EXISTENCE THEOREM FOR AN INITIAL VALUE PROBLEM
OF FRACTIONAL ORDER IN $L_1(0,1)$

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Abstract: In this paper we study the existence of solution $x \in L_1(0,1)$
for a functional integral equation. As an application we prove the existence of
solution for an initial value problem of the fractional order. The main tools used
are Schauder fixed point theorem, Lusin theorem and Scorza Dragoni theorem.

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1. Introduction

The class of the integral equations, arises in several fields such as mathematical
analysis, nonlinear functional analysis and mathematical physics [8],[9],[11],[15].

In this work, we use Schauder fixed point theorem and a set of conditions
on $f_1, f_2, k_1, k_2$ and $g$ to prove the existence of $L_1$–solution of equation

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\[ x(t) = g(t) + \int_{0}^{1} k_{1}(t,s) f_{1} \left( s, f_{2} \left( s, \int_{0}^{s} k_{2}(s, \tau) x(\tau) \, d\tau \right) \right) \, ds, \quad t \in (0,1). \quad (1) \]

The set of conditions we imposed to prove the existence of \( L_{1} \)-solution of equation (1), turn to be naturally satisfied in some applications. In particular, we can apply our existence result to establish the existence of \( L_{1} \)-solution of the initial value problem of fractional order

\[ \frac{dy(t)}{dt} = g(t) + \int_{0}^{1} k_{1}(t,s) f_{1} \left( s, f_{2} \left( s, D^{\beta}y(s) \right) \right) \, ds, \quad \text{and} \quad t > 0 \quad (2) \]

\[ y(0) = y_{0} \]

where \( D^{\beta}y(s) \) is the fractional derivative of \( y \) and \( \beta \in (0,1] \).

Sections 2, 3 cover the relevant material needed in our work. The main result is given in Section 4. We discuss a special case (convolution type) of our integral equation (1) in Section 5. Finally, our application in initial value problem of fractional order is presented in Section 6.

2. Preliminaries

In this section we recall some notations and results that will be needed to establish the main result.

Define \( L_{p} = L_{p}(0,1) \), \( 1 \leq p < \infty \) as the set of all measurable functions on the interval \( (0,1) \) where \( |x|^{p} \) is Lebesgue integrable function on the interval \( (0,1) \), for such function we define its norm

\[ \|x\|_{p} = \left\{ \int_{0}^{1} |x(t)|^{p} \, dt \right\}^{\frac{1}{p}}. \]

**Definition 1.** Assume that \( f : (0,1) \times R \rightarrow R \) satisfies Carathéodory conditions, that is \( f(t,x) \) is measurable in \( t \) for any \( x \in R \) and continuous in \( x \) for almost all \( t \in (0,1) \), \( x \in R \). Then to every function \( x \) that is measurable on the interval \( (0,1) \) we may assign the function

\[ (Fx)(t) = f(t,x(t)), t \in (0,1) \]

The operator \( F \) defined in this way is called the superposition operator generated by the function \( f \).
We remark that this operator is one of the simplest and most important operators that is investigated in nonlinear functional analysis [1],[2],[4],[5],[6],[7]. Furthermore, we have the following theorem [10],[12].

**Theorem 2.** Let the function $f$ satisfy the conditions in Definition 1. The superposition operator $F$ generated by the function $f$ maps continuously the space $L_1(0,1)$ into itself if and only if

$$|f(t,x)| \leq a(t) + b|x|,$$

for all $t \in (0,1)$ and $x \in R$, where $a$ is a function belong to $L_1(0,1)$ and $b$ is a nonnegative constant.

**Definition 3.** Suppose $E$ is a real Banach space. A mapping $H : E \to E$ is said to be completely continuous if $H$ is continuous and $H(Y)$ is relatively compact for every bounded subset $Y$ of $E$.

The next theorems will be used to prove our result [16],[17],[18].

**Theorem 4** (Schauder). Let $C$ be a nonempty, convex, closed, and bounded subset of a Banach space $E$. Let $H : C \to C$ be a completely continuous mapping. Then $H$ has at least one fixed point in $C$.

