AFFINE MAPPINGS ON SIMPLICES

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Abstract: In this paper, we study affine mappings on domains in $\mathbb{R}^3$ that are unions of simplices. Let f_i be an affine mapping of the form $f_i(x) = A_i x$, where A_i is a 3×3 transformation matrix, on a simplex Q_i . We establish the condition of these matrices A_i in order to obtain a continuous piecewise affine mapping f on the domain Q that is the union of simplices Q_i such that $f|_{Q_i} = f_i$.

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1. Introduction

An affine mapping consists of a linear transformation followed by a translation. An affine mapping f in $\mathbb{R}^n$ is of the form

$$f(x) = Ax + b$$

where A is an $n \times n$ transformation matrix and b is a vector in $\mathbb{R}^n$. Geometrically, an affine mapping in the Euclidean space preserves the collinearity relation between points and the ratio of distances along a straight line.

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Let Q be a domain in $\mathbb{R}^3$ that is a union of simplices Q_i whose interiors are disjoint. Consider an affine mapping on each simplex Q_i of the form $f_i(x) = A_i x$, where A_i is a 3×3 transformation matrix. We want to study the condition of these matrices A_i in order to obtain a continuous piecewise affine mapping f on the domain Q such that $f|_{Q_i} = f_i$. By the Singular value decomposition theorem, each A_i can be written as $U_i D_i V_i^T$, where U_i and V_i are orthogonal matrices and D_i is a non-singular diagonal matrix. Notice that $A_i^T A_i = V_i D_i^2 V_i^T$ is independent from U_i . To obtain the necessary and sufficient conditions for A_i so that f is continuous, we will investigate the conditions for D_i and V_i that affirm the existence of U_i such that the matrix $A_i = U_i D_i V_i^T$ satisfies the desired property.

2. Two Simplices Sharing a Common Face

We study first the simplest case when the domain Q is a union of two simplices sharing a common face. Let Q_1 and Q_2 be simplices sharing a common face which lies in a plane T . We call the plane T *the common plane*. For convenience, we may assume that the origin 0 is a common vertex of Q_1 and Q_2 .

Let $f_i(x) = A_i x$ be an affine mapping on Q_i and $U_i D_i V_i^T$ be the decomposition of A_i . To obtain a continuous piecewise affine mapping f on the domain $Q = Q_1 \cup Q_2$ such that $f|_{Q_i} = f_i$, f_1 and f_2 must satisfy $f_1(x) = f_2(x)$ for all x on the plane T . That is, for all x on T

$$U_1 D_1 V_1^T x = U_2 D_2 V_2^T x. \quad (1)$$

If A_1 is given, we want to know whether there exists a matrix A_2 that satisfies (1). In [1], Sinthaveelert gave the necessary and sufficient condition for the existence of A_2 .

Theorem 1. Suppose that U_1 , D_1 , D_2 , V_1 , and V_2 are given. There exists an orthogonal matrix U_2 satisfying (1) for all $x \in T$ if, and only if, for all $h_1, h_2 \in T$

$$h_2^T V_2 D_2^2 V_2^T h_1 = h_2^T V_1 D_1^2 V_1^T h_1.$$

Proof. We suppose first that there exists a matrix U_2 satisfying the requirement. Therefore, for any $h_1, h_2 \in T$,

$$(U_2 D_2 V_2^T h_2)^T U_2 D_2 V_2^T h_1 = (U_1 D_1 V_1^T h_2)^T U_1 D_1 V_1^T h_1,$$

which gives $h_2^T V_2 D_2^2 V_2^T h_1 = h_2^T V_1 D_1^2 V_1^T h_1$ because $U_i U_i^T = I = U_i^T U_i$.

Next suppose that $h_2^T V_2 D_2^2 V_2^T h_1 = h_2^T V_1 D_1^2 V_1^T h_1$ for all $h_1, h_2 \in T$. We want to show that we can find an orthogonal matrix U_2 that satisfies (1).

Let t_1 and t_2 be vectors in $\mathbb{R}^3$ such that $\text{span}\{t_1, t_2\} = T$. Set $x = D_2 V_2^T t_1$ and $y = D_2 V_2^T t_2$. Notice that $\{x, y, x \times y\}$ is a basis for $\mathbb{R}^3$ and $\text{span}\{x, y\} = D_2 V_2^T T$. To satisfy (1), we need to define U_2 so that for all $z \in D_2 V_2^T T$

$$U_2 z = U_1 D_1 V_1^T V_2 D_2^{-1} z.$$

Indeed we need only the condition that

$$U_2 x = U_1 D_1 V_1^T V_2 D_2^{-1} x \quad \text{and} \quad U_2 y = U_1 D_1 V_1^T V_2 D_2^{-1} y.$$

Let $r = U_1 D_1 V_1^T V_2 D_2^{-1} x$, $s = U_1 D_1 V_1^T V_2 D_2^{-1} y$, and let $\gamma = \frac{\pm|x \times y|}{|r \times s|}$. Then let

$$U_2 = \begin{bmatrix} r & s & \gamma(r \times s) \end{bmatrix} \begin{bmatrix} x & y & (x \times y) \end{bmatrix}^{-1}.$$

