SOME EFFICIENT IMPLEMENTATION SCHEMES 
FOR IMPLICIT RUNGE-KUTTA METHODS

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Abstract: Several iteration schemes have been proposed to solve the non-linear equations arising in the implementation of implicit Runge-Kutta methods. As an alternative to the modified Newton scheme, some iteration schemes with reduced linear algebra costs have been proposed. A scheme of this type proposed in [9] avoids expensive vector transformations and is computationally more efficient. The rate of convergence of this scheme is examined in [9] when it is applied to the scalar test differential equation \( x' = qx \) and the convergence rate depends on the spectral radius of the iteration matrix \( M(z) \), a function of \( z = hq \), where \( h \) is the step-length. In this scheme, we require the spectral radius of \( M(z) \) to be zero at \( z = 0 \) and at \( z = \infty \) in the \( z \)-plane in order to improve the rate of convergence of the scheme. New schemes with parameters are obtained for three-stage and four-stage Gauss methods. Numerical experiments are carried out to confirm the results obtained here.

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1. Background

Let us consider an initial value problem for stiff systems of \( n(\geq 1) \) ordinary
differential equations

\[ x' = f(x(t)), \quad x(t_0) = x_0, \quad f : \mathbb{R}^n \rightarrow \mathbb{R}^n, \] (1)

where \( f \) is assumed to be as smooth as necessary. An \( s \)-stage implicit Runge-Kutta method computes an approximation \( x_{r+1} \) to the solution \( x(t_{r+1}) \) at grid point \( t_{r+1} = t_r + h \) by

\[ x_{r+1} = x_r + h \sum_{i=1}^{s} b_i f(y_i) \]

where the internal approximations \( y_1, y_2, \cdots, y_s \) satisfy the \( sn \) equations

\[ y_i = x_r + h \sum_{j=1}^{s} a_{ij} f(y_j), \quad i = 1, 2, \cdots, s \] (2)

\( A = [a_{ij}] \) is the real coefficient matrix and \( b = (b_1, b_2, \cdots, b_s)^T \) is the column vector of the Runge-Kutta method. Let \( Y = y_1 \oplus y_2 \oplus \cdots \oplus y_s \in \mathbb{R}^{sn} \) and let \( F(Y) = f(y_1) \oplus f(y_2) \oplus \cdots \oplus f(y_s) \in \mathbb{R}^{sn} \). Then equation (2) may be represented by the compact form

\[ Y = e \otimes x_r + h(A \otimes I_n)F(Y) \] (3)

where \( e = (1, 1, \cdots, 1)^T \) and \( A \otimes I_n \) is the Kronecker product of the matrix \( A \) with \( n \times n \) identity matrix \( I_n \) and, in general \( A \otimes B = [a_{ij}B] \). This article deals with methods suitable for stiff systems so that the matrix \( A \) is not strictly lower triangular and, in particular, is concerned with Gauss methods since they have highest order and good stability properties.

Equation (3) may be solved by a modified Newton iteration. Let \( J \) be the Jacobian of \( f \) evaluated at some recent point \( x_r \), updated infrequently. The modified Newton scheme evaluates \( Y^1, Y^2, Y^3, \cdots, \) to satisfy

\[ (I_{sn} - hA \otimes J)(Y^m - Y^{m-1}) = D(Y^{m-1}), \quad m = 1, 2, \cdots \] (4)

where \( D \) is the approximation defect, \( D(Z) = e \otimes x_r - Z + h(A \otimes I_n)F(Z) \). In each step of this iteration, a set of \( sn \) linear equations has to be solved. Schemes have been developed, to solve equation (4), which use the fact that \( J \) is constant [1], [6], [7]. In other schemes advantage is taken of the special forms of some implicit methods [2], [4], [5], [12].

