + 2\langle \nabla f, \nabla \partial_t f \rangle$$

$$\begin{aligned}
&= -2R_{ij}f_i f_j + 2\langle \nabla f, \nabla(\Delta f + |\nabla f|^2 - R) \rangle \\
&= -2R_{ij}\nabla_i f \nabla_j f + 2\langle \nabla f, \nabla \Delta f \rangle + 2\langle \nabla f, \nabla |\nabla f|^2 \rangle - 2\langle \nabla f, \nabla R \rangle.
\end{aligned}$$

By Bochner-Weitzenböck identity (1.6), we have

$$\Delta(|\nabla f|^2) = 2|\nabla \nabla f|^2 + 2\langle \nabla f, \nabla \Delta f \rangle + 2R_{ij}f_i f_j. \quad (2.6)$$

Then

$$(\partial_t - \Delta)|\nabla f|^2 = -2f_{ij}^2 + 2\langle \nabla f, \nabla |\nabla f|^2 \rangle - 2\langle \nabla f, \nabla R \rangle - 4R_{ij}\nabla_i f \nabla_j f. \quad (2.7)$$

Next we compute

$$\begin{aligned}
w_t &= \frac{\partial}{\partial t} \left(\frac{|\nabla f|^2}{(1-f)^2} \right) = \frac{\partial_t |\nabla f|^2}{(1-f)^2} + \frac{2|\nabla f|^2 \partial_t f}{(1-f)^3} \\
&= \frac{-2R_{ij}\nabla_i f \nabla_j f + 2\langle \nabla f, \nabla \Delta f \rangle + 2\langle \nabla f, \nabla |\nabla f|^2 \rangle - 2\langle \nabla f, \nabla R \rangle}{(1-f)^2} \\
&\quad + \frac{2|\nabla f|^2(\Delta f + |\nabla f|^2 - R)}{(1-f)^3} \\
&= -\frac{2R_{ij}\nabla_i f \nabla_j f}{(1-f)^2} + \frac{2f_j f_{jji}}{(1-f)^2} + \frac{2\langle \nabla f, \nabla |\nabla f|^2 \rangle}{(1-f)^2} - \frac{2\langle \nabla f, \nabla R \rangle}{(1-f)^2} \\
&\quad + \frac{2|\nabla f|^2 \Delta f}{(1-f)^3} + \frac{2|\nabla f|^4}{(1-f)^3} - \frac{2R|\nabla f|^2}{(1-f)^3}.
\end{aligned}$$

Compute Δw

$$\begin{aligned}
\nabla w &= \nabla \left(\frac{|\nabla f|^2}{(1-f)^2} \right) = \frac{2\langle \nabla f, \nabla \nabla f \rangle}{(1-f)^2} + \frac{2|\nabla f|^2 \nabla f}{(1-f)^3} \\
&= \frac{2f_i f_{ij}}{(1-f)^2} + \frac{2f_i^2 f_j}{(1-f)^3}.
\end{aligned}$$

$$\begin{aligned}
\Delta w &= \left(\frac{2f_i f_{ij}}{(1-f)^2} \right)_j + \left(\frac{2f_i^2 f_j}{(1-f)^3} \right)_j \\
&= \frac{2(f_{ij} f_{ij} + f_i f_{ijj})}{(1-f)^2} + \frac{4(1-f)f_i f_{ij} f_j}{(1-f)^4} + \frac{4f_i f_{ij} f_j + 2f_i^2 f_{jj}}{(1-f)^3} + \frac{6f_i^2 f_j^2 (1-f)}{(1-f)^6}
\end{aligned}$$

$$= \frac{2f_{ij}^2}{(1-f)^2} + \frac{2f_i f_{ijj}}{(1-f)^2} + \frac{4f_i f_{ij} f_j}{(1-f)^3} + \frac{4f_i f_{ij} f_j}{(1-f)^3} + \frac{2f_i^2 f_{jj}}{(1-f)^3} + \frac{6f_i^2 f_j^2}{(1-f)^4}.$$

Hence

$$\begin{aligned} (\partial_t - \Delta)w = & -\frac{2R_{ij} f_i f_j}{(1-f)^2} + \frac{2f_j f_{jji}}{(1-f)^2} + \frac{2\langle \nabla f, \nabla |\nabla f|^2 \rangle}{(1-f)^2} - \frac{2\langle \nabla f, \nabla R \rangle}{(1-f)^2} + \frac{2|\nabla f|^4}{(1-f)^3} \\ & - \frac{2R|\nabla f|^2}{(1-f)^3} - \frac{2f_{ij}^2}{(1-f)^2} - \frac{2f_i f_{ijj}}{(1-f)^2} - \frac{4f_i f_{ij} f_j}{(1-f)^3} - \frac{4f_i f_{ij} f_j}{(1-f)^2} - \frac{6f_i^2 f_j^2}{(1-f)^4}. \end{aligned}$$

Rearranging

$$\begin{aligned} (\partial_t - \Delta)w = & -\underbrace{\frac{2}{(1-f)^2} \left(f_{ij}^2 + \frac{2f_i f_{ij} f_j}{1-f} + \frac{f_i^2 f_j^2}{(1-f)^2} \right)}_{(2.8)} - \underbrace{\frac{2R_{ij} f_i f_j}{(1-f)^2} + \frac{2f_j f_{jji}}{(1-f)^2} - \frac{2f_i f_{ijj}}{(1-f)^2}}_{(2.8)} \\ & - \underbrace{\frac{4f_i f_{ij} f_j}{(1-f)^3} - \frac{4f_i^2 f_j^2}{(1-f)^4} + \frac{2\langle \nabla f, \nabla |\nabla f|^2 \rangle}{(1-f)^2} - \frac{2\langle \nabla f, \nabla R \rangle}{(1-f)^2} - \frac{2R|\nabla f|^2}{(1-f)^3} + \frac{2|\nabla f|^4}{(1-f)^3}}_{(2.8)} \end{aligned}$$

Note that we can simplify some terms in the last equation further. The first term becomes

$$-\frac{2}{(1-f)^2} \left(f_{ij}^2 + \frac{2f_i f_{ij} f_j}{1-f} + \frac{f_i^2 f_j^2}{(1-f)^2} \right) = -\frac{2}{(1-f)^2} \left(f_{ij} + \frac{f_i f_j}{1-f} \right)^2.$$

Using the Ricci identity on the next three terms we have

$$-\frac{2R_{ij} f_i f_j}{(1-f)^2} + \frac{2f_j f_{jji}}{(1-f)^2} - \frac{2f_i f_{ijj}}{(1-f)^2} = -\frac{4R_{ij} f_i f_j}{(1-f)^2},$$

since the Ricci identity (1.7) implies

$$f_j f_{jji} - f_i f_{ijj} = f_j (f_{jji} - f_{ijj}) = -R_{ij} f_i f_j.$$

Also

$$-\frac{4f_i f_{ij} f_j}{(1-f)^3} - \frac{4f_i^2 f_j^2}{(1-f)^4} = \frac{-2}{1-f} f_j \left(\frac{2f_i f_{ij}}{(1-f)^2} + \frac{2f_i^2 f_j}{(1-f)^3} \right) = \frac{-2}{1-f} \langle \nabla f, \nabla w \rangle,$$

similarly,

$$\frac{2\langle \nabla f, \nabla |\nabla f|^2 \rangle}{(1-f)^2} = 2\langle \nabla f, \nabla w \rangle,$$

then the second to the last three terms gives

$$\begin{aligned} -\frac{4f_i f_{ij} f_j}{(1-f)^3} - \frac{4f_i^2 f_j^2}{(1-f)^4} + \frac{2\langle \nabla f, \nabla |\nabla f|^2 \rangle}{(1-f)^2} &= \frac{-2}{1-f} \langle \nabla f, \nabla w \rangle + 2\langle \nabla f, \nabla w \rangle \\ &= \frac{-2f}{1-f} \langle \nabla f, \nabla w \rangle. \end{aligned}$$