It is easy to see that $U_2 x = r$ and $U_2 y = s$, hence U_2 fulfills the first part of the desired condition. We still have to verify that U_2 is orthogonal. Since $\{x, y, x \times y\}$ is a basis for $\mathbb{R}^3$, any vectors a and b in $\mathbb{R}^3$ can be represented as linear combinations of x , y , and $x \times y$. For instance, let $a = a_1 x + a_2 y + a_3 (x \times y)$ and $b = b_1 x + b_2 y + b_3 (x \times y)$. Note that we have

$$\begin{aligned} r^T r &= x^T (D_2^{-1})^T V_2^T V_1 D_1^2 V_1^T V_2 D_2^{-1} x \\ &= x^T (D_2^{-1})^T V_2^T V_2 D_2^2 V_2^T V_2 D_2^{-1} x = x^T x. \end{aligned}$$

Similarly, $r^T s = x^T y$, $s^T s = y^T y$, and $s^T r = y^T x$. Therefore

$$\begin{aligned} a \cdot b &= (a_1 x + a_2 y + a_3 (x \times y)) \cdot (b_1 x + b_2 y + b_3 (x \times y)) \\ &= a_1 b_1 (x^T x) + a_2 b_2 (y^T y) + a_3 b_3 ((x \times y)^T (x \times y)) \\ &\quad + a_1 b_2 (x^T y) + a_2 b_1 (y^T x) \\ &= a_1 b_1 (r^T r) + a_2 b_2 (s^T s) + a_3 b_3 \gamma^2 ((r \times s)^T (r \times s)) \\ &\quad + a_1 b_2 (r^T s) + a_2 b_1 (s^T r) \\ &= (a_1 r + a_2 s + a_3 \gamma (r \times s)) \cdot (b_1 r + b_2 s + b_3 \gamma (r \times s)) \\ &= U_2 a \cdot U_2 b. \end{aligned}$$

Hence, U_2 preserves dot products and then it also preserves lengths. Thus U_2 is orthogonal, which completes the proof. $\square$

Remark that the existence of U_2 is independent from U_1 , but to find such U_2 it definitely involves U_1 .

3. Four Simplices Sharing a Common Edge

In this section, we turn our attention to a domain that is a union of four simplices. For the case when a domain is a union of three simplices, we will discuss it in the next section. It is indeed a special case of Theorem 3 below. We want find the conditions on the given matrices D_i and V_i for f to be continuous on the domain. We first introduce the notations which will be used for the rest of this paper. Let $Q(a, b, c, d)$ denote a simplex with vertices a, b, c , and d , and let $\vec{a}_i$ denote the vector from 0 to the point a_i in $\mathbb{R}^3$.

For nonzero and distinct points a_1, a_2, a_3, a_4, a_5 in $\mathbb{R}^3$, let $Q_1 = Q_1(0, a_1, a_4, a_5)$, $Q_2 = Q_2(0, a_1, a_2, a_5)$, $Q_3 = Q_3(0, a_2, a_3, a_5)$, and $Q_4 = Q_4(0, a_3, a_4, a_5)$. Note that Q_1, Q_2, Q_3 , and Q_4 share a common edge $\vec{a}_5$. Furthermore, for $k \in \{1, 2, 3, 4\}$, $T_k = \text{span}\{\vec{a}_k, \vec{a}_5\}$ is the common plane shared by Q_k and Q_{k+1} , except for T_4 which is the common plane shared by Q_4 and Q_1 . Let $f_i(x) = A_i x$ be an affine mapping on Q_i and $U_i D_i V_i^T$ be the decomposition of A_i . For $i, j \in \{1, 2, 3, 4\}$, let

$$X_i = D_i V_i^T \vec{a}_5 \quad \text{and} \quad Y_{ij} = D_i V_i^T \vec{a}_j.$$

The following theorem gives the necessary and sufficient condition for the function f defined on $\bigcup_{i=1}^4 Q_i$ and satisfying $f|_{Q_i} = f_i$ to be a continuous function in $\bigcup_{i=1}^4 Q_i$.

Theorem 2. For $1 \leq i \leq 4$, let an orthogonal matrix V_i and a diagonal matrix D_i with positive diagonal entries be given. The following two statements are equivalent.

(i) There exist orthogonal matrices U_1, U_2, U_3 , and U_4 such that the function f defined by $f|_{Q_i} = f_i$ is continuous in $\bigcup_{i=1}^4 Q_i$, where $f_i(x) = A_i x = U_i D_i V_i^T x$ for all x in Q_i .