In another approach, schemes based directly on iterative procedure have been developed [3], [8], [9], [10],[13],[21]. For a singly implicit method, there is a non-singular matrix \( S \) so that \( S^{-1}AS = \lambda(I_s - L)^{-1} \), where \( L \) is zero except
for some ones on the sub-diagonal. On applying this transformation, the scheme (4) becomes

\[
[I_s \otimes (I_n - h\lambda J)]E^m = [(I_s - L)S^{-1} \otimes I_n]D(Y^{m-1}) + (L \otimes I_n)E^m,
\]

\[
Y^m = Y^{m-1} + (S \otimes I_n)E^m, \quad m = 1, 2, 3, \ldots
\]

Cooper and Butcher [8] proposed an iterative scheme, sacrificing superlinear convergence for reduced linear algebra cost, which may be regarded as a generalization of the scheme (5) for singly implicit methods. They considered the scheme

\[
[I_s \otimes (I_n - h\lambda J)]E^m = (B_1S^{-1} \otimes I_n)D(Y^{m-1}) + (L_1 \otimes I_n)E^m,
\]

\[
Y^m = Y^{m-1} + (S \otimes I_n)E^m, \quad m = 1, 2, 3, \ldots
\]

where \(B_1\) and \(S\) are real \(s \times s\) non-singular matrices and \(L_1\) is strictly lower triangular matrix of order \(s\), and \(\lambda\) is a real constant. Cooper and Butcher [8] showed that successive over-relaxation may be applied to improve the rate of convergence for scalar test problem. Peat and Thomas [19], after extensive numerical experiments, concluded that the schemes proposed by Cooper and Butcher are, in general, the most efficient schemes for integration of stiff problems. Gladwell and Thomas [15] recommended this scheme for the two-stage Gauss method. Each step of the scheme (6) requires \(s\) function evaluations and the solution of \(s\) sets of \(n\) linear equations. These \(s\) sub-steps are performed in sequence and it is not possible to compute elements of \(Y^m = y_1^m \oplus y_2^m \oplus \cdots \oplus y_s^m\) until all sub-steps are completed. Cooper and Vignesvaran [9] considered a scheme where these elements are obtained in sequence and the approximation defect is updated after each sub-step completed. Only one vector transformation is needed for each full step so that this scheme is more efficient. Another scheme was proposed by Cooper and Vignesvaran [10] in order to obtain improved rate of convergence, by adding extra sub-steps. Vigneswaran [20] obtained further improvement in the rate of convergence of the iteration scheme proposed in [10]. Gonzalez, Gonzalez and Montijano [16] proposed a scheme for Gauss methods using an iterative procedure of semi-implicit type in which the Jacobian does not appear explicitly. A scheme of this type was proposed in [17] in which convergence and stability properties of the scheme are discussed in detail.
2. Efficient Iteration Scheme

Cooper and Vignesvaran [9] proposed the scheme

\[
[I_s \otimes (I_n - h\lambda J)]E^m = (L \otimes I_n)(e \otimes x_r - Y^m) \\
+ (U \otimes I_n)(e \otimes x_r - Y^{m-1}) \\
+ h(T \otimes I_n)F(Y^m) \\
+ h(R \otimes I_n)F(Y^{m-1}) \\
Y^m = Y^{m-1} + E^m, m = 1, 2, \ldots ,
\]  

(7)

where \( B \) is a real non-singular matrix such that \( B = L + U \) and \( BA = T + R \), \( L \) and \( T \) are strictly lower triangular matrices, \( U \) and \( R \) are upper triangular matrices, and \( \lambda \) is a real constant. Cooper and Vignesvaran [9] showed that \( D(Y) = 0 \) if the sequence \( \{Y^m\} \) has a limit \( Y \) and \( f \) is continuous on \( \mathbb{R}^n \). They observed that the scheme can be implemented efficiently by updating \( Y^{m-1} \) and \( F(Y^{m-1}) \) as soon as each element of \( Y^m = y^m_1 \oplus y^m_2 \oplus \cdots \oplus y^m_s \) is computed. The work involved is no more than is needed to carry out an evaluation of \( D(Y^{m-1}) \) followed by a transformation to \( (B \otimes I_n)D(Y^{m-1}) \).

