SOME REGULAR ELEMENTS, IDEMPOTENTS AND RIGHT UNITS OF COMPLETE SEMIGROUPS OF BINARY RELATIONS DEFINED BY SEMILATTICES OF THE CLASS LOWER INCOMPLETE NETS

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Abstract: In this paper, we investigate such a regular elements $\alpha$ and idempotents of the complete semigroup of binary relations $B_X(D)$ defined by semilattices of the class lower incomplete nets, for which $V(D, \alpha) = Q$.

Also we investigate right units of the semigroup $B_X(Q)$. For the case where $X$ is a finite set we derive formulas by means of which we can calculate the numbers of regular elements, idempotents and right units of the respective semigroup.

AMS Subject Classification: 20M30, 20M10, 20M15
Key Words: semigroups, binary relation, regular element, idempotents, right units

Received: January 30, 2014

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1. Introduction

Let $X$ be an arbitrary nonempty set. Let $D$ be some nonempty set of subsets of the set $X$, closed with respect to the operation of set-theoretic union of elements of the set $D$, i.e., $\cup D' \in D$ for any nonempty subset $D'$ of the set $D$. In that case, the set $D$ is called complete $X$–semilattice of unions. The union of all elements of the set $D$ is denoted by the symbol $\tilde{D}$. Clearly, $\tilde{D} \in D$ is the largest element.

Recall that a binary relation on the set $X$ is a subset of the cartesian product $X \times X$. If $\alpha$ and $\beta$ are binary relations on the set $X$ with the elements $x, y, z \in X$ the condition $(x, y) \in \alpha$ is denoted as $x \alpha y$ and $x \alpha y \beta z$ means the conditions $x \alpha y$ and $y \beta z$ are satisfied simultaneously. The binary relation $\alpha^{-1} = \{(x, y) : y \alpha x\}$ is usually called the binary relation inverse to $\alpha$. The empty binary relation which is empty subset of $X \times X$ is denoted by $\emptyset$. The binary relation $\delta = \alpha \circ \beta$ is called product of binary relations $\alpha$ and $\beta$. A pair $(x, y)$ belongs to $\delta$ if only if there exists $y \in X$ such that $x \alpha y \beta z$. The binary operation $\circ$ is associative. So, $B_X$, the set of all binary relations on $X$, is therefore a semigroup with respect to the operation $\circ$. This semigroup is called the semigroup of all binary relations on the set $X$.

Let $f$ be an arbitrary mapping from $X$ into $D$. Then one can construct such a mapping $f$ with a binary relation $\alpha_f$ on $X$ provided by the condition below, $\alpha_f = \bigcup_{x \in X} (\{x\} \times f(x))$. The set of all such binary relation is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the product operation of binary relations. This semigroup, $B_X(D)$, is called a complete semigroup of binary relations defined by an $X$–semilattice of unions $D$.

Further, let $x, y \in X$, $Y \subseteq X$, $\alpha \in B_X(D)$, $T \in D$, $\emptyset \neq D' \subseteq D$ and $t \in \tilde{D}$. Then we have the following notation,

\[
y \alpha = \{x \in X : y \alpha x\}, \quad Y \alpha = \bigcup_{y \in Y} y \alpha, 2^X = \{Y : Y \subseteq X\}, \quad X^* = 2^X \setminus \{\emptyset\}
\]

\[
V(D, \alpha) = \{Y \alpha : Y \in D\}, \quad D_t = \{Z' : t \in Z'\},
\]

\[
D'_T = \{Z' \in D' : T \subseteq Z'\}, \quad \tilde{D}'_T = \{Z' \in D' : Z' \subseteq T\}.
\]