(ii) The matrices D_i and V_i satisfy the following face conditions : for $k = 1, 2, 3$

$$x^T V_k D_k^2 V_k^T y = x^T V_{k+1} D_{k+1}^2 V_{k+1}^T y \quad \text{for all } x, y \in T_k, \quad (2)$$

$$x^T V_1 D_1^2 V_1^T y = x^T V_4 D_4^2 V_4^T y \quad \text{for all } x, y \in T_4, \quad (3)$$

and two edge conditions :

$$\begin{aligned} & |X_2 \times Y_{22}|^2 |X_3 \times Y_{33}|^2 \{ |X_1|^2 (Y_{11} \cdot Y_{14}) - (X_1 \cdot Y_{11})(X_1 \cdot Y_{14}) \} \\ &= \left\{ |X_2|^2 (Y_{21} \cdot Y_{22}) - (X_2 \cdot Y_{21})(X_2 \cdot Y_{22}) \right\} \left\{ |X_3|^2 (Y_{32} \cdot Y_{33}) \right. \\ &\quad \left. - (X_3 \cdot Y_{32})(X_3 \cdot Y_{33}) \right\} \{ |X_4|^2 (Y_{43} \cdot Y_{44}) - (X_4 \cdot Y_{43})(X_4 \cdot Y_{44}) \} \end{aligned}$$

$$\begin{aligned}
 & -|X_3|^2(X_3 \times Y_{32} \cdot Y_{33})(X_4 \times Y_{43} \cdot Y_{44}) \Big\} \\
 & -|X_2|^2(X_2 \times Y_{21} \cdot Y_{22}) \Big\{ (X_3 \times Y_{32} \cdot Y_{33}) \{ |X_4|^2(Y_{43} \cdot Y_{44}) \\
 & - (X_4 \cdot Y_{43})(X_4 \cdot Y_{44}) \} + (X_4 \times Y_{43} \cdot Y_{44}) \{ |X_3|^2(Y_{32} \cdot Y_{33}) \\
 & - (X_3 \cdot Y_{32})(X_3 \cdot Y_{33}) \} \Big\}
 \end{aligned}$$

and

$$\begin{aligned}
 & |X_2 \times Y_{22}|^2 |X_3 \times Y_{33}|^2 (X_1 \times Y_{11} \cdot Y_{14}) \\
 & = \{ |X_4|^2(Y_{43} \cdot Y_{44}) - (X_4 \cdot Y_{43})(X_4 \cdot Y_{44}) \} \{ |X_3|^2(Y_{32} \cdot Y_{33}) \\
 & - (X_3 \cdot Y_{32})(X_3 \cdot Y_{33}) \} (X_2 \times Y_{21} \cdot Y_{22}) \\
 & + \{ |X_2|^2(Y_{21} \cdot Y_{22}) - (X_2 \cdot Y_{21})(X_2 \cdot Y_{22}) \} \{ |X_4|^2(Y_{43} \cdot Y_{44}) \\
 & - (X_4 \cdot Y_{43})(X_4 \cdot Y_{44}) \} (X_3 \times Y_{32} \cdot Y_{33}) \\
 & + \{ |X_2|^2(Y_{21} \cdot Y_{22}) - (X_2 \cdot Y_{21})(X_2 \cdot Y_{22}) \} \{ (|X_3|^2(Y_{32} \cdot Y_{33}) \\
 & - (X_3 \cdot Y_{32})(X_3 \cdot Y_{33})) \} (X_4 \times Y_{43} \cdot Y_{44}) \\
 & - |X_3|^2(X_2 \times Y_{21} \cdot Y_{22})(X_3 \times Y_{32} \cdot Y_{33})(X_4 \times Y_{43} \cdot Y_{44}).
 \end{aligned}$$

Proof. Suppose that (i) holds. By Theorem 1, we obtain (2) and (3). It remains to be showed that the matrices D_i and V_i satisfy the edge conditions. From (2), we obtain

$$X_1 \cdot Y_{11} = \vec{a}_5^T V_1 D_1^2 V_1^T \vec{a}_1 = \vec{a}_5^T V_2 D_2^2 V_2^T \vec{a}_1 = X_2 \cdot Y_{21}.$$

Similarly, we have

$$X_2 \cdot Y_{22} = X_3 \cdot Y_{32}, \quad X_4 \cdot Y_{44} = X_1 \cdot Y_{14}, \quad X_3 \cdot Y_{33} = X_4 \cdot Y_{43}$$

$$|Y_{11}|^2 = |Y_{21}|^2, \quad |Y_{22}|^2 = |Y_{32}|^2, \quad |Y_{33}|^2 = |Y_{43}|^2, \quad |Y_{44}|^2 = |Y_{14}|^2$$

$$\text{and } |X_1|^2 = |X_2|^2 = |X_3|^2 = |X_4|^2.$$

If $B = [b_1 \ b_2]$ is a 3×2 matrix, where b_1 and b_2 are column vectors, let $\widetilde{B}$ denote the 3×3 matrix $\begin{bmatrix} b_1 & b_2 & \frac{b_1 \times b_2}{|b_1 \times b_2|} \end{bmatrix}$.