Cooper and Vignesvaran [9] tested the rate of convergence of this scheme when it is applied to the scalar test problem \( x' = qx \) with rapid convergence required for all \( z \in \mathbb{C}^- \), where \( \mathbb{C}^- = \{ z \in \mathbb{C} : \text{Re} \leq 0 \} \). For this test problem, the scheme gives (7) gives

\[
Y - Y^m = M(z)(Y - Y^{m-1}), \quad m = 1, 2, \ldots ,
\]

and the rate of convergence depends on the spectral radius \( \rho[M(z)] \) of the iteration matrix

\[
M(z) = I_s - [(I_s + L - z(\lambda I_s + T)]^{-1}B(I_s - zA).
\]  

(8)

Cooper and Vignesvaran[9] imposed the condition that the iteration matrix \( M \) has only one non-zero eigenvalue \( \phi \),

\[
\phi(z) = 1 - \beta \frac{\det(I_s - zA)}{(1 - \lambda z)^s},
\]  

(9)

so that the spectral radius, \( \rho[M(z)] \), given by \( \rho[M(z)] = |\phi(z)| \) and \( \lambda \) and \( \beta(= \det B) \) can be chosen to solve the problem

\[
\min_{\lambda, \beta} \max_{z \in \mathbb{C}^-} \rho[M(z)].
\]  

(10)
To solve the minimization problem (10), when \( \lambda > 0 \) it follows from (9) that \( \phi \) is analytic and bounded on \( \mathbb{C}^- \) and hence \( |\phi| \) attains its maximum on the imaginary axis \( z = iy, y \) real. The polynomial \( p \), defined by

\[
p(\omega) = |\phi(iy)|^2, \quad \omega = \frac{1}{1 + (\lambda y)^2},
\]

is a polynomial of degree \( s \). For a given method, the coefficients of \( p \) depends on \( \lambda \) and \( \beta \) only and Cooper and Vignesvaran[9] obtained these parameters to minimize the maximum of \( p \) on \([0, 1]\) for the Gauss methods of order 4, 6 and 8 respectively.

Consider the three-stage Gauss method with matrix of coefficients

\[
A = \begin{bmatrix}
\frac{5}{36} & \frac{2}{9} - \frac{\sqrt{15}}{15} & \frac{5}{36} - \frac{\sqrt{15}}{30} \\
\frac{5}{36} + \frac{\sqrt{15}}{24} & \frac{2}{9} & \frac{5}{36} - \frac{\sqrt{15}}{24} \\
\frac{5}{36} + \frac{\sqrt{15}}{30} & \frac{2}{9} + \frac{\sqrt{15}}{15} & \frac{5}{36}
\end{bmatrix}
\]

(12)

and \( \det(I-zA) = 1 - \frac{1}{2}z + \frac{1}{10}z^2 - \frac{1}{120}z^3 \).

Cooper and Vignesvaran[9] obtained the optimum values \( \lambda = 0.202740067 \) and \( \beta = 1.159572736 \) when solving the problem (10). For these values of \( \lambda \) and \( \beta \), \( \rho[M(z)] < 0.1599 \) for all \( z \in \mathbb{C}^- \).

Next it remains to choose the elements of \( B = [b_{ij}] \) so that the iteration matrix \( M(z) = [m_{ij}(z)] \) is strictly upper triangular matrix except that \( m_{ss}(z) = \phi \), a non-zero eigenvalue. For the three-stage Gauss method, the condition on \( M(z) \) gives

\[
\begin{align*}
b_{11} &= 1, \\
b_{12}a_{21} + b_{13}a_{31} &= \lambda - a_{11}, \\
b_{12}(a_{22} - \lambda) + b_{13}a_{32} &= -a_{12}, \\
b_{21}b_{12} - b_{22} &= -1, \\
b_{21}(a_{12} - b_{12}a_{11}) + b_{22}(a_{22} - a_{21}b_{12}) + b_{23}(a_{32} - a_{31}b_{12}) &= \lambda, \\
b_{31}b_{12} &= 0, \\
b_{31}a_{11} + b_{32}a_{21} + b_{33}a_{31} &= 0.
\end{align*}
\]