**Theorem 5** (Lusin). Let $\varphi : [0,1] \to R$ be measurable function. for any $\epsilon > 0$ there is a closed subset $A_\epsilon$ of $[0,1]$, with $\text{meas.}(A_\epsilon^c) < \epsilon$, such that $\varphi|_{A_\epsilon}$ is continuous.

**Theorem 6** (Scorza Dragoni). Let $f : [0,1] \times R \to R$ be a function satisfying Carathéodory hypothesis. Then for each $\epsilon > 0$ there exists a closed subset $D_\epsilon$ of $[0,1]$, with $\text{meas.}([0,1] - D_\epsilon) = \text{meas.}D_\epsilon^c < \epsilon$ and such that $f \mid_{D_\epsilon \times R}$ is continuous.

### 3. Fractional Calculus

We give the definition of both differential and integral operator of fractional order. Furthermore, we give some results that are needed in the development of our main result [13],[14].

**Definition 7.** Let $f \in L_1[0,1]$ and $\beta \in R_+ = [0,\infty)$. The Riemman-Liouville (R-L) fractional order integral of the function $f$ of order $\beta$ is defined as

$$I_0^\beta f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) \, ds, \quad 0 \leq t \leq 1.$$
Definition 8. Let \( g(t) \) be an absolutely continuous function on \([0,1]\). Then the fractional derivative of order \( \alpha \in (0,1) \) of \( g(t) \) is defined as

\[
D_\alpha^a g(t) = I_{\alpha}^{1-a} D g(t), \quad D = \frac{d}{dt}.
\]

Next, we list some basic properties of differential and integral operators of fractional order that will be used frequently in this work [5].

1. The operator \( I_\alpha^a \) maps \( L_1 \) into itself continuously.

2. Let \( f \in L_1 \) and \( \alpha, \beta \in (0,1] \), then \( I_\alpha^a I_\beta^a f(t) = I_{\alpha+\beta}^a f(t) \). Furthermore, \( (I_\alpha^n f(t) = I_n^{\alpha} f(t), n = 1, 2, 3, \ldots \).

3. Let \( \alpha, \beta \in (0,1] \) and \( f, Df \in L_1 \), we have \( DI_\alpha^a f(t) = I_\alpha^a D f(t) \), when \( f(a) = 0 \).

4. Existence of Solution

In this section, we present our main result by proving the existence of solutions of the nonlinear integral equation of mixed type (1). Let us suppose the operator,

\[
T x(t) = g(t) + \int_0^1 k_1(t,s) f_1 \left( s, f_2 \left( s, \int_0^s k_2(s,\tau) x(\tau) d\tau \right) \right) ds \quad (3)
\]

First we impose the following conditions on the functions appearing in equation (3).

\( (i) \) \( g \in L_1 = L_1(0,1) \), a.e. nonnegative on (0,1).

\( (ii) \) \( f_i : (0,1) \times R \rightarrow R, \ i = 1, 2 \) satisfy Carathéodory condition i.e \( f \) is measurable in \( t \) for any \( x \in R \) and continuous in \( x \) for almost all \( t \in (0,1) \).

\( (iii) \) There are two functions \( a_i \in L_1, \ i = 1, 2 \) and two positive constants \( b_i, \ i = 1, 2 \) such that

\[
|f_i(t,x(t))| \leq a_i(t) + b_i |x|, \quad i = 1, 2
\]

for all \( t \in (0,1) \).
(iv) $k_i : (0,1) \times (0,1) \to R, i = 1,2$ are measurable and the integral operators $K_i, i = 1,2$ maps the space $L_1$ into itself.

(v) The positive constant $c$ satisfies $c = b_1 b_2 \|K_1\|\|K_2\| < 1$.

Now we have the following lemma which is needed in proving our main result.