Let $c_j = [\vec{a}_5 \ \vec{a}_j]$. Then we have $D_i V_i^T c_j = [X_i \ Y_{ij}]$ and

$$\widetilde{D_i V_i^T c_j} = \begin{bmatrix} X_i & Y_{ij} & \frac{X_i \times Y_{ij}}{|X_i \times Y_{ij}|} \end{bmatrix}.$$

Note that the inverse matrix of $\widetilde{D_i V_i^T c_j}$ is

$$\left(\widetilde{D_i V_i^T c_j} \right)^{-1} = \frac{1}{|X_i \times Y_{ij}|^2} \begin{bmatrix} (Y_{ij} \times (X_i \times Y_{ij}))^T \\ ((X_i \times Y_{ij}) \times X_i)^T \\ |X_i \times Y_{ij}| (X_i \times Y_{ij})^T \end{bmatrix}.$$

Without loss of generality, we may assume that $U_1 = I$. Since $U_1 D_1 V_1^T x = U_2 D_2 V_2^T x$ for all $x \in T_1$, we have

$$U_2 = \widetilde{D_1 V_1^T c_1} \left(\widetilde{D_2 V_2^T c_1} \right)^{-1}. \quad (4)$$

Similarly, we obtain

$$U_3 = \left(\widetilde{U_2 D_2 V_2^T c_2} \right) \left(\widetilde{D_3 V_3^T c_2} \right)^{-1}, \quad (5)$$

$$U_4 = \left(\widetilde{U_3 D_3 V_3^T c_3} \right) \left(\widetilde{D_4 V_4^T c_3} \right)^{-1}. \quad (6)$$

Let $W = U_3 D_3 V_3^T c_3$. From (4) and (5), we get

$$W = \left(\widetilde{D_1 V_1^T c_1} \right) \left(\widetilde{D_2 V_2^T c_1} \right)^{-1} \left(\widetilde{D_2 V_2^T c_2} \right) \left(\widetilde{D_3 V_3^T c_2} \right)^{-1} D_3 V_3^T c_3.$$

Since $U_1 D_1 V_1^T x = U_4 D_4 V_4^T x$ for all $x \in T_4$, from (6) we obtain

$$D_1 V_1^T c_4 = U_4 D_4 V_4^T c_4 = \widetilde{W} P, \quad (7)$$

where $P = \left(\widetilde{D_4 V_4^T c_3} \right)^{-1} D_4 V_4^T c_4$. The equation (7) gives a relation between D_i and V_i , which leads to the edge conditions of the part (ii). Let

$$S = \left(\widetilde{D_1 V_1^T c_1} \right) \left(\widetilde{D_2 V_2^T c_1} \right)^{-1} D_2 V_2^T c_2 \text{ and } R = \left(\widetilde{D_3 V_3^T c_2} \right)^{-1} \widetilde{D_3 V_3^T c_3}.$$

Notice that $\widetilde{W} = \widetilde{S} R$. Next we want to simplify the matrices S and R . We have

$$\begin{aligned} S &= \begin{bmatrix} X_1 & Y_{11} & \frac{X_1 \times Y_{11}}{|X_1 \times Y_{11}|} \end{bmatrix} \frac{1}{|X_2 \times Y_{21}|^2} \begin{bmatrix} ((Y_{21} \times (X_2 \times Y_{21}))^T \\ ((X_2 \times Y_{21}) \times X_2)^T \\ |X_2 \times Y_{21}| (X_2 \times Y_{21})^T \end{bmatrix} \begin{bmatrix} X_2 & Y_{22} \end{bmatrix} \\ &= \frac{1}{|X_2 \times Y_{21}|^2} \begin{bmatrix} X_1 & Y_{11} & \frac{X_1 \times Y_{11}}{|X_1 \times Y_{11}|} \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_4 \\ 0 & \alpha_5 \\ 0 & \alpha_6 \end{bmatrix} \\ &= \frac{1}{|X_2 \times Y_{21}|^2} \begin{bmatrix} \alpha_1 X_1 & \alpha_4 X_1 + \alpha_5 Y_{11} + \frac{\alpha_6}{|X_1 \times Y_{11}|} X_1 \times Y_{11} \end{bmatrix}, \end{aligned}$$

where $\alpha_1 = (Y_{21} \times (X_2 \times Y_{21})) \cdot X_2 = |X_2 \times Y_{21}|^2 = |X_1 \times Y_{11}|^2$,

$$\alpha_4 = (Y_{21} \times (X_2 \times Y_{21})) \cdot Y_{22} = |Y_{21}|^2 (X_2 \cdot Y_{22}) - (X_2 \cdot Y_{21})(Y_{21} \cdot Y_{22}),$$

$$\begin{aligned}\alpha_5 &= ((X_2 \times Y_{21}) \times X_2) \cdot Y_{22} = |X_2|^2(Y_{21} \cdot Y_{22}) - (X_2 \cdot Y_{21})(X_2 \cdot Y_{22}), \\ \alpha_6 &= |X_2 \times Y_{21}|(X_2 \times Y_{21} \cdot Y_{22}).\end{aligned}$$