(13)
From (13), it happens that $b_{31} = 0$. Again the equations (13) together with 
\[
\det B = \beta \n\]
may be solved by choosing $b_{21} = 0$ and this gives
\[
B = \begin{bmatrix}
1 & 0.151290053 & 0.068750541 \\
0 & 1 & 0.058981649 \\
0 & -0.983175783 & 1.101583408
\end{bmatrix}.
\] (14)

Consider the four-stage Gauss method with matrix of coefficents \( A = [a_{ij}] \) obtained by solving the sets of equations
\[
\sum_{j=1}^{4} a_{ij} c_j^{r-1} = \frac{c_i^r}{r}, \quad r = 1, 2, 3, 4,
\]
for each \( i = 1, 2, 3, 4 \), where \( c_1, c_2, c_3, c_4 \) are the zeros of \( P_4(2x - 1) \), the transformed legendre polynomial of degree 4. For this method,
\[
\det(I - zA) = 1 - \frac{1}{2} z + \frac{3}{28} z^2 - \frac{1}{84} z^3 + \frac{1}{1680} z^4.
\]
The condition on \( M(z) \) with the choices $b_{31} = 0$ and $b_{41} = b_{42} = 0$ give a system of equations which may be ordered as a sequence of sets of linear equations given below:
\[
\begin{align*}
 b_{11} &= 1, \\
b_{12}a_{21} + b_{13}a_{31} + b_{14}a_{41} &= (\lambda - a_{11}), \\
b_{12}(a_{22} - \lambda) + b_{13}a_{32} + b_{14}a_{42} &= -a_{12}, \\
b_{12}a_{23} + b_{13}(a_{33} - \lambda) + b_{14}a_{43} &= -a_{13}, \\
b_{12}b_{21} - b_{22} &= -1, \\
b_{13}b_{21} - b_{23} &= 0, \\
(b_{12}a_{11} - a_{12})b_{21} + (b_{12}a_{21} - a_{22})b_{22} + (b_{12}a_{31} - a_{32})b_{23} + (b_{12}a_{41} - a_{42})b_{24} &= -\lambda, \\
(a_{13} - b_{13}a_{11})b_{21} + (a_{23} - b_{13}a_{21})b_{22} + (a_{33} - b_{13}a_{31})b_{23} + (a_{43} - b_{13}a_{41})b_{24} &= 0,
\end{align*}
\] (15) (16)
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\begin{align*}
  b_{33} &= 1, \\
  b_{32}a_{21} + b_{34}a_{41} &= -a_{31}, \\
  b_{32}a_{23} + b_{34}a_{43} &= \lambda - a_{33}, \\
  b_{43}a_{31} + b_{44}a_{41} &= 0.
\end{align*}

(17)

(18)

Cooper and Vignesvaran[9] showed that these equations can be solved only for one positive value of \(\lambda\), \(\lambda = 0.146840443\) and they obtained the optimum value \(\beta = 1.034\) to solve the problem (10). In this case, \(\rho[M(z)] < 0.3467\) for \(\text{Re}(z) \leq 0\). With these values of \(\lambda\) and \(\beta\), the set of equations (15),(16),(17),(18) and the equation \(\det B = \beta\) give

\[
B = \begin{bmatrix}
1 & 0.265166833 & 0.079402432 & -0.018488567 \\
0.124164683 & 1.032924356 & 0.009858978 & 0.124164683 \\
0 & -0.786754443 & 1 & -0.108118541 \\
0 & 0 & -1.109340683 & 1.045019753
\end{bmatrix}.
\]

(19)

3. Schemes with Improving Rates of Convergence

In this section, additional constraints, which require super-linear convergence at the origin and infinity, are imposed on the spectral radius of the iteration matrix \(M(z)\) in addition to the condition that \(M(z)\) has only one non-zero eigenvalue. The results were obtained for the two-stage Gauss method in [22]. In this paper, new schemes corresponding to the iteration scheme (7) for three-stage and four-stage Gauss methods are obtained respectively.