Now, let’s take $\alpha \in B_X(D)$. If $\beta \circ \alpha = \beta$ for any $\beta \in B_X(D)$, then $\alpha$ is called a right unit of semigroup $B_X(D)$. If $\alpha \circ \alpha = \alpha$ then $\alpha$ is called an idempotent element of semigroup $B_X(D)$. And if $\alpha \circ \beta \circ \alpha = \alpha$ for some $\beta \in B_X(D)$, then a binary relation $\alpha$ is called a regular element of semigroup $B_X(D)$.
\(D\) is partially ordered with respect to the set-theoretic inclusion. Let \(\emptyset \neq D' \subseteq D\) and \(N(D, D') = \{Z \in D : Z \subseteq Z'\ \text{for any} \ Z' \in D'\}\). It is clear that \(N(D, D')\) is the set of lower bounds of a nonempty subset \(D'\) included in \(D\). If \(N(D, D') \neq \emptyset\) then \(\cup N(D, D')\) belongs to \(D\) and it is the greatest lower bound of \(D'\) and is denoted by \(\land (D, D') = \cup N(D, D')\).

Let \(l(D', T) = \cup (D' \setminus D'_T)\). We say that a nonempty element \(T\) is a non-limiting element of \(D'\) if \(T \setminus l(D', T) \neq \emptyset\). A nonempty element \(T\) is a limiting element of \(D'\) if \(T \setminus l(D', T) = \emptyset\).

Now, we continue with some essential definitions and theorems given by the cited references.

**Definition 1.1.** [2, Definition 1] Let \(\alpha \in B_X\), \(T \in V (X^*, \alpha)\) and \(Y^\alpha_T = \{y \in X : y\alpha = T\}\). Then a representation of a binary relation \(\alpha\) of the form \(\alpha = \bigcup_{T \in V (X^*, \alpha)} (Y^\alpha_T \times T)\) is called quasinormal.

Note that, if \(\alpha = \bigcup_{T \in V (X^*, \alpha)} (Y^\alpha_T \times T)\) is a quasinormal representation of the binary relation \(\alpha\) then the following are true,

1. \(X = \bigcup_{T \in V (X^*, \alpha)} Y^\alpha_T\)
2. \(Y^\alpha_T \cap Y^\alpha_{T'} = \emptyset\) for \(T, T' \in V (X^*, \alpha)\) and \(T \neq T'\).

**Definition 1.2.** [3, Definition 2] Let \(\tilde{D}\) and \(D'\) be some nonempty subsets of the complete \(X\)–semilattices of unions. We say that a subset \(\tilde{D}\) generates a set \(D'\) if any element from \(D'\) is a set-theoretic union of the elements from \(\tilde{D}\).

**Definition 1.3.** [4, Definition 1.14.2] We say that a complete \(X\)–semilattice of unions \(D\) is an \(XI\)–semilattice of unions if it satisfies the following two conditions:

a) \(\land (D, D_t) \in D\) for any \(t \in \tilde{D}\),

b) \(Z = \bigcup_{t \in Z} \land (D, D_t)\) for any nonempty element \(Z\) of \(D\).

**Theorem 1.1.** [4, Corollary 1.18.1] Let \(Y = \{y_1, y_2, \ldots, y_k\}\) and \(D_j = \{T_1, \ldots, T_j\}\) be some sets, where \(k \geq 1\) and \(j \geq 1\). Then the numbers \(s(k, j)\) of all possible mappings of the sets \(Y\) on any subset \(D'_j\) of the set \(D_j\) and \(T_j \in D'_j\) can be calculated by the formula \(s(k, j) = j^k - (j - 1)^k\).
Lemma 1.1. [1, Lemma 3.1] Let $D$ complete $X$–semilattices of unions. If a binary relation $\varepsilon$ having the form 

$$\varepsilon = \varepsilon(D, f) = \bigcup_{t \in \hat{D}} (\{t\} \times \wedge(D, D_t)) \cup \left((X \setminus \hat{D}) \times \hat{D}\right)$$

is a right unit of the semigroup $B_x(D)$, then it is the largest right unit of this semigroup.

Theorem 1.2. [1, Theorem 2.5] Let $D'$ be a complete subsemilattice of the complete $X$–semilattice of unions $D$, $\hat{D}' = \cup D'$ and $f$ be an arbitrary mapping of the set $X \setminus \hat{D}'$ in the semilattice $D'$. If $D'$ is a complete $XI$–semilattice of unions then the binary relation 

$$\alpha = \alpha(D', f) = \bigcup_{t \in \hat{D}'} (\{t\} \times \wedge(D', D'_t)) \cup \bigcup_{t' \in X \setminus \hat{D}'} (\{t'\} \times f(t'))$$

is an idempotent element of semigroup $B_x(D)$ and $V(D', \alpha) = D'$.