Hence,

$$\tilde{S} = \frac{1}{|X_1 \times Y_{11}|^2} \begin{bmatrix} \alpha_1 X_1 & \alpha_4 X_1 + \alpha_5 Y_{11} + \frac{\alpha_6}{|X_1 \times Y_{11}|} X_1 \times Y_{11} & \alpha_1 \frac{Z}{|Z|} \end{bmatrix},$$

where

$$Z = \frac{\alpha_6(X_1 \cdot Y_{11})}{|X_1 \times Y_{11}|} X_1 - \frac{\alpha_6 |X_1|^2}{|X_1 \times Y_{11}|} Y_{11} + \alpha_5 (X_1 \times Y_{11})$$

and

$$|Z| = |X_1 \times Y_{11}|^2 |X_2 \times Y_{22}|.$$

Note that

$$\begin{aligned}\alpha_1 \frac{Z}{|Z|} &= \frac{\alpha_6(X_1 \cdot Y_{11})}{|X_1 \times Y_{11}| |X_2 \times Y_{22}|} X_1 - \frac{\alpha_6 |X_1|^2}{|X_1 \times Y_{11}| |X_2 \times Y_{22}|} Y_{11} \\ &\quad + \frac{\alpha_5}{|X_2 \times Y_{22}|} (X_1 \times Y_{11}).\end{aligned}$$

We also get

$$\begin{aligned}R &= \frac{1}{|X_3 \times Y_{32}|^2} \begin{bmatrix} (Y_{32} \times (X_3 \times Y_{32}))^T \\ ((X_3 \times Y_{32}) \times X_3)^T \\ |X_3 \times Y_{32}|(X_3 \times Y_{32})^T \end{bmatrix} \begin{bmatrix} X_3 & Y_{33} & \frac{X_3 \times Y_{33}}{|X_3 \times Y_{33}|} \end{bmatrix} \\ &= \frac{1}{|X_2 \times Y_{22}|^2} \begin{bmatrix} \gamma_1 & \gamma_4 & \gamma_7 \\ 0 & \gamma_5 & \gamma_8 \\ 0 & \gamma_6 & \gamma_9 \end{bmatrix},\end{aligned}$$

where

$$\begin{aligned}\gamma_1 &= (Y_{32} \times (X_3 \times Y_{32})) \cdot X_3 = |X_3 \times Y_{32}|^2 = |X_2 \times Y_{22}|^2, \\ \gamma_4 &= (Y_{32} \times (X_3 \times Y_{32})) \cdot Y_{33} = |Y_{32}|^2(X_3 \cdot Y_{33}) - (X_3 \cdot Y_{32})(Y_{32} \cdot Y_{33}), \\ \gamma_5 &= ((X_3 \times Y_{32}) \times X_3) \cdot Y_{33} = |X_3|^2(Y_{32} \cdot Y_{33}) - (X_3 \cdot Y_{32})(X_3 \cdot Y_{33}), \\ \gamma_6 &= |X_3 \times Y_{32}|(X_3 \times Y_{32} \cdot Y_{33}), \\ \gamma_7 &= (Y_{32} \times (X_3 \times Y_{32})) \cdot \frac{(X_3 \times Y_{33})}{|(X_3 \times Y_{33})|} = \frac{(X_3 \cdot Y_{32})}{|X_3 \times Y_{33}|} (X_3 \times Y_{32} \cdot Y_{33}), \\ \gamma_8 &= ((X_3 \times Y_{32}) \times X_3) \cdot \frac{(X_3 \times Y_{33})}{|(X_3 \times Y_{33})|} = -\frac{|X_3|^2}{|X_3 \times Y_{33}|} (X_3 \times Y_{32} \cdot Y_{33}), \\ \gamma_9 &= |X_3 \times Y_{32}|(X_3 \times Y_{32}) \cdot \frac{(X_3 \times Y_{33})}{|(X_3 \times Y_{33})|}\end{aligned}$$

$$= \frac{|X_3 \times Y_{32}|}{|X_3 \times Y_{33}|} (|X_3|^2 (Y_{32} \cdot Y_{33}) - (X_3 \cdot Y_{32})(X_3 \cdot Y_{33})).$$

Therefore

$$\begin{aligned} |X_1 \times Y_{11}|^2 |X_2 \times Y_{22}|^2 \widetilde{W} &= |X_1 \times Y_{11}|^2 |X_2 \times Y_{22}|^2 \widetilde{S}R \\ &= \begin{bmatrix} \alpha_1 X_1 & \alpha_4 X_1 + \alpha_5 Y_{11} + \frac{\alpha_6}{|X_1 \times Y_{11}|} X_1 \times Y_{11} & \alpha_1 \frac{Z}{|Z|} \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_4 & \gamma_7 \\ 0 & \gamma_5 & \gamma_8 \\ 0 & \gamma_6 & \gamma_9 \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1 \gamma_1 X_1 & \gamma_4 F_1 + \gamma_5 F_2 + \gamma_6 F_3 & \gamma_7 F_1 + \gamma_8 F_2 + \gamma_9 F_3 \end{bmatrix}, \end{aligned} \quad (8)$$

where $F_1 = \alpha_1 X_1$, $F_2 = \alpha_4 X_1 + \alpha_5 Y_{11} + \frac{\alpha_6}{|X_1 \times Y_{11}|} X_1 \times Y_{11}$, and $F_3 = \alpha_1 \frac{Z}{|Z|}$. Note that