**Lemma 9.** Let the above hypotheses $(i) \to (v)$ be satisfied. Then there exists a unique a.e. nonnegative function $x_0$ belonging to $L_1$, such that

$$x_0(t) = g(t) + \int_0^1 k_1(t,s) \left\{ a_1(s) + b_1 \left( a_2(s) + b_2 \int_0^s k_2(s,\tau)x_0(\tau)d\tau \right) \right\} ds.$$  \hspace{1cm} (4)

**Proof.** Set

$$\psi(t) = g(t) + \int_0^1 k_1(t,s) \{a_1(s) + b_1 a_2(s)\} ds$$

clearly $\psi \in L_1$ according to condition $(iv)$. Let $B_r = \{x : x \in L_1, \|x\| \leq r\}$ where,

$$r = \frac{\|\psi\|}{1 - c} = \frac{\|\psi\|}{1 - b_1 b_2 \|K_1\|\|K_2\|}.$$

Next, we consider the operator $A$ that is defined by

$$Ax(t) = g(t) + \int_0^1 k_1(t,s) \left\{ a_1(s) + b_1 \left( a_2(s) + b_2 \int_0^s k_2(s,\tau)x(\tau)d\tau \right) \right\} ds$$

clearly, the operator $A$ maps the space $L_1$ into itself . Moreover we can get, for $x \in B_r$

$$\|Ax\| \leq \int_0^1 \left\{ \left| g(t) \right| + \int_0^1 \left| k_1(t,s) \{a_1(s) + b_1 a_2(s)\} \right| ds \right\} dt + \left\{ \int_0^1 \left| b_1 b_2 \left| k_1(t,s) \right| \left( \int_0^s \left| k_2(s,\tau) \right| |x(\tau)| d\tau \right) ds \right\} dt$$

$$\leq \|\psi\| + b_1 b_2 \|K_1\|\|K_2\|\|x\|$$

$$\leq \|\psi\| + b_1 b_2 \|K_1\|\|K_2\| r$$
≤ \|\psi\| + b_1 b_2 \|K_1\| \|K_2\| \left( \frac{\|\psi\|}{1 - b_1 b_2 \|K_1\| \|K_2\|} \right) = r

This shows that \( A \) maps \( B_r \) into itself.

Next, let \( Q_r \) be subset of \( B_r \) consisting of all functions that are a.e. nonnegative on \((0,1)\), \( Q_r = \{ x : x \in B_r, x(t) \geq 0 \text{ a.e.} \} \). Furthermore, \( Q_r \) is closed subset of \( L_1 \) to show this, let us take \( \{ x_n \} \) be a convergent sequence of elements in \( Q_r \) converging to \( x \) (in norm of \( L_1(0,1) \), \( \| x_n - x \|_{L_1} \to 0 \)) this implies that the sequence \( \{ x_n \} \) converges in measure to \( x \). Using Vitali theorem \([19]\), we deduce that the sequence \( \{ x_n \} \) contains a subsequence \( \{ x_{n_k} \} \) that converges a.e. to \( x \) on \((0,1)\). Since \( \{ x_n \} \) is nonnegative sequence, then its limit \( x \) is also nonnegative on \((0,1)\). Which mean that \( x \in Q_r \) and hence \( Q_r \) is closed subset of \( L_1 \) and so it is a complete metric space.

Now, we prove that \( A|_{Q_r} \) is contraction mapping. Let \( x_1, x_2 \in Q_r \) we have,

\[
\|Ax_1 - Ax_2\| = \int_{0}^{1} |Ax_1(t) - Ax_2(t)| \, dt \\
\leq \int_{0}^{1} \int_{0}^{1} |k_1(t,s)| \left\{ b_1 b_2 \int_{0}^{1} |k_2(s,\tau)| \, d\tau \int_{0}^{1} |x_1(\tau) - x_2(\tau)| \, d\tau \right\} ds \, dt \\
\leq b_1 b_2 \|K_1\| \|K_2\| \|x_1 - x_2\|.
\]

Since \( 0 < b_1 b_2 \|K_1\| \|K_2\| < 1 \), then we deduce that \( A|_{Q_r} \) is contraction mapping. Then there exists a unique a.e. nonnegative function \( x_0 \) belonging to \( L_1 \) that satisfied equation (4).

**Theorem 10.** Assume that the conditions \((i) \rightarrow (v)\) are satisfied and \( k \) satisfies the Carathédory condition. Then the equation (1) has at least one solution \( x \in L_1 \).