3.1. The Case \(\rho[M(z)] = 0\) at \(z = 0\)

For the three-stage Gauss method, the additional constraint \(\rho[M(z)] = 0\) at \(z = 0\) gives \(\beta = 1\). Therefore, the other parameter \(\lambda\) has to be chosen to solve
the problem (10). It follows from (11) that the polynomial $p$ is given by
\[ p(\omega) = a_0\omega(1 - \omega)^2 + (1 - \omega)[a_1\omega - a_2(1 - \omega)]^2, \]
where $a_0 = 3 - \frac{1}{10}\lambda^2$, $a_1 = 3 - \frac{1}{2}\lambda$, $a_2 = 1 - \frac{1}{120}\lambda^2$.

A simple grid search procedure shows that good approximation to the optimum value of $\lambda$ to minimize the maximum of $p$ on $[0,1]$ is given by $\lambda = 0.191729022$. Again the condition on $M(z)$ gives the set of equations (13) and these equations together with $\det B = \beta$ may be solved by choosing $b_{21} = 0$. This gives
\[
B = \begin{bmatrix}
1 & 0.115697224 & 0.067542178 \\
0 & 1 & 0.009448755 \\
0 & -0.885047715 & 0.991637400
\end{bmatrix}. \tag{20}
\]

In this case $\rho[M(z)] < 0.2326$ for all $z \in \mathbb{C}^-$.

For the four-stage Gauss method, the additional constraint $\rho[M(z)] = 0$ at $z = 0$ gives $\beta = 1$. Again from (11), the polynomial $p$ is given by
\[ p(\omega) = (1 - \omega)^2[a_4(1 - \omega) - a_2\omega]^2 + \omega(1 - \omega)[a_1\omega - a_3(1 - \omega)]^2, \]
where $a_1 = 4 - \frac{1}{2}\lambda$, $a_2 = 6 - \frac{3}{28}\lambda^2$, $a_3 = 4 - \frac{1}{84}\lambda^3$, $a_4 = 1 - \frac{1}{1680}\lambda^4$. Again the system of equations (15),(16),(17) and (18) can be solved only for $\lambda = 0.146840443$ and for these fixed values of $\lambda$ and $\beta$, the equations (15), (16), (17), (18) and $\det B = \beta$ gives
\[
B = \begin{bmatrix}
1 & 0.265166833 & 0.079402432 & -0.018488567 \\
0.124164683 & 1.032924356 & 0.009858978 & 0.124164683 \\
0 & -0.786754443 & 1 & -0.108118541 \\
0 & 0 & -1.072863330 & 1.010657402
\end{bmatrix}. \tag{21}
\]

In this case $\rho[M(z)] < 0.3542$ for all $z \in \mathbb{C}^-$.

The equation $|\phi(z)| = c$ describes a closed curve in the $z$-plane. Typical curves are plotted for different values of $c$ and sketched in Figures 1 and 2 for three-stage and four-stage Gauss methods respectively. In this case, $\rho[M(z)] \leq c$ on and interior to the curve. Since $\rho[M(0)] = 0$, these schemes are expected to perform well as typical stiff problems have Jacobian with some eigenvalues of small modulus.
3.2. The Case $\rho[M(z)] = 0$ at $z = \infty$

The constraint $\rho[M(\infty)] = 0$ for the three-stage Gauss method gives $\lambda = \frac{3\sqrt{\beta}}{120}$ and the polynomial $p$, given by (11), is

$$p(\omega) = \omega[a_0 \omega^2 - a_2(1 - \omega)]^2 + a_1^2 \omega^2 (1 - \omega),$$

where $a_0 = 1 - \beta$, $a_1 = 3 - \frac{\beta}{2\lambda}$, $a_2 = 3 - \frac{\beta}{10\lambda^2}$. By search procedure, a good approximation to the optimum value of $\beta$ is obtained by $\beta = 1.181387098$ and the corresponding $\lambda$ is given by $\lambda = 0.214323763$. In this case $\rho[M(z)] < 0.2359$ for all $z \in \mathbb{C}^-$. With these values of $\lambda$ and $\beta$, the equations (13) with $\det B = \beta$ may be solved by choosing $b_{21} = 0$. This gives

$$B = \begin{bmatrix} 1 & 0.187138824 & 0.071808998 \\ 0 & 1 & 0.112237507 \\ 0 & -0.958395854 & 1.073819136 \end{bmatrix}.$$  \hspace{1cm} (22)