$$P = \frac{1}{|X_4 \times Y_{43}|^2} \begin{bmatrix} (Y_{43} \times (X_4 \times Y_{43}))^T \\ ((X_4 \times Y_{43}) \times X_4)^T \\ |X_4 \times Y_{43}| (X_4 \times Y_{43})^T \end{bmatrix} \begin{bmatrix} X_4 & Y_{44} \end{bmatrix} = \frac{1}{|X_3 \times Y_{33}|^2} \begin{bmatrix} \beta_1 & \beta_4 \\ 0 & \beta_5 \\ 0 & \beta_6 \end{bmatrix},$$

where

$$\begin{aligned} \beta_1 &= (Y_{43} \times (X_4 \times Y_{43})) \cdot X_4 = |X_4 \times Y_{43}|^2 = |X_3 \times Y_{33}|^2, \\ \beta_4 &= (Y_{43} \times (X_4 \times Y_{43})) \cdot Y_{44} = |Y_{43}|^2 (X_4 \cdot Y_{44}) - (X_4 \cdot Y_{43})(Y_{43} \cdot Y_{44}), \\ \beta_5 &= ((X_4 \times Y_{43}) \times X_4) \cdot Y_{44} = |X_4|^2 (Y_{43} \cdot Y_{44}) - (X_4 \cdot Y_{43})(X_4 \cdot Y_{44}), \\ \beta_6 &= |X_4 \times Y_{43}| (X_4 \times Y_{43} \cdot Y_{44}). \end{aligned}$$

Since $\alpha_1 = |X_1 \times Y_{11}|^2$, $\gamma_1 = |X_2 \times Y_{22}|^2$, and $\beta_1 = |X_3 \times Y_{33}|^2$, by (8) we obtain

$$\alpha_1 \gamma_1 \beta_1 \widetilde{W}P = [\beta_1 \alpha_1 \gamma_1 X_1 \quad \beta_4 B_1 + \beta_5 B_2 + \beta_6 B_3],$$

where $B_1 = \alpha_1 \gamma_1 X_1$, $B_2 = \gamma_4 F_1 + \gamma_5 F_2 + \gamma_6 F_3$, and $B_3 = \gamma_7 F_1 + \gamma_8 F_2 + \gamma_9 F_3$. Since, by (7), $[X_1 \quad Y_{14}] = D_1 V_1^T c_4 = U_4 D_4 V_4^T c_4 = \widetilde{W}P$, we have

$$\alpha_1 \beta_1 \gamma_1 Y_{14} = \beta_4 B_1 + \beta_5 B_2 + \beta_6 B_3. \quad (9)$$

Since $\{X_1, Y_{11}, X_1 \times Y_{11}\}$ is a basis for $\mathbb{R}^3$, we can write Y_{14} uniquely as a linear combination of the elements of this basis. Indeed

$$Y_{14} = \frac{X_1 \times Y_{11} \cdot Y_{14}}{|X_1 \times Y_{11}|^2} (X_1 \times Y_{11}) + \alpha X_1 + \beta Y_{11}, \quad (10)$$

where

$$\alpha = \frac{1}{|X_1|^2 |Y_{11}|^2 - (X_1 \cdot Y_{11})^2} (|Y_{11}|^2 (X_1 \cdot Y_{14}) - (X_1 \cdot Y_{11})(Y_{11} \cdot Y_{14}))$$

and

$$\beta = \frac{1}{|X_1|^2 |Y_{11}|^2 - (X_1 \cdot Y_{11})^2} \left(|X_1|^2 (Y_{11} \cdot Y_{14}) - (X_1 \cdot Y_{11}) (X_1 \cdot Y_{14}) \right).$$

By substituting the values of F_1, F_2 , and F_3 into B_1, B_2 , and B_3 , we obtain

$$\begin{aligned} & \beta_4 B_1 + \beta_5 B_2 + \beta_6 B_3 \\ &= \left\{ \beta_4 \gamma_1 \alpha_1 + \beta_5 \gamma_4 \alpha_1 + \beta_5 \gamma_5 \alpha_4 + \beta_6 \gamma_7 \alpha_1 + \beta_6 \gamma_8 \alpha_4 \right. \\ & \quad \left. + \frac{(\beta_5 \gamma_6 + \beta_6 \gamma_9) \alpha_6 (X_1 \cdot Y_{11})}{|X_1 \times Y_{11}| |X_2 \times Y_{22}|} \right\} X_1 \\ & \quad + \left(\beta_5 \gamma_5 \alpha_5 + \beta_6 \gamma_8 \alpha_5 - \frac{(\beta_5 \gamma_6 + \beta_6 \gamma_9) \alpha_6 |X_1|^2}{|X_1 \times Y_{11}| |X_2 \times Y_{22}|} \right) Y_{11} \\ & \quad + \left(\frac{(\beta_5 \gamma_5 + \beta_6 \gamma_8) \alpha_6}{|X_1 \times Y_{11}|} + \frac{(\beta_5 \gamma_6 + \beta_6 \gamma_9) \alpha_5}{|X_2 \times Y_{22}|} \right) (X_1 \times Y_{11}). \end{aligned}$$