Putting all these together, we get

$$\begin{aligned} (\partial_t - \Delta)w &= -\frac{2}{(1-f)^2} \left(f_{ij} + \frac{f_i f_j}{1-f} \right)^2 - \frac{2f}{1-f} \langle \nabla f, \nabla w \rangle - \frac{4R_{ij} f_i f_j}{(1-f)^2} \\ &\quad - \frac{2\langle \nabla f, \nabla R \rangle}{(1-f)^2} - \frac{2R|\nabla f|^2}{(1-f)^3} - \frac{2|\nabla f|^4}{(1-f)^3}, \end{aligned}$$

then using the curvature conditions we are left with the following inequality

$$(\partial_t - \Delta)w \leq -\frac{2f}{1-f} \langle \nabla f, \nabla w \rangle + \frac{2}{1-f} k_1 w + 4k_2 w + 2k_3 w^{\frac{1}{2}} - 2(1-f)w^2. \quad (2.9)$$

We now apply a cut-off function in order to derive the desired estimate. Now, let $\psi(s)$ be a smooth cut function defined on $[0, \infty)$ such that $0 \leq \psi(s) \leq 1$, with

$$\psi'(s) \leq 0, \quad \psi''(s) \geq -C_1 \quad \text{and} \quad \frac{|\psi'|^2}{\psi} \leq C_2$$

for some constants $C_1, C_2 > 0$ depending on the dimension of the manifold only. Define a distance function $d(p, x)$ between points p and x such that

$$\varphi(x, t) = \varphi(d(p, x, t)) = \psi\left(\frac{d_{g(t)}(p, x)}{\rho}\right)$$

for smooth function $\varphi : M \times [0, T] \rightarrow \mathbb{R}$.

It is easily seen that $\varphi(x, t)$ has its support in the closure of $\mathcal{Q}_{2\rho, T}$. We note that $\varphi(x, t)$ is smooth at $(y, s) \in M \times [0, T]$ whenever point y does not either coincide with p or fall in the cut locus of p , with respect to the metric $g(y, s)$. In what follows, we consider the function φw supported in $\mathcal{Q}_{2\rho, T} \times [0, \infty)$, though, this is in general Lipchitz continuous. For the purpose of the application of the maximum principle, we may therefore, without loss of generality assume that $\varphi(x, t)$ with support in $\mathcal{Q}_{2\rho, T}$ is C^2 at the maxima. The assumption is made possible by a standard argument called Calabi's trick (due to Calabi 1958 [3]). This approach is used in [6], see also [10, 20, 21]. Therefore we can calculate

$$\frac{|\nabla \varphi|^2}{\varphi} = \frac{|\psi'|^2 \cdot |\nabla d|^2}{\rho^2 \varphi} \leq \frac{C_2}{\rho^2}, \quad \frac{\partial \varphi}{\partial t} \leq C_2 k_1 \quad (2.10)$$

and by the Laplacian comparison theorem we have

$$\Delta\varphi = \frac{\psi'\Delta d}{\rho} + \frac{\psi''|\nabla d|^2}{\rho} \quad (2.11)$$

$$\geq -\frac{C_1}{\rho}(n-1)\sqrt{k_2}\coth(\sqrt{k_2}\rho) - \frac{C_2}{\rho^2} \quad (2.12)$$

$$\implies -\Delta\varphi \leq \frac{1}{\rho^2}(C_1\sqrt{k_2}\rho\coth(\sqrt{k_2}\rho) + C_2). \quad (2.13)$$

Let (x_0, t_0) be a point in $\mathcal{Q}_{2\rho, T}$ at which $F = \varphi w$ attains its maximum value. At this point we have to assume that F is positive, since, if $F = 0$, implies $\varphi w(x_0, t_0) = 0 \implies \varphi w(x, t) = 0 \implies w(x, t) = 0$ for all $x \in M$ such that the distance $d(x, x_0, t) < 2\rho$, this yields $\nabla u(x, t) = 0$ and the theorem will follow trivially at (x, t) . The approach here is to estimate $(\partial_t - \Delta)(tF)$ and do some analysis on the result at the maximum point. The argument is as follows

$$(\partial_t - \Delta)(tF) = F + t(\partial_t - \Delta)(\varphi w) \quad (2.14)$$

$$= F - 2t\langle \nabla\varphi, \nabla w \rangle + t\varphi(\partial_t - \Delta)w + tw(\partial_t - \Delta)\varphi. \quad (2.15)$$

Note that at the maximum point (x_0, t_0) , we have by derivative test that

$$\nabla F(x_0, t_0) = 0, \quad \frac{\partial}{\partial t}F(x_0, t_0) \geq 0 \quad \text{and} \quad \Delta F(x_0, t_0) \leq 0. \quad (2.16)$$

Taking tF on $M \times [0, T]$, we have $(\partial_t - \Delta)(tF) \geq 0$ whenever (tF) achieves its maximum. Similarly, by this argument we have

$$\nabla(\varphi w)(x_0, t_0) - w\nabla\varphi(x_0, t_0) = \varphi\nabla w(x_0, t_0),$$

which means $\varphi\nabla w$ can always be replaced by $-w\nabla\varphi$. By (2.9), (2.14) and (2.16)

$$\begin{aligned} 0 \leq & F - 2t\langle \nabla\varphi, \nabla w \rangle \\ & + t\varphi\left\{ -\frac{2f}{1-f}\langle \nabla f, \nabla w \rangle + \frac{2}{1-f}k_1w + 4k_2w + 2k_3w^{\frac{1}{2}} + 2(1-f)w^2 \right\} \\ & + tw(\partial_t - \Delta)\varphi. \end{aligned}$$

Taking $0 \leq \varphi \leq 1$ and noticing that $\frac{1}{1-f} \leq 1$, then the last inequality becomes

$$0 \leq F - 2t\frac{\nabla\varphi}{\varphi} \cdot \nabla(\varphi w) - 2t\frac{|\nabla\varphi|^2}{\varphi}w$$

$$\begin{aligned}
& + t\{2k_1F + 4k_2F + 2k_3F^{\frac{1}{2}} + 2(1-f)F^2\} + 2t\left(-\frac{f}{1-f}\right)|\nabla f||\nabla\varphi|w \\
& + tw(\partial_t - \Delta)\varphi.
\end{aligned}$$

Using the following relation as noticed in [10];

$$2k_3F^{\frac{1}{2}} \leq K_3F + k_3$$

and by the Young's inequality

$$\begin{aligned}
2t\left(-\frac{f}{1-f}\right)|\nabla f||\nabla\varphi|w & \leq tw\left(\frac{|\nabla f|^2}{1-f}\varphi + \frac{|\nabla\varphi|^2}{\varphi}\frac{f^2}{1-f}\right) \\
& = t\varphi(1-f)w^2 + tw\frac{|\nabla\varphi|^2}{\varphi}\frac{f^2}{1-f} \\
& = t(1-f)F^2 + tF\frac{C_2}{\rho^2}\frac{f^2}{1-f}.
\end{aligned}$$

Notice also that by bounds given in (2.10) and (2.11) we have

$$\begin{aligned}
tw(\partial_t - \Delta)\varphi & \leq tw(C_2k_1 + \frac{1}{\rho^2}(C_1\sqrt{k_2}\coth(\sqrt{k_2}\rho) + C_2)) \\
& \leq tw\left(C_2k_1 + \frac{1}{\rho^2}(\rho C_1\sqrt{k_2} + C_2)\right).
\end{aligned}$$