For the four-stage Gauss method, the additional constraint $\rho[M(\infty)] = 0$ gives $\beta = 1680\lambda^4$. It follows from (11) that the polynomial $p$ is given by

$$p(\omega) = [a_0 \omega^2 - a_2 \omega(1 - \omega)]^2 + \omega(1 - \omega)[a_1 \omega - a_3(1 - \omega)]^2,$$

where $a_0 = 1 - \beta$, $a_1 = 4 - \frac{\beta}{2\lambda}$, $a_2 = 6 - \frac{3\beta}{28\lambda^2}$, $a_3 = 4 - \frac{\beta}{84\lambda^3}$. With the value $\lambda = 0.146840443$, which solves the sets of equations 15),(16),(17),(18), and the corresponding value of $\beta$, those sets of equations and $\det B = \beta$ give
In this case $\rho[M(z)] < 0.2189$ for all $z \in \mathbb{C}^{-}$.

As per the plotted curves for $\rho[M(z)] = c$ for different values of $c$ in \( s = 3 \) and \( s = 4 \) for three-stage and four-stage Gauss methods, these schemes are expected to perform well as typical stiff problems have Jacobian with some eigenvalues of large negative real parts and $\rho[M(\infty)] = 0$.

4. Numerical Results

To evaluate the efficiency of the schemes obtained here, a range of numerical experiments was carried out. For each experiment, a single step was carried out, in each case, using the Jacobian evaluated at the initial point. For each scheme tested, the initial iterate $Y_{0}$ is chosen as $Y_{0} = e \otimes x$, where $x$ is the true solution at the initial point.

**Problem 1** denotes the non-linear system given by [14]

\[
\begin{align*}
    x_{1}' &= -0.013x_{1} + 1000x_{1}x_{3}, & x_{1}(0) &= 1, \\
    x_{2}' &= 2500x_{2}x_{3}, & x_{2}(0) &= 1, \\
    x_{3}' &= 0.013x_{1} - 1000x_{1}x_{3} - 2500x_{2}x_{3}, & x_{3}(0) &= 0,
\end{align*}
\]

where the eigenvalues of the Jacobian at the initial point are $0$, $-0.0093$ and $-3500$. 
Problem 2 is the elliptic two-body problem, with eccentricity 0.6,

\[
x'_1 = x_3,
\]
\[
x'_2 = x_4,
\]
\[
x'_3 = -x_1 (x_1^2 + x_2^2)^{-3/2},
\]
\[
x'_4 = -x_2 (x_1^2 + x_2^2)^{-3/2},
\]

The eigenvalues of the Jacobian at the initial point are 0 and transient components,

Problem 3 is the HIRES problem given by [18],

\[
x'_1 = -1.71x_1 + 0.43x_2 + 8.32x_3 + 0.0007,
\]
\[
x'_2 = 1.71x_1 - 8.75x_2,
\]
\[
x'_3 = -10.03x_3 + 0.43x_4 + 0.035x_5,
\]
\[
x'_4 = 8.32x_2 + 1.71x_3 - 1.12x_4,
\]
\[
x'_5 = -1.745x_5 + 0.43x_6 + 0.43x_7,
\]
\[
x'_6 = -280x_6x_8 + 0.69x_4 + 1.71x_5) - 0.43x_6 + 0.69x_7,
\]
\[
x'_7 = 280x_8x_8 - 1.81x_7,
\]
\[
x'_8 = -x'_7,
\]

The eigenvalues of the Jacobian at the initial point are ±5.5902 and ±3.9528i.