After multiplying (10) by $\alpha_1 \beta_1 \gamma_1$ and comparing the coefficients of Y_{11} and also the coefficients of $X_1 \times Y_{11}$, we obtain

$$\begin{aligned} & |X_2 \times Y_{22}|^2 |X_3 \times Y_{33}|^2 \left(|X_1|^2 (Y_{11} \cdot Y_{14}) - (X_1 \cdot Y_{11}) (X_1 \cdot Y_{14}) \right) \\ &= \beta_5 \gamma_5 \alpha_5 + \beta_6 \gamma_8 \alpha_5 - \frac{(\beta_5 \gamma_6 + \beta_6 \gamma_9) \alpha_6 |X_1|^2}{|X_1 \times Y_{11}| |X_2 \times Y_{22}|} \end{aligned}$$

and

$$\beta_1 \gamma_1 (X_1 \times Y_{11} \cdot Y_{14}) = \frac{(\beta_5 \gamma_5 + \beta_6 \gamma_8) \alpha_6}{|X_1 \times Y_{11}|} + \frac{(\beta_5 \gamma_6 + \beta_6 \gamma_9) \alpha_5}{|X_2 \times Y_{22}|}.$$

After substituting the values of α_i, β_i , and γ_i , these two equations can be simplified into two edge conditions, respectively. This proves that (i) implies (ii), as needed.

Suppose next that (ii) holds. Let $U_1 = I$. From (2) and Theorem 1, there exist orthogonal matrices U_2, U_3 , and U_4 such that

$$U_k D_k V_k^T x = U_{k+1} D_{k+1} V_{k+1}^T x, \quad \text{for all } x \in T_k,$$

where $k = 1, 2, 3$. Also from (3) and Theorem 1, there exists an orthogonal matrix U_4^* such that

$$D_1 V_1^T w = U_4^* D_4 V_4^T w, \quad \text{for all } w \in T_4.$$

Let $A_i = U_i D_i V_i^T$ for all $1 \leq i \leq 4$ and $A_4^* = U_4^* D_4 V_4^T$. On each simplex Q_i , we define $f_i(x) = A_i x$. Furthermore, we also define $f_4^*(x) = A_4^* x$ for $x \in Q_4$.

Now we have the simplices $f_1(Q_1)$ and $f_2(Q_2)$ sharing a common face $f_1(T_1) = f_2(T_1)$, the simplices $f_2(Q_2)$ and $f_3(Q_3)$ sharing a common face $f_2(T_2) = f_3(T_2)$, the simplices $f_3(Q_3)$ and $f_4(Q_4)$ sharing a common face $f_3(T_3) = f_4(T_3)$, and the simplices $f_1(Q_1)$ and $f_4^*(Q_4)$ sharing a common face $f_1(T_4) = f_4^*(T_4)$. In order to obtain a continuous piecewise affine mapping f on $\bigcup_{i=1}^4 Q_i$ such that $f|_{Q_i} = f_i$, we need to show that

$$U_4 D_4 V_4^T(T_4) = U_4^* D_4 V_4^T(T_4) = D_1 V_1^T(T_4). \quad (11)$$

Let the angle θ_1 be the angle between $f_1(T_1)$ and $f_1(T_4)$, the angle θ_2 be the angle between $f_2(T_1)$ and $f_2(T_2)$, θ_3 be the angle between $f_3(T_2)$ and $f_3(T_3)$, and θ_4 be the angle between $f_4(T_3)$ and $f_4(T_4)$. To achieve the equation (11), we only need to show that $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2\pi$. Since θ_1 is the angle between $f_1(T_1)$ and $f_1(T_4)$, we have

$$\begin{aligned} \sin \theta_1 &= \frac{|(X_1 \times Y_{11}) \times (X_1 \times Y_{14})|}{|X_1 \times Y_{11}| |X_1 \times Y_{14}|} = \frac{|X_1| |X_1 \times Y_{11} \cdot Y_{14}|}{|X_1 \times Y_{11}| |X_1 \times Y_{14}|} \quad \text{and} \\ \cos \theta_1 &= \frac{(X_1 \times Y_{11}) \cdot (X_1 \times Y_{14})}{|X_1 \times Y_{11}| |X_1 \times Y_{14}|} \\ &= \frac{|X_1|^2 (Y_{11} \cdot Y_{14}) - (X_1 \cdot Y_{11})(X_1 \cdot Y_{14})}{|X_1 \times Y_{11}| |X_1 \times Y_{14}|}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \sin \theta_2 &= \frac{|X_2| |X_2 \times Y_{21} \cdot Y_{22}|}{|X_2 \times Y_{21}| |X_2 \times Y_{22}|}, \quad \sin \theta_3 = \frac{|X_3| |X_3 \times Y_{32} \cdot Y_{33}|}{|X_3 \times Y_{32}| |X_3 \times Y_{33}|}, \\ \sin \theta_4 &= \frac{|X_4| |X_4 \times Y_{43} \cdot Y_{44}|}{|X_4 \times Y_{43}| |X_4 \times Y_{44}|}, \\ \cos \theta_2 &= \frac{|X_2|^2 (Y_{21} \cdot Y_{22}) - (X_2 \cdot Y_{21})(X_2 \cdot Y_{22})}{|X_2 \times Y_{21}| |X_2 \times Y_{22}|}, \\ \cos \theta_3 &= \frac{|X_3|^2 (Y_{32} \cdot Y_{33}) - (X_3 \cdot Y_{32})(X_3 \cdot Y_{33})}{|X_3 \times Y_{32}| |X_3 \times Y_{33}|}, \\ \cos \theta_4 &= \frac{|X_4|^2 (Y_{43} \cdot Y_{44}) - (X_4 \cdot Y_{43})(X_4 \cdot Y_{44})}{|X_4 \times Y_{43}| |X_4 \times Y_{44}|}. \end{aligned}$$