Putting these together and dividing through by $(1-f)$, while noticing that $\frac{1}{1-f} \leq 1$ and $\frac{-f}{1-f} \leq 1$, we have

$$\begin{aligned}
0 & \leq F - 2t\frac{C_2}{\rho^2}F + t(2k_1 + 4k_2 + k_3)F + tk_3 + tF\frac{C_2}{\rho^2} - tF^2 \\
& + tw\left(C_2k_1 + \frac{1}{\rho^2}(\rho C_1\sqrt{k_2} + C_2)\right).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
tF^2 & \leq F + t(2k_1 + 4k_2 + k_3)F \\
& + t\left(C_2k_1 + \frac{1}{\rho^2}(\rho C_1\sqrt{k_2} + C_2)\right)F - t\frac{C_2}{\rho^2}F + tk_3 \\
F^2 & \leq F\left\{\frac{1}{t} + C_2K_1 + 4k_2 + k_3 + \frac{1}{\rho^2}(\rho C_1\sqrt{k_2} + C_2) + k_3\right\} + k_3.
\end{aligned}$$

From here we can conclude that

$$F \leq \frac{1}{t} + C_2K_1 + 4k_2 + k_3 + \sqrt{k_3} + \frac{1}{\rho^2}(\rho C_1\sqrt{k_2} + C_2)$$

at (x_0, t_0) . Therefore

$$|\nabla f|^2 \leq (1-f)^2 \left(\frac{1}{t} + C_2 K_1 + 4k_2 + 2k_3 + \frac{1}{\rho^2} (\rho C_1 \sqrt{k_2} + C_2) \right). \quad (2.17)$$

By standard scaling argument we can assume $A \equiv 1$, hence the proof is completed. $\square$

Proof. (of Theorem 2.2)

The proof of Theorem 2.2 is similar to that of Theorem 2.1, the disparity between the estimates (2.4) and (2.5) arises in some calculation which we briefly point out here.

Similarly, set $f = \log u$ and $w = \frac{|\nabla f|^2}{(1-f)^2}$. Notice that $g(t)$ evolves by the Ricci flow (2.2), where the inverse metric evolves as $\partial_t(g^{ij}) = 2R_{ij}$ and then

$$\partial_t(|\nabla f|^2) = 2R_{ij}\nabla_i f \nabla_j f + 2\langle \nabla f, \nabla \Delta f \rangle + 2\langle \nabla f, \nabla |\nabla f|^2 \rangle - 2\langle \nabla f, \nabla R \rangle$$

hence, the counterpart of (2.8) is

$$\begin{aligned} & (\partial_t - \Delta)w \\ &= - \underbrace{\frac{2}{(1-f)^2} \left(f_{ij}^2 + \frac{2f_i f_{ij} f_j}{1-f} + \frac{f_i^2 f_j^2}{(1-f)^2} \right)}_{\text{braced}} + \underbrace{\frac{2R_{ij} f_i f_j}{(1-f)^2} + \frac{2f_j f_{jji}}{(1-f)^2} - \frac{2f_{ij}^2}{(1-f)^2}}_{\text{braced}} \\ & - \underbrace{\frac{4f_i f_{ij} f_j}{(1-f)^3} - \frac{4f_i^2 f_j^2}{(1-f)^4} + \frac{2\langle \nabla f, \nabla |\nabla f|^2 \rangle}{(1-f)^2}}_{\text{braced}} - \underbrace{\frac{2\langle \nabla f, \nabla R \rangle}{(1-f)^2} - \frac{2R|\nabla f|^2}{(1-f)^3} + \frac{2|\nabla f|^4}{(1-f)^3}}_{\text{braced}} \end{aligned} \quad (2.18)$$

Using the Ricci identity, the second braced terms vanish and we are left with following inequality as in (2.9)

$$(\partial_t - \Delta)w \leq -\frac{2f}{1-f} \langle \nabla f, \nabla w \rangle + \frac{2}{1-f} k_1 w + 2k_3 w^{\frac{1}{2}} - 2(1-f)w^2. \quad (2.19)$$

We then proceed as in the the rest of the proof of theorem (2.1). $\square$

This can be seen as an improvement since the curvature assumption is weakened, we only require boundedness of scalar curvature. Global estimates can be obtained from the two gradient estimates, simply by sending ρ to ∞ . For instance, when u is globally defined and $g(t)$ has nonnegative curvature, the estimate simply read as

$$\frac{|\nabla u|^2}{u^2} \leq \left(1 + \frac{A}{u} \right)^2 \left(\frac{1}{t} + 2k_3 \right). \quad (2.20)$$

3. Gradient Estimates on Forward Heat equation

Let M be an n -dimensional complete Riemannian manifold without boundary, we discuss space-time gradient estimates for positive solutions of the heat equation along the Ricci flow

$$\frac{\partial}{\partial t}g(x, t) = -2R(x, t), \quad (x, t) \in M \times [0, T] \quad (3.1)$$

coupled to

$$\left(\Delta - \frac{\partial}{\partial t}\right)u(x, t) = 0, \quad (x, t) \in M \times [0, T]. \quad (3.2)$$

In general, our estimate is a local one as we obtain it in the interior of geodesic cube. We will show how this local estimate can lead to achieving a global one. We first proof a lemma which is very crucial to our derivation. Define the geodesic cube

$$\mathcal{Q}_{2\rho, T} := \{(x, t) \in M \times (0, T] : d(x, x_0, t) \leq \rho\}. \quad (3.3)$$

Lemma 3.1. *Let $(M, g(t))$ be a complete solution to the Ricci flow in some time interval $[0, T]$. Suppose that $-k_1g \leq Rc \leq k_2g$ for some positive constants k_1 and k_2 and for all $t \in [0, T]$. For any smooth positive solution $u \in C^{2,1}(M \times [0, T])$ to the heat equation in the geodesic cube $\mathcal{Q}_{2\rho, T}$, it holds that*

$$(\Delta - \partial_t)G \geq -2\langle \nabla f, \nabla G \rangle - (|\nabla f|^2 - \alpha \partial_t f) - 2\alpha t k_1 |\nabla f|^2 \quad (3.4)$$

$$+ \frac{2\alpha}{np} t (|\nabla f|^2 - \partial_t f)^2 - \frac{\alpha n q}{2} t (k_1 + k_2)^2, \quad (3.5)$$

where $f = \log u$, $G = t(|\nabla f|^2 - \alpha \partial_t f)$ and $\alpha \geq 1$ are given such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$ for any real numbers $p, q > 0$.

Proof. We have

$$\begin{aligned} \Delta G &= t(\Delta |\nabla f|^2 - \alpha \Delta \partial_t f) \\ \partial_t G &= (|\nabla f|^2 - \alpha \partial_t f) + t(|\nabla f|^2 - \alpha \partial_t f)_t. \end{aligned}$$

Working in local coordinates system at any point $x \in M$ and using Einstein summation convention where repeated indices are summed up. We have by Bochner-Weitzenböck's identity (1.6)

$$\Delta |\nabla f|^2 = 2f_{ij}^2 + 2f_j f_{jji} + 2R_{ij} f_i f_j.$$

By the hypothesis of the lemma that $g(x, t)$ evolves by the Ricci flow we have

$$\begin{aligned}\partial_t(|\nabla f|^2) &= (\partial_t g^{ij})\partial_i f \partial_j f + 2g^{ij}\partial_i f \partial_j \partial_t f \\ &= 2R_{ij}f_i f_j + 2f_i f_{t,i},\end{aligned}$$

similarly,

$$\begin{aligned}\partial_t(\Delta f) &= (\partial_t g^{ij})\partial_i \partial_j f + g^{ij}\partial_i \partial_j \partial_t f \\ &= 2R_{ij}f_{ij} + \Delta f_t, \\ \text{which } \implies \Delta f_t &= (\Delta f)_t - 2R_{ij}f_{ij}.\end{aligned}$$

With the above computations we obtain the following at an arbitrary point $(x, t) \in \mathcal{Q}_{2\rho, T}$

$$\begin{aligned}\Delta G &= t\left(2f_{ij}^2 + 2f_i f_{jji} + 2R_{ij}f_i f_j + 2\alpha R_{ij}f_{ij} - \alpha(\Delta f)_t\right) \\ &\geq t\left[\left(2f_{ij}^2 + 2\alpha R_{ij}f_{ij}\right) + 2f_j f_{jji} - 2k_1|\nabla f|^2 - \alpha(\Delta f)_t\right].\end{aligned}$$