Theorem 1.3. [4, Theorem 4.1.3] A binary relation $\varepsilon \in B_X(D)$ is right units of this semigroup iff $\varepsilon$ is idempotent and $V(D, \varepsilon) = D$.

Definition 1.4. [6, Definition 7] A one-to-one mapping $\varphi$ between the complete $X$–semilattices of unions $D'$ and $D''$ is called a complete isomorphism if the condition $\varphi(\cup D_1) = \bigcup_{T' \in D_1} \varphi(T')$ is fulfilled for each nonempty subset $D_1$ of the semilattice $D'$.

Definition 1.5. [6, Definition 8] Let $\alpha$ be some binary relation of the semigroup $B_X(D)$. We say that a complete isomorphism $\varphi$ between $XI$–semilattice of unions $Q$ and $D'$ is a complete $\alpha$–isomorphism if

a) $Q = V(D, \alpha)$

b) $\varphi(\emptyset) = \emptyset$ for $\emptyset \in V(D, \alpha)$ and $\varphi(T)\alpha = T$ for any $T \in V(D, \alpha)$.

Theorem 1.4. [4, Theorem 6.3.3] Let $D$ be a finite $X$–semilattice of unions and $\alpha \circ \sigma \circ \alpha = \alpha$ for some elements $\alpha, \sigma \in B_X(D)$; $D(\alpha)$ the set those elements $T$ of the semilattice $D = V(D, \alpha) \setminus \{\emptyset\}$ which are nonlimiting elements of the set $\hat{D}(\alpha)$. Then binary relation $\alpha$ having a quasinormal representation of the form $\alpha = \bigcup_{T \in V(D, \alpha)} (Y_T \times T)$ is a regular element of the semigroup $B_X(D)$ iff the set $V(D, \alpha)$ is a $XI$–semilattice of unions and for some $\alpha$–isomorphism $\varphi$ of the semilattice $V(D, \alpha)$ on some $X$–subsemilattice $D'$ of the semilattice $D$ the following conditions are fulfilled:
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\[ \bigcup_{T \in \mathcal{D}(\alpha)_T} Y_T^\alpha \supseteq \varphi(T) \text{ for any } T \in D(\alpha); \]

\[ Y_T^\alpha \cap \varphi(T) \neq \emptyset \text{ for any nonlimiting element } T \text{ of the set } \mathcal{D}(\alpha)_T. \]

**Theorem 1.5.** [4, Theorem 6.3.7] A regular element \( \alpha \) of the semigroup \( B_X(D) \) is idempotent iff the mapping \( \varphi \) satisfying the condition \( \varphi(T) = T\alpha \) for any \( T \in V(D, \alpha) \) is an identity mapping of the semilattice \( V(D, \alpha) \).

**Definition 1.6.** [4, Definition 6.3.4] Let \( Q \) and \( D' \) be respectively some \( XI \) and \( X \)–subsemilattices of the complete \( X \)–semilattice of unions \( D \). Then \( R_\varphi(Q, D') \) is a subset of the semigroup \( B_X(D) \) such that \( \alpha \in R_\varphi(Q, D') \) only if the following conditions are fulfilled for the elements of \( \alpha \) and \( \varphi \),

a) The binary relation \( \alpha \) be regular element of the semigroup \( B_X(D) \),

b) \( V(D, \alpha) = Q \),

c) \( \varphi \) is a \( \alpha \)–isomorphism between the complete semilattices of unions \( Q \) and \( D' \) satisfying the conditions i) and ii) of the Theorem 1.4.

**Definition 1.7.** [4, Definition 6.3.4] Let \( \Phi(D, D') \) be the set of all complete isomorphism \( \varphi \) between \( XI \)–semilattice of unions \( D \) and \( D' \) such that \( \varphi \in \Phi(D, D') \) only if \( \varphi \) is a \( \alpha \)–isomorphism for some \( \alpha \in B_X(D) \) and \( V(D, \alpha) = D \).