Problem 4 denotes the system

\[
x'_1 = x_2,
\]
\[
x'_2 = 10^6((1 - x_1^2)x_2) - x_1,
\]

derived from the Van der Pol’s equation and given by [11]. The eigenvalues of the Jacobian at the initial point are close to 0 and −3000000.

Problem 5 denotes the system, with non-linear coupling between smooth and transient components,

\[
x'_1 = -10^5x_1 + 2,
\]
\[
x'_2 = -10^6x_2 + 0.1x_1^2,
\]
\[
x'_3 = -40 \times 10^5x_3 + 0.4 (x_1^2 + x_2^2),
\]
\[
x'_4 = -10^7x_4 + x_1^2 + x_2^2 + x_3^2,
\]

where the Jacobian has constant eigenvalues −10^5, −10^6, −40 \times 10^5 and −10^7.

For each problem, a single step was carried out, in each case, using the Jacobian evaluated at the initial point. For each scheme tested, the initial iterate \( Y^0 \) is chosen as \( Y^0 = e \otimes x \), where \( x \) is the true solution at the initial point.
Method 1 denotes the three-stage Gauss method implemented according to the iteration scheme (7) with \( \lambda = 0.202740067 \) and the matrix \( B \) given by (14). Method 1* is the same method implemented using the scheme (7) with \( \lambda = 0.191729022 \) and \( B \) given by (20) for the case \( \rho[M(z)] = 0 \) at \( z = 0 \). Method 1** is also the same method implemented using the scheme (7) with \( \lambda = 0.214323763 \), \( B \) given by (22) for the case \( \rho[M(z)] = 0 \) at \( z = \infty \).

Method 2 denotes the four-stage Gauss method implemented according to the scheme (7) with \( \lambda = 0.146840443 \) and \( B \) given by (19). Method 2* is the same method implemented using the scheme (7) with \( \lambda = 0.146840443 \) and \( B \) given by (21) for \( \rho[M(0)] = 0 \). Method 2** is also the same method implemented using the scheme (7) with the same value of \( \lambda \) and \( B \) given by (23) for \( \rho[M(\infty)] = 0 \).

For each method and problem, the quantities

\[
e_m = \|E^m\|, \quad m = 1, 2, 3, \ldots
\]

were computed using the maximum norm on \( \mathbb{R}^{n_s} \). The values \( e_m \) for which \( e_m \leq TOL = 10^{-9} \) are tabulated for each problem and method. Similar results are obtained for different values of TOL. The results are given below for each problem for three-stage and four-stage Gauss methods.

<table>
<thead>
<tr>
<th>( e_m )</th>
<th>Method 1</th>
<th>Method 1*</th>
<th>Method 2</th>
<th>Method 2*</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_1 )</td>
<td>0.000956220</td>
<td>0.000824833</td>
<td>0.000895782</td>
<td>0.000866327</td>
</tr>
<tr>
<td>( e_2 )</td>
<td>0.000152341</td>
<td>0.000110398</td>
<td>0.000142783</td>
<td>0.000143328</td>
</tr>
<tr>
<td>( e_3 )</td>
<td>0.000024273</td>
<td>0.000000910</td>
<td>0.000028768</td>
<td>0.00000033</td>
</tr>
<tr>
<td>( e_4 )</td>
<td>0.000003867</td>
<td>0.000000031</td>
<td>0.000001011</td>
<td>0.000000127</td>
</tr>
<tr>
<td>( e_5 )</td>
<td>0.000000616</td>
<td>0.000000005</td>
<td>0.000000054</td>
<td>0.000000002</td>
</tr>
<tr>
<td>( e_6 )</td>
<td>0.000000098</td>
<td>0.000000001</td>
<td>0.000000016</td>
<td>0.000000008</td>
</tr>
<tr>
<td>( e_7 )</td>
<td>0.000000002</td>
<td>0.000000000</td>
<td>0.000000005</td>
<td>0.000000002</td>
</tr>
<tr>
<td>( e_8 )</td>
<td>0.000000000</td>
<td>0.000000000</td>
<td>0.0000000001</td>
<td>0.0000000001</td>
</tr>
<tr>
<td>( e_9 )</td>
<td>0.000000000</td>
<td>0.000000000</td>
<td>0.0000000000</td>
<td>0.0000000000</td>
</tr>
</tbody>
</table>