On the other hand

$$\partial_t G = (|\nabla f|^2 - \alpha \partial_t f) + t(2R_{ij}f_i f_j + 2f_i f_{t,i} - \alpha f_{tt}).$$

Noticing that $f = \log u$ implies the evolution

$$\partial_t f = \Delta f + |\nabla f|^2, \quad (x, t) \in \mathcal{Q}_{2\rho, T}.$$

Therefore

$$\begin{aligned}(\Delta - \partial_t)G &\geq t\left[\left(2f_{ij}^2 + 2\alpha R_{ij}f_{ij}\right) + 2f_j f_{jji} - 2k_1|\nabla f|^2 - \alpha(\Delta f)_t\right] \\ &\quad - \left[(|\nabla f|^2 - \alpha \partial_t f) + t(|\nabla f|^2 - \alpha \partial_t f)_t \right] \\ &= t\left(2f_{ij}^2 + 2\alpha R_{ij}f_{ij}\right) + 2t\langle \nabla f, \nabla(\partial_t f - |\nabla f|^2) \rangle - 2tk_1|\nabla f|^2 \\ &\quad - \alpha(\partial_t f - |\nabla f|^2)_t - (|\nabla f|^2 - \alpha \partial_t f) - t(|\nabla f|^2 - \alpha \partial_t f)_t \\ &= t\left(2f_{ij}^2 + 2\alpha R_{ij}f_{ij}\right) - 2t\langle \nabla f, \nabla(|\nabla f|^2 - \alpha \partial_t f) \rangle - 2t\langle \nabla f, \nabla(\alpha \partial_t f - \partial_t f) \rangle \\ &\quad - 2tk_1|\nabla f|^2 - \alpha t(\partial_t f - |\nabla f|^2)_t - (|\nabla f|^2 - \alpha \partial_t f) - t(|\nabla f|^2 - \alpha \partial_t f)_t \\ &= t\left(2f_{ij}^2 + 2\alpha R_{ij}f_{ij}\right) - 2\langle \nabla f, \nabla G \rangle - (|\nabla f|^2 - \alpha \partial_t f) - 2tk_1|\nabla f|^2 \\ &\quad - 2(\alpha - 1)t\langle \nabla f, \nabla \partial_t f \rangle - \alpha t(\partial_t f - |\nabla f|^2)_t - t(|\nabla f|^2 - \alpha \partial_t f)_t\end{aligned}$$

$$\begin{aligned}
&= t \left(2f_{ij}^2 + 2\alpha R_{ij} f_{ij} \right) - 2\langle \nabla f, \nabla G \rangle - (|\nabla f|^2 - \alpha \partial_t f) - 2tk_1 |\nabla f|^2 \\
&\quad - 2(\alpha - 1)t \langle \nabla f, \nabla \partial_t f \rangle + (\alpha - 1)t (|\nabla f|^2)_t \\
&= t \left(2f_{ij}^2 + 2\alpha R_{ij} f_{ij} \right) - 2\langle \nabla f, \nabla G \rangle - (|\nabla f|^2 - \alpha \partial_t f) - 2tk_1 |\nabla f|^2 \\
&\quad - 2(\alpha - 1)t \langle \nabla f, \nabla \partial_t f \rangle - (\alpha - 1)t \left[2R_{ij} f_{ij} + 2\langle \nabla f, \nabla \partial_t f \rangle \right] \\
&\geq t \left(2f_{ij}^2 + 2\alpha R_{ij} f_{ij} \right) - 2\langle \nabla f, \nabla G \rangle - (|\nabla f|^2 - \alpha \partial_t f) - 2tk_1 |\nabla f|^2 \\
&\quad - 2(\alpha - 1)t \langle \nabla f, \nabla \partial_t f \rangle - 2(\alpha - 1)tk_1 |\nabla f|^2 + 2(\alpha - 1)t \langle \nabla f, \nabla \partial_t f \rangle \\
&= t \left(2f_{ij}^2 + 2\alpha R_{ij} f_{ij} \right) - 2\langle \nabla f, \nabla G \rangle - (|\nabla f|^2 - \alpha \partial_t f) - 2\alpha tk_1 |\nabla f|^2.
\end{aligned}$$

Now we can choose any two real numbers $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$, so that we can write

$$\begin{aligned}
2f_{ij}^2 + 2\alpha R_{ij} f_{ij} &= \frac{2\alpha}{p} f_{ij}^2 + 2\alpha \left(\frac{1}{q} f_{ij}^2 + R_{ij} f_{ij} \right) \\
&= \frac{2\alpha}{p} f_{ij}^2 + 2\alpha \left(\frac{1}{\sqrt{q}} f_{ij} + \frac{\sqrt{q}}{2} R_{ij} \right)^2 - \frac{\alpha q}{2} R_{ij}^2 \\
&\geq \frac{2\alpha}{p} f_{ij}^2 - \frac{\alpha q}{2} R_{ij}^2,
\end{aligned}$$

where we have used completing the square method to arrive at the last inequality. Also by Cauchy-Schwarz inequality

$$(\Delta f)^2 = |g^{ij} \partial_i \partial_j f|^2 \leq n f_{ij}^2$$

holds at an arbitrary point $(x, t) \in \mathcal{Q}_{2\rho, T}$, therefore we have $f_{ij}^2 \geq \frac{1}{n} (\Delta f)^2$. We can also write the boundedness condition on the Ricci curvature as $-(k_1 + k_2)g \leq R_{ij} \leq (k_1 + k_2)g$ so that

$$\sup_M |R_{ij}|^2 \leq n(k_1 + k_2)^2$$

since the Ricci curvature tensor is symmetric. Therefore

$$2t \left(f_{ij}^2 + \alpha R_{ij} f_{ij} \right) \geq \frac{2\alpha}{np} t (\Delta f)^2 - \frac{\alpha n q}{2} t (k_1 + k_2)^2. \quad (3.6)$$

Hence the result. Our calculation is valid in the cube $\mathcal{Q}_{2\rho, T}$. $\square$

In the next, we state and prove a result for local gradient estimate (space-time) for the positive solutions to the heat equation in the geodesic cube $\mathcal{Q}_{2\rho, T}$ of bounded Ricci curvature manifold evolving by the Ricci flow.

Theorem 3.2. Let $(M, g(t)), t \in [0, T]$ be a complete solution of the Ricci flow (3.1) such that the Ricci curvature is bounded in $\mathcal{Q}_{2\rho, T}$, i.e., $-k_1 g(x, t) \leq Rc(x, t) \leq k_2 g(x, t)$ for some positive constants k_1 and k_2 , with $(x, t) \in \mathcal{Q}_{2\rho, T} \subset (M \times [0, T])$. Suppose a smooth positive function $u \in C^{2,1}(M \times [0, t])$ solves the heat equation (3.2) in the cube $\mathcal{Q}_{2\rho, T}$. Then, for any given $\alpha > 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$ and all $(x, t) \in \mathcal{Q}_{2\rho} \times [0, T]$, the following estimate holds

$$\sup_{x \in \mathcal{Q}_{2\rho}} \left\{ |\nabla f|^2 - \alpha f_t \right\} \leq \frac{\alpha np}{4t} + C\alpha^2 \left(\frac{\alpha^2 p}{\rho^2(\alpha - 1)} + \frac{1}{t} + (k_1 + k_2) \right) + \frac{\alpha^2 np}{2(\alpha - 1)} k_1 + \frac{\alpha n}{2} (k_1 + k_2) \sqrt{pq} \quad (3.7)$$

where C is an arbitrary constant depending only on the dimension of the manifold.