\( \Omega(D) \) is the set of all \( XI \)–subsemilattices of the complete \( X \)–semilattice of unions \( D \) such that \( D' \in \Omega(D) \) iff there exists a complete isomorphism between the semilattices \( D' \) and \( D \).

Let us denote

\[ R(D, D') = \bigcup_{\varphi \in \Phi(D, D')} R_\varphi(D, D') \text{ and } R(D') = \bigcup_{D' \in \Omega(D)} R(D, D'). \]

**Theorem 1.6.** [4, Theorem 6.3.5] Let \( X \) is a finite set. If \( \varphi \) is a fixed element of the set \( \Phi(D, D') \) and \( \Omega(D) = m_0 \) and \( q \) is a number of all automorphisms of the semilattice \( D \), then \( |R(D')| = m_0 \cdot q \cdot |R_\varphi(D, D')| \).

**Theorem 1.7.** [6, Theorem 22] Let \( D = \{ \tilde{D}, T_1, T_2, \ldots, T_{m-1} \} \) be some finite \( X \)–semilattice of unions and \( C(D) = \{ P_0, P_1, P_2, \ldots, P_{m-1} \} \) be the family of sets of pairwise disjoint subsets of the set \( X \). If \( \varphi \) is a mapping of the semilattice \( D \) to the family sets \( C(D) \) that satisfies the condition \( \varphi(\tilde{D}) = P_0 \)
and $\varphi(T_i) = P_i$ ($i = 1, 2, \ldots, m - 1$) and $\hat{D}_Z = D \setminus \{T \in D : Z \subseteq T\}$ then the following equalities are valid:

$$
\check{D} = P_0 \cup P_1 \cup \cdots \cup P_{m - 1}, \quad T_i = P_0 \cup \bigcup_{T \in \hat{D}_{T_i}} \varphi(T).
$$

(1.1)

In the sequel these equalities will be called formal.

It is proved that if the elements of the semilattice $D$ are represented in the form (1.1), then among the parameters $P_i$ ($i = 0, 1, 2, \ldots, m - 1$) there exist such parameters that cannot be empty sets for $D$. Such sets $P_i$ ($0 < i \leq m - 1$) are called basis sources, whereas sets $P_j$ ($0 \leq j \leq m - 1$) which can be empty sets too are called completeness sources.

It is proved that under the mapping $\varphi$ the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping $\varphi$ the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one.

Note that the set $P_0$ is always considered to be a source of completeness.

**Lemma 1.2.** Let $D$ and $C(D) = \{P_0, P_1, \ldots, P_{n - 1}\}$ are the finite semilattice of unions and the family of sets of pairwise nonintersecting subsets of the set $X$; $\varphi$ is a mapping of the semilattice $D$ on the family of sets $C(D)$. If $\varphi(T) = P \in C(D) \setminus \{P_0\}$ for some $T \in D$, then $D_t = D \setminus \check{D}_T$ for all $t \in P$.

**Proof.** Let $t$ and $Z'$ are any elements of the set $P$ ($P \neq P_0$) and of the semilattice $D$ respectively. Then the equality $P \cap Z' = \emptyset$ (i.e., $Z' \notin D_t$ for any $t \in P$) is true if and only if $T \notin \check{D}_{Z'}$ (if $T \in \check{D}_{Z'}$, then $\varphi(T) \subseteq Z'$ by definition of the formal equalities of the semilattice $D$). Since $\check{D}_{Z'} = D \setminus \{T' \in D : Z' \subseteq T'\}$ by definition of the set $\check{D}_{Z'}$. Thus the condition $T \notin \check{D}_{Z'}$ hold iff $T \in \{T' \in D : Z' \subseteq T'\}$. So, $Z' \subseteq T$ and $Z' \in \check{D}_T$ by definition of the set $\check{D}_T$.