Hence, from the first edge conditions, we obtain

$$\cos \theta_1 = \cos \theta_2 (\cos \theta_3 \cos \theta_4 - \sin \theta_3 \sin \theta_4)$$

$$\begin{aligned}
 & -\sin \theta_2 (\cos \theta_3 \sin \theta_4 + \sin \theta_3 \cos \theta_4) \\
 = & \cos \theta_2 (\cos(\theta_3 + \theta_4)) - \sin \theta_2 (\sin(\theta_3 + \theta_4)) \\
 = & \cos(\theta_2 + \theta_3 + \theta_4)
 \end{aligned}$$

and, from the second edge condition,

$$\begin{aligned}
 -\sin \theta_1 &= \sin \theta_2 (\cos \theta_3 \cos \theta_4 + \sin \theta_3 \sin \theta_4) \\
 &\quad + \cos \theta_2 (\cos \theta_3 \sin \theta_4 + \sin \theta_3 \cos \theta_4) \\
 &= \sin \theta_2 (\cos(\theta_3 + \theta_4)) + \cos \theta_2 (\sin(\theta_3 + \theta_4)) \\
 &= \sin(\theta_2 + \theta_3 + \theta_4).
 \end{aligned}$$

It is readily verified that $\cos(\theta_1 + \theta_2 + \theta_3 + \theta_4) = \cos^2 \theta_1 + \sin^2 \theta_1 = 1$. Since $0 < \theta_1 + \theta_2 + \theta_3 + \theta_4 < 4\pi$, we must have $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2\pi$, as desired. This completes the proof of Theorem 3. $\square$

4. Three Simplices Sharing a Common Edge

In this section we will establish the analogous result to Theorem 3 for the case when a domain is a union of three simplices. In fact, Sinthaveelert already established a result for this case in [1]. Here we will induce the result by using Theorem 3. By using the same notations as in the previous section and considering $Q_3 = Q_4$ and $A_3 = A_4$, we get

$$X_3 = X_4, Y_{33} = Y_{43} = Y_{44}, \text{ and } Y_{13} = Y_{14}.$$

Then we derive the following theorem.

Theorem 3. *For $1 \leq i \leq 3$, let an orthogonal matrix V_i and a diagonal matrix D_i with positive diagonal entries be given. The following two statements are equivalent.*

(i) *There exist orthogonal matrices U_1, U_2 , and U_3 such that the function f defined by $f|_{Q_i} = f_i$ is continuous in $\bigcup_{i=1}^3 Q_i$, where $f_i(x) = A_i x = U_i D_i V_i^T x$ for all x in Q_i .*

(ii) *The matrices D_i and V_i satisfy the following face conditions : for $k = 1, 2$*

$$\begin{aligned}
 x^T V_k D_k^2 V_k^T y &= x^T V_{k+1} D_{k+1}^2 V_{k+1}^T y \quad \text{for all } x, y \in T_k, \\
 x^T V_1 D_1^2 V_1^T y &= x^T V_3 D_3^2 V_3^T y \quad \text{for all } x, y \in T_3,
 \end{aligned}$$

and two edge conditions :

$$|X_2 \times Y_{22}|^2 \{ |X_1|^2 (Y_{11} \cdot Y_{13}) - (X_1 \cdot Y_{11})(X_1 \cdot Y_{13}) \}$$

$$= \{ |X_2|^2 (Y_{21} \cdot Y_{22}) - (X_2 \cdot Y_{21})(X_2 \cdot Y_{22}) \} \{ |X_3|^2 (Y_{32} \cdot Y_{33}) - (X_3 \cdot Y_{32})(X_3 \cdot Y_{33}) \} - |X_2|^2 (X_2 \times Y_{21} \cdot Y_{22}) (X_3 \times Y_{32} \cdot Y_{33})$$

and

$$\begin{aligned} & |X_2 \times Y_{22}|^2 (X_1 \times Y_{11} \cdot Y_{13}) \\ &= (X_2 \times Y_{21} \cdot Y_{22}) \{ |X_3|^2 (Y_{32} \cdot Y_{33}) - (X_3 \cdot Y_{32})(X_3 \cdot Y_{33}) \} \\ & \quad + (X_3 \times Y_{32} \cdot Y_{33}) \{ |X_2|^2 (Y_{21} \cdot Y_{22}) - (X_2 \cdot Y_{21})(X_2 \cdot Y_{22}) \}. \end{aligned}$$

References

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- [2] D.C. Lay, *Linear Algebra and its Applications*, Second Edition, Addison Wesley Longman, USA (1996).