Proof. Note that as before we let $f = \log u$ and $G = t(|\nabla f|^2 - \alpha f_t)$ then $\frac{1}{t}G = \left(\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \right)$. The approach is also by using cut-off function and estimating $(\Delta - \partial_t)(t\varphi G)$ at the point where the maximum value for (φG) is attained as we did in Theorem 2.1. The argument follows;

$$(\Delta - \partial_t)(t\varphi G) = 2t\nabla\varphi\nabla G - \varphi G + t\varphi(\Delta - \partial_t)G + tG(\Delta - \partial_t)\varphi. \quad (3.8)$$

Suppose (φG) attains its maximum value at $(x_0, t_0) \in M \times [0, T]$, for $t_0 > 0$. Since $(\varphi G)(x, 0) = 0$ for all $x \in M$, we have by derivative test that

$$\nabla(\varphi G)(x_0, t_0) = 0, \quad \frac{\partial}{\partial t}(\varphi G)(x_0, t_0) \geq 0, \quad \Delta(\varphi G)(x_0, t_0) \leq 0, \quad (3.9)$$

where the function (φG) is being considered with support on $\mathcal{Q}_{2\rho} \times [0, T]$ and we have assumed that $(\varphi G)(x_0, t_0) > 0$ for $t_0 > 0$. By (3.9) we notice that

$$(\Delta - \partial_t)(\varphi G)(x_0, t_0) \leq 0.$$

Using (3.8)-(3.9) and the last lemma, we have

$$\begin{aligned} 0 &\geq (\Delta - \partial_t)(t\varphi G) \geq 2t\nabla\varphi\nabla G - \varphi G \\ &\quad + t\varphi \left\{ -2\langle \nabla f, \nabla G \rangle - (|\nabla f|^2 - \alpha \partial_t f) - 2\alpha t k_1 |\nabla f|^2 \right. \\ &\quad \left. + \frac{2\alpha}{np} t (|\nabla f|^2 - \partial_t f)^2 - \frac{\alpha n q}{2} t \widehat{K} \right\} \\ &\quad + tG(\Delta - \partial_t)\varphi, \end{aligned}$$

where $\widehat{K} = (k_1 + k_2)^2$. Noticing also that $\varphi \nabla G$ can be replaced by $-G \nabla \varphi$ by the condition $\nabla(\varphi G) = 0$. Therefore

$$\begin{aligned} 0 &\geq 2t \nabla \varphi \nabla G - \varphi G + 2t \nabla f \nabla \varphi \cdot G \\ &\quad + t \varphi \left\{ \frac{2\alpha}{np} t (|\nabla f|^2 - \partial_t f)^2 - 2\alpha t k_1 |\nabla f|^2 - \frac{\alpha n q}{2} t \widehat{K} \right\} \\ &\quad - t \varphi (|\nabla f|^2 - \alpha \partial_t f) + t G (\Delta - \partial_t) \varphi. \\ 0 &\geq -2t \frac{|\nabla \varphi|^2}{\varphi} G - 2\varphi G - 2t |\nabla f| |\nabla \varphi| G \\ &\quad + t \varphi \left\{ \frac{2\alpha}{np} t (|\nabla f|^2 - \partial_t f)^2 - 2\alpha t k_1 |\nabla f|^2 - \frac{\alpha n q}{2} t \widehat{K} \right\} \\ &\quad + t G (\Delta - \partial_t) \varphi. \end{aligned}$$

As we have noted earlier, Calabi's trick and Laplacian comparison theorem allows us to do the following calculation on the cut-off function depending on the geodesic distance, since we know that cut locus does not intersect with the geodesic cube

$$-\frac{|\nabla \varphi|^2}{\varphi} \geq -\frac{C_2}{\rho^2}, \quad \Delta \varphi \geq -\frac{C_1}{\rho} \sqrt{k_1} - \frac{C_2}{\rho^2} \quad \text{and} \quad -\frac{\partial \varphi}{\partial t} \geq -C_2 \widehat{K} - \frac{C_1}{\tau}, \quad \tau \in (0, T].$$

Hence

$$tG(\Delta - \partial_t)\varphi \geq tC_3 \left(-\frac{1}{\rho} \sqrt{k_1} - \frac{1}{\rho^2} - \frac{1}{\tau} - \widehat{K} \right),$$

where we have taken C_3 to be the maximum of C_1, C_2 , so our computation becomes

$$\begin{aligned} 0 &\geq -2t \frac{C_2}{\rho^2} G - 2\varphi G - 2t \frac{\sqrt{C_2}}{\rho} |\nabla f| \varphi^{\frac{1}{2}} G \\ &\quad + t \varphi \left\{ \frac{2\alpha}{np} t (|\nabla f|^2 - \partial_t f)^2 - 2\alpha t k_1 |\nabla f|^2 - \frac{\alpha n q}{2} t \widehat{K} \right\} \\ &\quad + C_3 t \left(-\frac{1}{\rho} \sqrt{k_1} - \frac{1}{\rho^2} - \frac{1}{\tau} - \widehat{K} \right) G. \end{aligned}$$

Multiplying through by $0 \leq \varphi \leq 1$ again we have

$$\begin{aligned} 0 &\geq -2\varphi^2 G - 2t \frac{\sqrt{C_2}}{\rho} |\nabla f| \varphi^{\frac{3}{2}} G \\ &\quad + \frac{2t^2}{n} \left\{ \frac{\alpha}{p} (\varphi |\nabla f|^2 - \varphi \partial_t f)^2 - \alpha n k_1 \varphi^2 |\nabla f|^2 - \frac{\alpha n^2 q}{4} \widehat{K} \varphi^2 \right\} \\ &\quad + t C_3 \left(-\frac{1}{\rho} \sqrt{k_1} - \frac{1}{\rho^2} - \frac{1}{\tau} - \widehat{K} \right) (\varphi G) \end{aligned} \quad \Bigg\}. \quad (3.10)$$

Using a standard argument from Li and Yau [17], see also Schoen and Yau [20], we let

$$y = \varphi |\nabla f|^2 \text{ and } Z = \varphi f_t \text{ to have } \varphi^2 |\nabla f|^2 = \varphi y \leq y$$

and

$$y^{\frac{1}{2}}(y - \alpha z) = \varphi^{\frac{1}{2}} |\nabla f| (\varphi |\nabla f|^2 - \varphi \partial_t f) = \frac{1}{t} |\nabla f| \varphi^{\frac{3}{2}} G.$$

Noticing that

$$\left. \begin{aligned} (y - z)^2 &= \left[\frac{1}{\alpha}(y - \alpha z) + \frac{\alpha - 1}{\alpha} y \right]^2 \\ &= \frac{1}{\alpha^2}(y - \alpha z)^2 + \frac{(\alpha - 1)^2}{\alpha^2} y^2 + \frac{2(\alpha - 1)}{\alpha^2} y(y - \alpha z) \end{aligned} \right\} \quad (3.11)$$

and

$$\left. \begin{aligned} (y - z)^2 - npk_1 y - np \frac{\sqrt{C_2}}{\rho} y^{\frac{1}{2}}(y - \alpha z) \\ &= \frac{1}{\alpha^2}(y - \alpha z)^2 + \left(\frac{(\alpha - 1)^2}{\alpha^2} y^2 - npk_1 y \right) \\ &\quad + \left(\frac{2(\alpha - 1)}{\alpha^2} y - np \frac{\sqrt{C_2}}{\rho} y^{\frac{1}{2}} \right) (y - \alpha z) \end{aligned} \right\}, \quad (3.12)$$

using the inequality of the form $ax^2 - bx \geq -\frac{b^2}{4a}$, ($a, b > 0$), we have

$$\frac{(\alpha - 1)^2}{\alpha^2} y^2 - npk_1 y \geq -\frac{\alpha^2 n^2 p^2 k_1^2}{4(\alpha - 1)^2}, \quad \frac{2(\alpha - 1)}{\alpha^2} y - np \frac{\sqrt{C_2}}{\rho} y^{\frac{1}{2}} \geq -C_2 \frac{\alpha^2 n^2 p^2}{8(\alpha - 1)\rho^2}. \quad (3.13)$$