Therefore, $\varphi(T) \cap Z' = \emptyset$ if and only if $Z' \in \check{D}_T$. Of this follows that the inclusion $\varphi(T) = P \subseteq Z'$ is true iff $D_t = D \setminus \check{D}_T$ for all $t \in \varphi(T) = P$. \qed
2. Results

Let $N_m = \{0, 1, 2, \ldots, m\}$ ($m \geq 1$) be some subset of the set of all natural numbers. A subsemilattice

$$Q = \{T_{ij} \subseteq X : i \in N_s, j \in N_k\} \setminus \{T_{00}\}$$

of the complete $X$–semilattice of unions $D$ is called lower incomplete net which contains two subsets $Q_1 = \{T_{10}, T_{20}, \ldots, T_{s0}\}$, $Q_2 = \{T_{01}, T_{02}, \ldots, T_{0k}\}$ and satisfies the following conditions:

a) $T_{10} \subset T_{20} \subset \cdots \subset T_{s0}$ and $T_{01} \subset T_{02} \subset \cdots \subset T_{0k}$;

b) $Q_1 \cap Q_2 = \varnothing$;

c) $T_{pq} \neq T_{ij}$ if $(p, q) \neq (i, j)$

d) the elements of the sets $Q_1$ and $Q_2$ are pairwise noncomparable;

e) $T_{ij} \cup T_{i'j'} = T_{pq}$, if $p = \max\{i, i'\}$ and $q = \max\{j, j'\}$.

Note that the diagram of the given $X$–semilattice of unions $Q$ is shown in Fig. 2.1.

Let $C(Q) = \{P_{01}, P_{10}, P_{02}, \ldots, P_{s-1k}, P_{ks-1}, P_{sk}\}$ is a family sets, where $P_{01}, P_{10}, P_{02}, \ldots, P_{ks-1}, P_{sk}$ are pairwise disjoint subsets of the set $X$ and

$$\varphi = \left( \begin{array}{cccccccc} T_{01} & T_{10} & T_{02} & T_{11} & T_{20} & \cdots & T_{s-1k} & T_{ks-1} & T_{sk} \\ P_{01} & P_{10} & P_{02} & P_{11} & P_{20} & \cdots & P_{s-1k} & P_{ks-1} & P_{sk} \end{array} \right)$$

is a mapping of the semilattice $Q$ onto the family sets $C(Q)$. Then for the formal equalities of $Q$ we have a form (see Theorem 1.7):

$$T_{sk} = P_{sk} \cup \bigcup_{T_{rt} \in \hat{D}_{sk}} \varphi(T_{rt}),$$

(2.1)

$$T_{s-1k} = P_{sk} \cup \bigcup_{T_{rt} \in \hat{D}_{s-1k}} \varphi(T_{rt}), T_{sk-1} = P_{sk} \cup \bigcup_{T_{rt} \in \hat{D}_{sk-1}} \varphi(T_{rt}),$$

$$\cdots$$

$$T_{20} = P_{sk} \cup \bigcup_{T_{rt} \in \hat{D}_{20}} \varphi(T_{rt}),$$

$$T_{11} = P_{sk} \cup \bigcup_{T_{rt} \in \hat{D}_{11}} \varphi(T_{rt}), T_{02} = P_{sk} \cup \bigcup_{T_{rt} \in \hat{D}_{02}} \varphi(T_{rt}),$$
\[ T_{10} = P_{sk} \cup \bigcup_{T_{rt} \in \mathcal{D}_{10}} \varphi(T_{rt}) = T_{01} \cup T_{02} \cup \cdots \cup T_{0k} \]
\[ T_{01} = P_{sk} \cup \bigcup_{T_{rt} \in \mathcal{D}_{01}} \varphi(T_{rt}) = T_{10} \cup T_{20} \cup \cdots \cup T_{s0}. \]

Here \( P_{0k}, P_{1k}, \ldots, P_{s-1k}, P_{s0}, P_{s1}, \ldots, P_{sk-1} \) are basis sources, the elements of the set \( C(Q) \setminus \{ P_{0k}, P_{1k}, P_{2k}, \ldots, P_{s0}, P_{s1}, \ldots, P_{sk-1} \} \) are sources of completeness of the semilattice \( Q \) (see Theorem 1.7).