**Proof.** We have two cases.

First, assume that \( x_0 \) the zero element of \( L_1 \) solves equation (4) and let,

\[
y(t) = g(t) + \int_{0}^{1} k_1(t,s) f_1 \left( s, f_2 \left( s, \int_{0}^{s} k_2(s,\tau) x_0(\tau) \, d\tau \right) \right) \, ds \quad t \in (0,1)
\]

then from condition \((iii)\) we get,

\[
|y(t)| \leq |g(t)| + \int_{0}^{1} |k_1(t,s)| \left\{ a_1(s) + b_1(a_2(s) + b_2 \int_{0}^{s} k_2(s,\tau) x_0(\tau) \, d\tau \right\} ds.
\]
\[ |y(t)| \leq x_0(t), \text{ a.e.} \]

From Lemma 9, we get \( y = 0 \) and hence \( x_0 \) solve equation (1).

Second, we assume that \( x_0 \) is not the zero element of \( L_1 \) and we consider the set \( Q = \{ y : y \in L_1, |y(t)| \leq x_0(t), \text{ a.e.} \} \). This set \( Q \) can be shown to be nonempty, closed, bounded and convex in \( L_1 \).

Now let be the nonlinear operator associated with equation (1) and defined by

\[
Tx(t) = g(t) + \int_0^1 k_1(t,s) f_1 \left( s, f_2 \left( s, \int_0^s k_2(s,\tau) x(\tau) \, d\tau \right) \right) \, ds, \quad t \in (0,1).
\]

Based on our assumptions, we can see that \( T \) maps \( L_1 \) into itself continuously, we will prove that \( T : Q \to Q \). Let \( x \in Q \) then

\[
|Tx| = |g(t) + \int_0^1 k_1(t,s) f_1 \left( s, f_2 \left( s, \int_0^s k_2(s,\tau) x(\tau) \, d\tau \right) \right) \, ds| \\
\leq |g(t)| + \int_0^1 |k_1(t,s)| \{ \|a_1(s)\| + b_1(\|a_2(s)\| + b_2 \int_0^s |k_2(s,\tau)||x_0(\tau)||d\tau|) \} ds \\
\leq x_0(t), \text{ a.e.}
\]

Hence \( Tx \) belong to \( Q \).

In the following we show that the set \( T(Q) \) is relatively compact in \( L_1 \). So the Schauder fixed point principle works to get our aim. To achieve this we assume that \( \{y_n\} \) is a sequence in \( Q \) and fixed \( \epsilon > 0 \). Upon using Theorem 5 and Theorem 6 imply that there is a closed subset \( D_\epsilon \) of \([0,1]\) with means.\( D_\epsilon^c < \epsilon(D_\epsilon^c = [0,1] - D_\epsilon) \) such that all the restriction \( g|_{D_\epsilon}, x_0|_{D_\epsilon} \) and \( k_1|_{D_\epsilon \times [0,1]}, k_2|_{D_\epsilon \times [0,1]} \) are uniformly continuous. In what follows we show that \( \{y_n\} \) is equicontinuous on \( D_\epsilon \), for that let \( \epsilon_1 > 0 \) be given since \( g \) is uniformly continuous on \( D_\epsilon \), there exists \( \delta > 0 \) such that \( t', t'' \in D_\epsilon, |t'' - t'| < \delta \) implies that

\[
|g(t'') - g(t')| < \frac{\epsilon_1}{2}
\]

also \( k_1|_{D_\epsilon \times [0,1]} \) is uniformly continuous, there exists \( \delta > 0 \) such that \( t', t'' \in D_\epsilon, |t'' - t'| < \delta \) we have

\[
|k_1(t'',s) - k_1(t',s)| < \frac{\epsilon_1}{2\gamma}
\]
where 
\[ \gamma = \|a_1\| + b_1 \|a_2\| + b_1 b_2 \|K_2\| \|x_0\|. \]

Finally, for \( t', t'' \in D_\epsilon, |t'' - t'| < \delta \) we get

\[
|Ty_n(t'') - Ty_n(t')| \leq |g(t'') - g(t')| + \int_0^1 |k_1(t'', s) - k_1(t', s)| \cdot \left\{ a_1(s) + b_1(a_2(s) + b_2 \int_0^s k_2(s, \tau) x_0(\tau) d\tau) \right\} ds
\]

\[
\leq \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2\gamma} \gamma = \epsilon_1.
\]