Putting together (3.11) - (3.13), the second and the third terms in the right hand side of (3.10) are further computed as follows, using $t(y - \alpha z) = \varphi G$;

$$\begin{aligned} &\frac{2t^2}{n} \left\{ \frac{\alpha}{p}(y - z)^2 - \alpha n k_1 y - \frac{\alpha n^2 q}{4} \widehat{K} \varphi^2 - \frac{n \sqrt{C_2}}{\rho} y^{\frac{1}{2}}(y - \alpha z) \right\} \\ &= \frac{2t^2}{n} \left\{ \frac{\alpha}{p} \left((y - z)^2 - npk_1 y - \frac{np \sqrt{C_2}}{\rho} y^{\frac{1}{2}}(y - \alpha z) \right) - \frac{\alpha n^2 q}{4} \widehat{K} \varphi^2 \right\} \\ &= \frac{2t^2}{n} \left\{ \frac{\alpha}{p} \left[\frac{1}{\alpha^2}(y - \alpha z)^2 + \left(\frac{(\alpha - 1)^2}{\alpha^2} y^2 - npk_1 y \right) \right. \right. \\ &\quad \left. \left. + \left(\frac{2(\alpha - 1)}{\alpha^2} y - np \frac{\sqrt{C_2}}{\rho} y^{\frac{1}{2}} \right) (y - \alpha z) \right] - \frac{\alpha n^2 q}{4} \widehat{K} \varphi^2 \right\} \end{aligned}$$

$$\geq \frac{2t^2}{n} \left\{ \left(\frac{1}{\alpha p} (y - \alpha z)^2 - \frac{\alpha^3 n^2 p k_1^2}{4(\alpha - 1)^2} \right) - \left(\frac{C_2 \alpha^3 n^2 p}{8\rho^2(\alpha - 1)} (y - \alpha z) + \frac{\alpha n^2 q}{4} \widehat{K} \varphi^2 \right) \right\}.$$

Hence by (3.10)

$$0 \geq -2\varphi^2 G + \left\{ \frac{2t^2}{n} \left(\frac{1}{\alpha p} (y - \alpha z)^2 - \frac{\alpha^3 n^2 p k_1^2}{4(\alpha - 1)^2} \right) - \frac{2t}{n} \left(\frac{C_2 \alpha^3 n^2 p}{8\rho^2(\alpha - 1)} (\varphi G) \right) \right. \\ \left. - \frac{\alpha n q}{2} t^2 \widehat{K} \varphi^2 \right\} + C_3 t \left(-\frac{1}{\rho} \sqrt{k_1} - \frac{1}{\rho^2} - \frac{1}{\tau} - \widehat{K} \right) (\varphi G)$$

since $t(y - \alpha z) = \varphi G$, we have

$$0 \geq \frac{2}{\alpha n p} (\varphi G)^2 + \left[\frac{C_4}{\rho^2} t \left(-\frac{p \alpha^3}{(\alpha - 1)} - \rho \sqrt{k_1} - 1 - \frac{\rho^2}{\tau} - \rho^2 \widehat{K} \right) - 1 \right] (\varphi G) \\ - \left(\frac{\alpha^3 n p k_1^2}{2(\alpha - 1)^2} t^2 + \frac{\alpha n q}{2} t^2 \widehat{K} \varphi^2 \right),$$

where C_4 depends on n . We see that the left hand side of the last inequality is a quadratic polynomial in (φG) , then using the quadratic formula and elementary inequality of the form $\sqrt{1 + a^2 + b^2} \leq 1 + a + b$ yields

$$\varphi G \leq \frac{\alpha n p}{4} + \frac{\alpha n p}{4\rho^2} C_4 t \left(\frac{p \alpha^3}{(\alpha - 1)} + \rho \sqrt{k_1} + \frac{\rho^2}{\tau} + \rho^2 \widehat{K} \right) \\ + \frac{\alpha n p}{4} \left(\frac{2\alpha k_1}{\alpha - 1} t + 2\widehat{K}^{\frac{1}{2}} \varphi t \sqrt{\frac{q}{p}} \right) \\ = \frac{\alpha n p}{4} + \frac{\alpha n p}{4\rho^2} C_4 t \left(\frac{p \alpha^3}{(\alpha - 1)} + \frac{\rho^2}{\tau} + \rho^2 \widehat{K} \right) + \frac{\alpha^2 n p k_1}{2(\alpha - 1)} t + \frac{\alpha n}{2} t \widehat{K}^{\frac{1}{2}} \sqrt{p q}.$$

Recall that we picked up $\varphi(x, t)$ such that $0 \leq \varphi \leq 1$ and particularly $\varphi(x, t) = 1$ in $\mathcal{Q}_{2\rho}$ and since (x_0, t_0) is a maximum point for (φG) in $\mathcal{Q}_{2\rho}$, we have $G(x, \tau) = (\varphi G)(x, \tau) \leq (\varphi G)(x_0, t_0)$. Hence

$$\frac{1}{t} G(x, t) \leq \frac{\alpha n p}{4t} + \frac{\alpha n}{4\rho^2} C_4 \left(\frac{\alpha p}{(\alpha - 1)} + \frac{\rho^2}{\tau} + \rho^2 \widehat{K} \right) + \frac{\alpha^2 n p}{2(\alpha - 1)} k_1 + \frac{\alpha n}{2} \widehat{K}^{\frac{1}{2}} \sqrt{p q} \quad (3.14)$$

for all $x \in M$ such that $\text{dist}(x, x_0, \tau) < \rho$ and since $\tau \in (0, T]$ was arbitrarily chosen. This formula implies the estimate (3.7) which was to be proved. $\square$

Let M be an n -dimensional compact (or noncompact without boundary) manifold with bounded Ricci curvature, using previous Lemma 3.1 and local gradient Theorem 3.2, we now present global estimates for the positive solutions to the heat equation when the metric evolves by the Ricci flow.

Theorem 3.3. (Global Estimates)

Let $(M, g(t))$ be a complete manifold and $g(t)$ solves the Ricci flow (3.1) such that its Ricci curvature is bounded for all $(x, t) \in M \times [0, T]$. Suppose $u = u(x, t) > 0$ is any positive solution to the heat equation (3.2), then we have for $-k_1g(x, t) \leq Rc(x, t) \leq k_2g(x, t)$ and $\alpha > 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{\alpha np}{4t} + \frac{\alpha^2 np}{2(\alpha - 1)} k_1 + \frac{\alpha n}{2} (k_1 + k_2) \sqrt{pq} \quad (3.15)$$

for all $(x, t) \in M \times [0, T]$.

Moreover, if we assume M has nonnegative Ricci curvature in the case of compact manifold, i.e., for $0 \leq Rc(x, t) \leq kg(x, t)$, we have

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{\alpha np}{4t} + \frac{\alpha n}{2} k \sqrt{pq} \quad (3.16)$$

for all $(x, t) \in M \times [0, T]$ and $\alpha \geq 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$.