**Lemma 2.1.** Let \( Q \) be a lower incomplete net. Then \( Q \) is \( XI \)-semilattice of unions iff it satisfies the condition \( T_{0k} \cap T_{s0} = \emptyset \).

**Proof.** Let \( t \in \tilde{Q}, Q_t = \{ Z \in Q : t \in Z \} \) and \( \land(Q, Q_t) \) is the exact lower bound of the set \( Q_t \) in \( Q \). Then from Lemma 1.2 and from the formal equalities (2.1) we have:

\[
\begin{align*}
&\quad \text{if } t \in P_{sk}, \quad Q_t = Q \\
&\quad \text{if } t \in P_{s-1k}, \quad Q_t = Q \setminus Q^\ast T_{s-1k} \\
&\quad \text{if } t \in P_{s-2k}, \quad Q_t = Q \setminus Q^\ast T_{s-2k} \\
&\quad \text{if } t \in P_{sk-1}, \quad Q_t = Q \setminus Q^\ast T_{sk-1} \\
&\quad \text{if } t \in P_{sk-2}, \quad Q_t = Q \setminus Q^\ast T_{sk-2} \\
&\quad \vdots \\
&\quad \text{if } t \in P_{0k}, \quad Q_t = Q \setminus Q^\ast T_{0k}, \\
&\quad \text{if } t \in P_{s0}, \quad Q_t = Q \setminus Q^\ast T_{s0}.
\end{align*}
\]

If \( T_{ij} \subseteq T_{s-1k-1}, \varphi(T_{ij}) = P_{ij} \) and \( t \in P_{ij} \), then \( Q_t = Q \setminus Q^\ast T_{ij} \) and \( \land(Q, Q_t) \notin Q \).

We have \( Q^\ast = \{ \land(Q, Q_t) : \land(Q, Q_t) \in Q \} = \{ T_{10}, \ldots, T_{s0}, T_{01}, \ldots, T_{0k} \} \) and \( \land(Q, Q_t) \notin Q \), when \( t \notin \cup \{ P_{0k}, P_{1k}, \ldots, P_{s-1k}, P_{s0}, P_{s1}, \ldots, P_{sk-1} \} \), i.e., \( \land(Q, Q_t) \notin Q \), when \( t \in T_{0k} \cap T_{s0} \). Therefore the semilattice \( Q \) is not \( XI \)-semilattice of unions, if \( T_{0k} \cap T_{s0} \neq \emptyset \).

\((*)\) If \( T_{0k} \cap T_{s0} = \emptyset \), i.e., \( t \in \cup \{ P_{0k}, P_{1k}, \ldots, P_{s-1k}, P_{s0}, \ldots, P_{sk-1} \} \) then \( \land(Q, Q_t) \in Q \) for all elements \( t \) of the set \( \tilde{D} \) and \( T_{ij} = \bigcup_{t \in T_{ij}} \land(Q, Q_t) \). Therefore the semilattice \( Q \) is \( XI \)-semilattice of unions, if \( T_{0k} \cap T_{s0} = \emptyset \).

If the equality \( T_{0k} \cap T_{s0} = \emptyset \) is true, then by \((*)\) follows that \( Q \) is \( XI \)-semilattice of unions.

**Lemma 2.2.** Let \( Q \) be a \( XI \)-lower incomplete net. Then following equalities are true:

\[
P_{s-1k} = T_{s0} \setminus T_{s-1k}, \quad P_{s-2k} = T_{s-10} \setminus T_{s-2k}, \quad \ldots, \quad P_{0k} = T_{10} \setminus T_{0k}
\]
the following inclusions:

\[ P_{sk-1} = T_{0k} \setminus T_{sk-1}, \, P_{sk-2} = T_{0k-1} \setminus T_{sk-2}, \ldots, \, P_{s0} = T_{01} \setminus T_{s0}. \]