Also, the sequence \( \{Ty_n\} \) is a sequence of uniformly bounded function, since

\[
|Ty_n(t)| \leq |g(t)| + \int_0^1 |k_1(t, s)| \left\{ |a_1(s)| + b_1(|a_2(s)| + b_2 \int_0^s |k_2(s, \tau)| |x_0(\tau)| d\tau) \right\} ds \leq x_0(t), a.e.
\]

Hence

\[
\|Ty_n\|_{C(D_\epsilon)} \leq \|x_0\|_{C(D_\epsilon)}, \forall n.
\]

This implies that the sequence \( \{Ty_n\} \) is a sequence of uniformly bounded and equicontinuous functions on \( D_\epsilon \). Hence, in view of Ascoli-Arzëla theorem [3] we deduce that \( \{Ty_n\} \) is relatively compact subset of \( C(D_\epsilon) \). Now, let us observe that the above proof does not depend on the choice of \( \epsilon \). Thus we construct a sequence \( \{D_k\} \) of closed subsets of \([0, 1]\) with \( \text{meas.} D_k \to 0 \) such that \( \{Ty_n\} \) is relatively compact subset of \( C(D_k) \). This implies that \( \{Ty_n\} \) has a convergent subsequence in each \( C(D_k), k = 1, 2, \ldots \). But \( C(D_k) \) is a complete metric space, hence this subsequence \( \{Ty_n\} \) is Cauchy sequence in each \( C(D_k), k = 1, 2, \ldots \), for any \( \epsilon > 0 \), and \( n, m \) arbitrary large \( \|Ty_m - Ty_n\|_{C(D_k)} < \epsilon \)

But we want to prove that the set \( T(Q) \) is relatively compact in \( L_1 \) that is \( \overline{T(Q)} \) is compact in \( L_1 \). To do this, we will prove that the sequence \( \{Ty_n\} \) is convergent sequence in \( L_1 \). Since \( L_1 \) is complete metric space, then it is sufficient to prove that the sequence \( \{Ty_n\} \) is Cauchy sequence in \( L_1 \).

Let \( \eta > 0 \) be given, we will show that there exists \( N(\eta) \) such that
\[ \|Ty_m - Ty_n\|_{L_1} < \eta, \text{ whenever } n, m > N(\eta). \] Given \( \eta > 0 \), there is a \( \delta > 0 \) such that \( \text{meas.} D_\delta < \delta \) implies

\[ \int_{D_\delta} x_0(s) \, ds < \frac{\eta}{4}. \]

Choose \( k^* \in \mathbb{N} \) with \( \text{meas.} D_{k^*} < \delta \) and

\[
\int_0^1 |(Ty_m)(t) - (Ty_n)(t)| \, dt = \int_{D_{k^*}} |(Ty_m)(t) - (Ty_n)(t)| \, dt + \\
+ \int_{D_{k^*}} |(Ty_m)(t) - (Ty_n)(t)| \, dt \\
\leq \int_{D_{k^*}} |(Ty_m)(t)| + |(Ty_n)(t)| \, dt + \\
+ \|Ty_m - Ty_n\|_{C(D_{k^*})} \int_{D_{k^*}} \, dt
\]

\[
\int_0^1 |(Ty_m)(t) - (Ty_n)(t)| \, dt \leq \frac{\eta}{4} + \frac{\eta}{4} + \|Ty_m - Ty_n\|_{C(D_{k^*})} (1 - \delta) \\
\leq \frac{\eta}{2} + \|Ty_m - Ty_n\|_{C(D_{k^*})}.
\]

Choose \( N \) such that \( n, m > N \) implies \( \|Ty_m - Ty_n\|_{C(D_{k^*})} < \frac{\eta}{2} \). Then

\[
\int_0^1 |(Ty_m)(t) - (Ty_n)(t)| \, dt = \frac{\eta}{2} + \frac{\eta}{2} = \eta.
\]

This means that \( \{T(y_n)\} \) is a Cauchy sequence in the space \( L_1 \) and hence the set \( T(Q) \) is relatively compact in \( L_1 \). Then \( T \) has at least one fixed point. Hence there exists at least one solution \( x \in L_1 \) of equation (1).
5. Convolution Type Integral Equation

Assume that \( k : (0,1) \rightarrow R_+ \) is an measurable function. For an arbitrary function \( x \in L_1 \) put

\[
(Kx) (t) = \int_0^t k(t-s)x(s)
\]

This operator \( K \) is a linear integral operator of convolution type and maps \( L_1 \) into itself continuously.