The extreme case when $\alpha = 1$ is the cheapest to prove where we choose $p = q = 2\alpha$ and the result immediately yields

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{n}{2t} + nk. \quad (3.17)$$

Proof. Set $f = \log u$ and allow G to remain as before, calculating at the maximum point $(x_0, t_0) \in M \times (0, T]$, we will show that a new function

$$\tilde{G} = G - \frac{\alpha^2 np}{2(\alpha - 1)} tk_1 - \frac{\alpha n}{2} t(k_1 + k_2) \sqrt{pq} \quad (3.18)$$

satisfies the inequality

$$\tilde{G} \leq \frac{\alpha np}{4} + \frac{\alpha^2 np}{2(\alpha - 1)} tk_1 + \frac{\alpha n}{2} t(k_1 + k_2) \sqrt{pq}. \quad (3.19)$$

It suffices to prove that $\tilde{G} \leq \alpha np$ for any $(x, t) \in M \times (0, T]$ and the inequality (4.4) will follow at once. We now show this by contradiction. Suppose $\tilde{G} > \alpha np$ and that $\tilde{G}$ has its maximum at the point (x_0, t_0) , then we know that

$$\nabla \tilde{G}(x_0, t_0) = 0, \quad \Delta \tilde{G}(x_0, t_0) \leq 0, \quad \frac{\partial}{\partial t} \tilde{G}(x_0, t_0) \geq 0, \quad \text{and} \quad (\Delta - \partial_t) \tilde{G}(x_0, t_0) \leq 0$$

then by Lemma 3.1, we have $0 \geq (\Delta - \partial_t)\tilde{G} \geq (\Delta - \partial_t)G$. Noticing that

$$\begin{aligned} (|\nabla f|^2 - f_t)^2 &= \frac{1}{\alpha^2}(|\nabla f|^2 - \alpha f_t)^2 + \frac{2(\alpha - 1)}{\alpha^2}(|\nabla f|^2 - \alpha f_t)|\nabla f|^2 + \frac{(\alpha - 1)^2}{\alpha}|\nabla f|^4 \\ &= \frac{1}{\alpha^2}\left(\frac{G}{t_0}\right)^2 + \frac{2(\alpha - 1)}{\alpha^2}|\nabla f|^2\left(\frac{G}{t_0}\right) + \frac{(\alpha - 1)^2}{\alpha}|\nabla f|^4. \end{aligned}$$

By Lemma 3.1

$$\begin{aligned} 0 \geq (\Delta - \partial_t)G &\geq -2\nabla f, \nabla G + \frac{2\alpha}{np}t_0(|\nabla f|^2 - \partial_t f)^2 - (|\nabla f|^2 - \alpha \partial_t f) \\ &\quad - 2\alpha t_0 k_1 |\nabla f|^2 - \frac{\alpha n q}{2}t_0(k_1 + k_2)^2. \end{aligned}$$

Following the calculation in Theorem 3.2 we obtain

$$\frac{G}{t_0} \leq \frac{\alpha np}{4t_0} + \frac{\alpha^2 np}{2(\alpha - 1)}k_1 + \frac{\alpha n}{2}(k_1 + k_2)^2 \sqrt{pq}$$

after sending ρ to ∞ . Consequently, we have the following inequality

$$0 \geq \frac{2t_0}{\alpha np}\left(\frac{G}{t_0}\right)^2 - \frac{G}{t_0} - \left(\frac{\alpha^2 np}{2(\alpha - 1)}k_1 + \frac{\alpha n}{2}(k_1 + k_2)^2 \sqrt{pq}\right)$$

resulting into a quadratic inequality, and since from (3.18)

$$\frac{G}{t_0} = \frac{\tilde{G}}{t_0} + \frac{\alpha^2 np}{2(\alpha - 1)}k_1 + \frac{\alpha n}{2}(k_1 + k_2)^2 \sqrt{pq},$$

$$\frac{2t_0}{\alpha np}\left(\frac{G}{t_0}\right)^2 - \frac{G}{t_0} - \frac{\alpha^2 np}{2(\alpha - 1)}k_1 \leq 0. \quad (3.20)$$

Using the quadratic formula, we have

$$\frac{G}{t_0} \leq \frac{\alpha np}{4t_0} \left\{ 1 + \sqrt{1 + \frac{4\alpha np}{(\alpha - 1)}k_1} \right\},$$

which obviously implies that $\tilde{G} \leq \alpha np$, a contradiction by the assumption (3.19). By definition of $\tilde{G}$, we therefore have

$$\frac{G}{t_0} = \frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{\tilde{G}}{t_0} + \frac{\alpha^2 np}{2(\alpha - 1)}k_1 + \frac{\alpha n}{2}(k_1 + k_2)^2 \sqrt{pq}.$$

The desired estimate follows since t_0 was arbitrarily chosen. $\square$

Corollary 3.4. *Let $(M, g(t))$ be a complete manifold and $g(t)$ solves the Ricci flow (3.1) such that its Ricci curvature satisfies $-k_1 g(x, t) \leq Rc(x, t) \leq k_2 g(x, t)$ for all $(x, t) \in M \times [0, T]$. Let $u = u(x, t) > 0$ be any positive solution to the heat equation (3.2), then, for $\alpha > 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$ we have*

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{\alpha np}{4t} + C(n)\alpha^2(k_1 + k_2) \quad (3.21)$$

for all $(x, t) \in M \times [0, T]$ and absolute constant $C(n)$ depending on n only.

4. Harnack Inequalities and Estimates

As an application of our gradient estimate in the last section, we obtain some Harnack estimates for all positive solutions to the heat equation under the Ricci flow. We choose to follow the traditional approach of integrating Harnack quantity along a geodesic path connecting two points and exponentiating the result. Given $x_1, x_2 \in M$ and $t_1, t_2 \in [0, T]$ satisfying $t_1 < t_2$

$$\Theta(x_1, t_1; x_2, t_2) = \inf_{\gamma} \int_{t_1}^{t_2} \left(\left| \frac{d}{dt} \gamma(t) \right|^2 + \frac{4}{A} h \right) dt,$$

where the infimum is taken over all the smooth path $\gamma : [t_1, t_2] \rightarrow M$ connecting x_1 and x_2 . The norm $|\cdot|$ depends on t . We now present a lemma whose aim is to give an insight into how the approach goes. (See [2] and [20]).

Lemma 4.1. *Let $(M, g(t))$ be a complete solution to the Ricci flow. Let $u : M \times [0, T] \rightarrow \mathbb{R}$ be a smooth positive solution to the heat equation (1.1) and h be a C^2 -function on $M \times [0, T]$. Define $f = -\log u$ and assumed that*

$$-\frac{\partial f}{\partial t} \leq \frac{1}{A} \left(\frac{B}{t} + C - |\nabla f|^2 \right) + h, \quad (x, t) \in M \times [0, T]$$

for some $A, B, C > 0$. Then, the inequality

$$u(x_1, t_1) \leq u(x_2, t_2) \left(\frac{t_2}{t_1} \right)^{\frac{B}{A}} \exp \left(\frac{A}{4} \Theta(x_1, t_1; x_2, t_2) + \frac{C}{A} (t_2 - t_1) \right) \quad (4.1)$$

holds for all (x_1, t_1) and (x_2, t_2) such that $t_1 < t_2$.

Proof. Obtain the time differential of a function f depending on the path γ as follows ($t \in [t_1, t_2]$),

$$\frac{d}{dt} f(\gamma(t), t) = \nabla f(\gamma(t), t) \frac{d}{dt} \gamma(t) - \frac{\partial}{\partial s} f(\gamma(t), s) \Big|_{s=t}$$

$$\begin{aligned} &\leq \left| \nabla f \right| \left| \frac{d}{dt} \gamma(t) \right| + \frac{1}{A} \left(\frac{B}{t} + C - |\nabla f|^2 \right) + h \\ &\leq \frac{A}{4} \left| \frac{d}{dt} \gamma(t) \right|^2 + \frac{B}{At} + \frac{C}{A} + h. \end{aligned}$$

The last inequality was obtained by the application of completing the square method in form of a quadratic inequality satisfying $ax^2 - bx \geq -\frac{b^2}{4a}$, ($a, b > 0$). Then integrating over the path from t_1 to t_2 , we have

$$\begin{aligned} f(x_2, t_2) - f(x_1, t_1) &= \int_{t_1}^{t_2} \frac{d}{dt} f(\gamma(t), t) dt \\ &\leq \frac{A}{4} \int_{t_1}^{t_2} \left| \frac{d}{dt} \gamma(t) \right|^2 dt + \frac{B}{A} \log t \Big|_{t_1}^{t_2} + \frac{C}{A} (t_2 - t_1) + \int_{t_1}^{t_2} h dt. \end{aligned}$$