**Proof.** The given Lemma immediately follows from the formal equalities (2.1) of the semilattice \( Q \). For the largest right unit \( \varepsilon \) of the semigroup \( B_X(D) \) we have:

\[
\varepsilon = (P_{s-1k} \times T_{s0}) \cup (P_{s-2k} \times T_{s-10}) \cup (P_{sk-1} \times T_{0k}) \\
\cup ((T_{0k-1} \setminus Y_{sk-2}) \times T_{0k-1}) \cup \cdots \cup (P_{0k} \times T_{10}) \cup (P_{s0} \times T_{01}) \\
= ((T_{0k-1} \setminus T_{s-1k}) \times T_{s0}) \cup ((T_{s-10} \setminus T_{s-2k}) \times T_{s-10}) \cup ((T_{0k} \setminus T_{sk-1}) \times T_{0k}) \\
\cup ((T_{0k-1} \setminus T_{sk-2}) \times T_{0k-1}) \cup \cdots \cup ((T_{10} \times T_{0k}) \times T_{10}) \cup ((T_{01} \times T_{s0}) \times T_{01})
\]

(see, Lemma 1.1).

**Theorem 2.1.** Let \( Q \) be a \( XI \)-lower incomplete net. Then a binary relation \( \alpha \) of the semigroup \( B_X(D) \) having a quasinormal representation of the form \( \alpha = \bigcup_{T_{ij} \in Q} (Y_{ij}^\alpha \times T_{ij}) \) such that \( Q = V(D, \alpha) \), is a regular element of the semigroup \( B_X(D) \) iff for some \( \alpha \)-isomorphism \( \varphi \) of the semilattice \( Q \) on some subsemilattice \( D' \) of the semilattice \( D \) the following conditions are fulfilled:

\[
\begin{align*}
Y_{00}^\alpha & \supseteq \varphi(T_{0k}) \cap \varphi(T_{s0}), \quad Y_{00}^\alpha \cup Y_{01}^\alpha \supseteq \varphi(T_{01}), \quad Y_{00}^\alpha \cup Y_{01}^\alpha \cup Y_{02}^\alpha \supseteq \varphi(T_{02}), \ldots, \\
Y_{00}^\alpha \cup Y_{01}^\alpha \cup Y_{02}^\alpha \cup \cdots \cup Y_{0k}^\alpha & \supseteq \varphi(T_{0k}), \\
Y_{00}^\alpha \cup Y_{10}^\alpha \supseteq \varphi(T_{10}), \quad Y_{00}^\alpha \cup Y_{10}^\alpha \cup Y_{20}^\alpha \supseteq \varphi(T_{20}), \\
\cdots, \quad Y_{00}^\alpha \cup Y_{10}^\alpha \cup Y_{20}^\alpha \cup \cdots \cup Y_{s0}^\alpha & \supseteq \varphi(T_{s0}), \quad Y_{ij}^\alpha \cap \varphi(T_{ij}) \neq \emptyset
\end{align*}
\]

for any \( T_{ij} \in (Q_1 \cup Q_2) \setminus \{ \emptyset \} \).

**Proof.** It is easy to see that the set \( Q^\sim = \{ T_{10}, T_{20}, \ldots, T_{s0}, T_{01}, \ldots, T_{0k} \} \) is an irreducible generating set of the semilattice \( Q \). Moreover, all elements of the set \( Q^\sim = Q_1 \cup Q_2 \) are nonlimiting. Now by Theorem 1.4 we obtain

a) \( \bigcup_{T_{pq} \in Q^\sim} \supseteq \varphi(T_{pq}) \) for any \( T_{pq} \in Q^\sim \).

b) \( Y_{ij}^\alpha \cap \varphi(T_{ij}) \neq \emptyset \) for any \( T_{ij} \in Q^\sim \).