Now consider the following condition

\[ (vi) \ k : (0,1) \rightarrow R_+ \text{ and } k \in L_1. \]

Then we have the following corollary

**Corollary 11.** Let the hypotheses \((i) \rightarrow (vi)\) be satisfied. Then there exists a unique a.e. nonnegative function \( x_0 \) belonging to \( L_1 \) such that

\[
x_0(t) = g(t) + \int_0^1 k_1(t,s) \left\{ a_1(s) + b_2(a_2(s) + b_2 \int_0^s k_2(s-\tau)(a_2(\tau)+x_0(\tau))d\tau \right\} ds.
\]

Furthermore, the equation

\[
x(t) = g(t) + \int_0^1 k_1(t,s) f_1 \left( s, f_2 \left( s, \int_0^s k_2(s-\tau)x(\tau)d\tau \right) \right) ds, \quad t \in (0,1)
\]

has at least one integrable solution \( x \in L_1 \).

6. Initial Value Problems of Fractional Order

As a special case of equation (5), we consider

\[
x(t) = g(t) + \int_0^1 k_1(t,s) f_1 \left( s, f_2 \left( s, \int_0^s \frac{(\tau-s)^{-\beta}}{\Gamma(1-\beta)}x(\tau)d\tau \right) \right) ds, \quad t \in (0,1)
\]
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where \( k_2 (\tau - s) = \frac{(\tau - s)^{-\beta}}{\Gamma(1-\beta)} \). Equation (6) is an integral equation of fractional order that can be written in the form

\[
x(t) = g(t) + \int_0^1 k_1(t,s) f_1 \left( s, f_2 \left( s, I^{1-\beta} x(s) \right) \right) ds.
\]

(7)

Clearly, according to Corollary 11, equation (7) has at least one solution \( x \in L_1 \).

**Definition 12.** By a solution of the initial value problem (2) we mean an absolutely continuous function \( x \) satisfies the initial value problem (2).

**Theorem 13.** Let \( \beta \in (0,1] \). Let \( k_1 \) and \( f_1 \) satisfy the conditions of Theorem 10. If \( c = b_1 \|K_1\| < 1 \), then the initial value problem (2) has at least one solution \( x \in L_1 \).

**Proof.** Let \( x \) be a solution of equation (7). Putting,

\[
y(t) = \int_0^t x(\tau) d\tau.
\]

Since \( x \) is integrable on \((0,1)\), then,

\[
Dy(t) = D \int_0^t x(\tau) d\tau \quad a.e.
\]

where \( D = \frac{d}{dt} \).

Moreover, the integral \( \int_0^t x(\tau) d\tau \) of integrable function \( x \) is absolutely continuous then,

\[
Dy(t) = DI^1_0 x(t).
\]

Then we have,

\[
Dy(t) = x(t) \quad a.e.
\]

Furthermore, we have,

\[
I^{1-\beta} Dy(t) = I^{1-\beta} x(t) \\
D^\beta y(t) = I^{1-\beta} x(t).
\]
Consequently, we have,
\[
\frac{dy(t)}{dt} = Dy(t) = g(t) + \int_0^1 k_1(t,s) f_1\left(s, f_2\left(s, D^\beta y(s)\right)\right) ds.
\]

Since \(x\) is integrable and absolutely continuous, then
\[
Dy(\tau) = x(\tau)
\]
\[
\int_0^t \frac{dy(\tau)}{d\tau} d\tau = \int_0^t x(\tau) d\tau
\]
\[
y(t) - y_0 = \int_0^t x(\tau) d\tau.
\]

Clearly, \(y(0) = y_0\).

We deduced that this \(y\) is an absolutely continuous function satisfies the initial value problem (2). Hence our proof is complete.

References