The required estimate (4.1) follows immediately after exponentiation. $\square$

Theorem 4.2. *Let $(M, g(t))$ be a complete manifold and $g(t)$ solves the Ricci flow (3.1) such that $-k_1 g(x, t) \leq Rc(x, t) \leq k_2 g(x, t)$ for some constants $k_1, k_2 > 0$ and all $(x, t) \in M \times [0, T]$. Suppose $u(x, t)$ is any positive solution to the heat equation (3.2), then for any points (x_1, t_1) and (x_2, t_2) in $M \times [0, T]$ such that $0 < t_1 < t_2 \leq T$, we have for any $\alpha > 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$*

$$\frac{u(x_1, t_1)}{u(x_2, t_2)} \leq \left(\frac{t_2}{t_1} \right)^{\frac{np}{4}} \exp \left\{ \frac{\alpha \Theta(x_1, t_1; x_2, t_2)}{4(t_2 - t_1)} + \frac{(t_2 - t_1)}{2(\alpha - 1)} A(n, \alpha, p, q, k_1, k_2) \right\} \quad (4.2)$$

where

$$\Theta(x_1, t_1; x_2, t_2) = \inf \int_0^1 |\dot{\gamma}(s)|^2 ds,$$

the infimum is taken over all the smooth paths $\gamma : [t_1, t_2] \rightarrow M$ connecting x_1 and x_2 , and $A(n, \alpha, p, q, k_1, k_2) = n\alpha p k_1 + n(\alpha - 1)(k_1 + k_2)\sqrt{pq}$.

Theorem 4.3. *Let $(M, g(t))$ be a compact Riemannian manifold evolving by the Ricci flow. Let $u(x, t)$ be a positive solution to the heat equation for $t \in (0, T]$, if the Ricci curvature is nonnegatively bounded by $0 \leq Rc \leq Kg$, then for $0 < t_1 < t_2 \leq T$, we have for any $\alpha > 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$*

$$\frac{u(x_1, t_1)}{u(x_2, t_2)} \leq \left(\frac{t_2}{t_1} \right)^{\frac{np}{4}} \exp \int_0^1 \left(\frac{\alpha |\dot{\gamma}(s)|^2}{4(t_2 - t_1)} + C(n) \frac{(t_2 - t_1)}{2(\alpha - 1)} K \sqrt{pq} \right) ds \quad (4.3)$$

where $C(n)$ is some constant depending only on n .

Proof. (of Theorem 4.2)

Let $\gamma : [0, 1]$ be a minimizing geodesic connecting points x_1 and x_2 in M such that $\gamma(0) = x_2$ and $\gamma(1) = x_1$ with $|\gamma'(s)|$ being the length of the vector $\gamma'(s)$ at time $\tau(s) = ((1-s)t_2 + st_1)$, $0 \leq t_1 < t_2 \leq T$. Define $\eta(s) = \log u(\gamma(s), (1-s)t_2 + st_1)$, clearly, $\eta(1) = \log u(x_1, t_1)$ and $\eta(0) = \log u(x_2, t_2)$.

Integrating $(\frac{\partial}{\partial s} \log u)$ along $\eta(s)$, we obtain

$$\log u(x_1, t_1) - \log u(x_2, t_2) = \int_0^1 \left(\frac{\partial}{\partial s} \log u(s) \right) ds$$

i.e.,

$$\log \left(\frac{u(x_1, t_1)}{u(x_2, t_2)} \right) = \log(\gamma(t), t) \Big|_0^1.$$

By direct computation, we have on the path $\gamma(s)$ that

$$\begin{aligned} \frac{\partial}{\partial s} \eta(s) &= \frac{d}{ds} \log u = \nabla \log u \cdot \dot{\gamma}(s) - \frac{\partial}{\partial t} \log u \\ &= \frac{\nabla u}{u} \cdot \dot{\gamma}(s) - \frac{u_t(t_2 - t_1)}{u} \\ &= (t_2 - t_1) \left(\frac{\nabla u}{u} \cdot \frac{\dot{\gamma}(s)}{t_2 - t_1} - \frac{u_t}{u} \right). \end{aligned}$$

From Theorem 3.3, we have

$$-\frac{u_t}{u} \leq -\frac{|\nabla u|^2}{\alpha u^2} + \frac{\alpha np}{2(\alpha - 1)} k_1 + \frac{n}{2} (k_1 + k_2) \sqrt{pq} + \frac{np}{4t}.$$

Therefore

$$\begin{aligned} \frac{d}{ds} \log u &\leq \frac{(t_2 - t_1)}{\alpha} \left\{ \alpha \frac{\nabla u}{u} \cdot \frac{\dot{\gamma}(s)}{t_2 - t_1} - \frac{|\nabla u|^2}{u^2} \right\} \\ &\quad + \frac{(t_2 - t_1)}{2(\alpha - 1)} \left(n\alpha p k_1 + n(\alpha - 1)(k_1 + k_2) \sqrt{pq} \right) + (t_2 - t_1) \frac{np}{4t} \\ &= -\frac{(t_2 - t_1)}{\alpha} \left(\frac{\nabla u}{u} - \frac{\dot{\gamma}(s)}{(t_2 - t_1)} \right)^2 + \frac{(t_2 - t_1)}{\alpha} \cdot \frac{\alpha^2}{4(t_2 - t_1)^2} |\dot{\gamma}|^2 \\ &\quad + \frac{(t_2 - t_1)}{2(\alpha - 1)} \left(n\alpha p k_1 + n(\alpha - 1)(k_1 + k_2) \sqrt{pq} \right) + (t_2 - t_1) \frac{np}{4t} \\ &\leq \frac{\alpha |\dot{\gamma}|^2}{4(t_2 - t_1)} + \frac{(t_2 - t_1)}{2(\alpha - 1)} \left(n\alpha p k_1 + n(\alpha - 1)(k_1 + k_2) \sqrt{pq} \right) + (t_2 - t_1) \frac{np}{4t}. \end{aligned}$$

Integrating the last inequality over the path $\gamma(s)$, we get

$$\begin{aligned} \log \frac{u(x_1, t_1)}{u(x_2, t_2)} &= \int_0^1 \log u(\gamma(s), \tau(s)) ds \leq \int_0^1 \left\{ \frac{\alpha |\dot{\gamma}(s)|^2}{4(t_2 - t_1)} \right. \\ &\quad \left. + \frac{(t_2 - t_1)}{2(\alpha - 1)} \left(n\alpha p k_1 + n(\alpha - 1)(k_1 + k_2)\sqrt{pq} \right) \right\} ds \\ &\quad + (t_2 - t_1) \frac{np}{4} \int_0^1 \frac{ds}{(1-s)t_2 + st_1}, \end{aligned}$$

which at once yield the required Harnack estimate by simply exponentiating. $\square$

Proof. (of Theorem 4.3) We can follow Theorem 3.3 on condition $0 \leq Rc \leq Kg$ to obtain the Harnack inequality

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{\alpha np}{4t} + \frac{\alpha n}{2} k \sqrt{pq} \quad \text{with } \alpha \geq 1 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}. \quad (4.4)$$

with $\alpha \geq 1$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$. Calculating as in the proof of Theorem 4.2, the desired Harnack estimate follows immediately. $\square$

Corollary 4.4. *With the same hypothesis and notation as in Theorem 4.3, it holds for any $\alpha > 1$ that*

$$\frac{u(x_1, t_1)}{u(x_2, t_2)} \leq \left(\frac{t_2}{t_1} \right)^{\frac{\alpha n}{2}} \exp \int_0^1 \left(\frac{\alpha |\dot{\gamma}(s)|^2}{4(t_2 - t_1)} + C(n) \frac{\alpha(t_2 - t_1)}{2(\alpha - 1)} K \right) ds \quad (4.5)$$

where $C(n)$ is some constant depending only on n .

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