Table 1: Values of \( e_m \) for Problem 1 with \( h = 0.1 \)
### Table 2: Values of $e_m$ for Problem 2 with $h = 0.01$

<table>
<thead>
<tr>
<th>$e_m$</th>
<th>Method 1</th>
<th>Method 1$^*$</th>
<th>Method 2</th>
<th>Method 2$^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>0.064323263</td>
<td>0.055470109</td>
<td>0.060234720</td>
<td>0.058254081</td>
</tr>
<tr>
<td>$e_2$</td>
<td>0.010337141</td>
<td>0.007429666</td>
<td>0.009595467</td>
<td>0.009632142</td>
</tr>
<tr>
<td>$e_3$</td>
<td>0.001670882</td>
<td>0.00067048</td>
<td>0.001945151</td>
<td>0.001918104</td>
</tr>
<tr>
<td>$e_4$</td>
<td>0.000270379</td>
<td>0.000000270</td>
<td>0.000072013</td>
<td>0.000008450</td>
</tr>
<tr>
<td>$e_5$</td>
<td>0.000043831</td>
<td>0.000000002</td>
<td>0.000002754</td>
<td>0.000000149</td>
</tr>
<tr>
<td>$e_6$</td>
<td>0.000001157</td>
<td>0.000000000</td>
<td>0.000000016</td>
<td>0.000000000</td>
</tr>
<tr>
<td>$e_7$</td>
<td>0.000000189</td>
<td>0.000000000</td>
<td>0.000000000</td>
<td></td>
</tr>
<tr>
<td>$e_8$</td>
<td>0.000000031</td>
<td>0.000000000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e_9$</td>
<td>0.000000005</td>
<td>0.000000000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e_{10}$</td>
<td>0.000000001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e_{11}$</td>
<td>0.000000000</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 3: Values of $e_m$ for Problem 3 with $h = 0.01$

<table>
<thead>
<tr>
<th>$e_m$</th>
<th>Method 1</th>
<th>Method 1$^*$</th>
<th>Method 2</th>
<th>Method 2$^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>0.017382122</td>
<td>0.015000547</td>
<td>0.016278083</td>
<td>0.015742827</td>
</tr>
<tr>
<td>$e_2$</td>
<td>0.002728084</td>
<td>0.002012693</td>
<td>0.002608108</td>
<td>0.002618024</td>
</tr>
<tr>
<td>$e_3$</td>
<td>0.000428244</td>
<td>0.00013213</td>
<td>0.000523517</td>
<td>0.000516215</td>
</tr>
<tr>
<td>$e_4$</td>
<td>0.000067235</td>
<td>0.000000021</td>
<td>0.000000020</td>
<td>0.000000025</td>
</tr>
<tr>
<td>$e_5$</td>
<td>0.000010557</td>
<td>0.000000000</td>
<td>0.000000000</td>
<td></td>
</tr>
<tr>
<td>$e_6$</td>
<td>0.000001658</td>
<td>0.000000000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e_7$</td>
<td>0.000000260</td>
<td>0.000000000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e_8$</td>
<td>0.000000041</td>
<td>0.000000000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e_9$</td>
<td>0.000000006</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e_{10}$</td>
<td>0.000000001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e_{11}$</td>
<td>0.000000000</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. **Concluding Remarks**

According to the numerical results, for three-stage Gauss method, the method 1$^*$ performs better than method 1 for the problems whose Jacobian matrices have small eigenvalues and the method 1$^{**}$ performs better than method 1 for the problems whose Jacobian matrices have eigenvalues with large negative real part. For four-stage Gauss method, Method 2$^*$ is better than Method 2 for
problems with small eigenvalues and Method 2** is better than Method 2 for problems with eigenvalues which have large negative real parts. In overall, the numerical experiments confirm that the new schemes obtained for the Gauss methods perform well.
References