From the condition a) of this theorem we immediately have the validity of the following inclusions:

\[
\begin{align*}
Y_{10}^\alpha & \supseteq \varphi(T_{10}), \quad Y_{10}^\alpha \cup Y_{20}^\alpha \supseteq \varphi(T_{20}), \ldots, \quad Y_{10}^\alpha \cup Y_{20}^\alpha \cup \cdots \cup Y_{s0}^\alpha \supseteq \varphi(T_{s0}) \\
Y_{01}^\alpha & \supseteq \varphi(T_{01}), \quad Y_{01}^\alpha \cup Y_{02}^\alpha \supseteq \varphi(T_{02}), \ldots, \quad Y_{01}^\alpha \cup Y_{02}^\alpha \cup \cdots \cup Y_{0k}^\alpha \supseteq \varphi(T_{0k}) \\
Y_{20}^\alpha \cap \varphi(T_{20}) & \neq \emptyset, \quad \ldots, \quad Y_{s0}^\alpha \cap \varphi(T_{s0}) \neq \emptyset, \\
Y_{02}^\alpha \cap \varphi(T_{02}) & \neq \emptyset, \quad \ldots, \quad Y_{0k}^\alpha \cap \varphi(T_{0k}) \neq \emptyset
\end{align*}
\]

\( \square \)
Theorem 2.2. Let $Q$ be a $XI$–lower incomplete net. Then a binary relation $\alpha$ of the semigroup $B_X(Q)$, which has a quasinormal representation of the form $\alpha = \bigcup_{T_{ij} \in Q} \left( Y_{ij}^\alpha \times T_{ij} \right)$ such that $Q = V(D, \alpha)$, is an idempotent element of the semigroup $B_X(D)$ iff the following conditions are fulfilled:

\[
Y_{10}^\alpha \supseteq T_{10}, \ Y_{10}^\alpha \cup Y_{20}^\alpha \supseteq T_{20}, \ldots, \ Y_{10}^\alpha \cup Y_{20}^\alpha \cup \ldots \cup Y_{s0}^\alpha \supseteq T_{s0}, \\
Y_{01}^\alpha \supseteq T_{01}, \ Y_{01}^\alpha \cup Y_{02}^\alpha \supseteq T_{02}, \ldots, \ Y_{01}^\alpha \cup Y_{02}^\alpha \cup \ldots \cup Y_{0k}^\alpha \supseteq T_{0k}, \\
Y_{20}^\alpha \cap T_{20} \neq \varnothing, \ldots, \ Y_{s0}^\alpha \cap T_{s0} \neq \varnothing, \ Y_{02}^\alpha \cap T_{02} \neq \varnothing, \ldots, Y_{0k}^\alpha \cap T_{0k} \neq \varnothing,
\]

for any $T_{ij} \in Q^\ast$.

Proof. This theorem immediately follows from Lemma 2.1, from the Theorem 2.1 and Theorem 1.5.

Theorem 2.3. Let $Q$ be a $XI$–lower incomplete net. Then a binary relation $\alpha$ of the semigroup $B_X(Q)$ that has a quasinormal representation of the form $\alpha = \bigcup_{T_{ij} \in Q} \left( Y_{ij}^\alpha \times T_{ij} \right)$ such that $Q = V(D, \alpha)$, is a right unit of the semigroup $B_X(Q)$ iff the following conditions are fulfilled:

\[
Y_{10}^\alpha \supseteq T_{10}, \ Y_{10}^\alpha \cup Y_{20}^\alpha \supseteq T_{20}, \ldots, \ Y_{10}^\alpha \cup Y_{20}^\alpha \cup \ldots \cup Y_{s0}^\alpha \supseteq T_{s0}, \\
Y_{01}^\alpha \supseteq T_{01}, \ Y_{01}^\alpha \cup Y_{02}^\alpha \supseteq T_{02}, \ldots, \ Y_{01}^\alpha \cup Y_{02}^\alpha \cup \ldots \cup Y_{0k}^\alpha \supseteq T_{0k}, \\
Y_{20}^\alpha \cap T_{20} \neq \varnothing, \ldots, \ Y_{s0}^\alpha \cap T_{s0} \neq \varnothing, \ Y_{02}^\alpha \cap T_{02} \neq \varnothing, \ldots, Y_{0k}^\alpha \cap T_{0k} \neq \varnothing,
\]

for any $T_{ij} \in Q^\ast$.

Proof. This theorem immediately follows from Theorem 1.3.

Theorem 2.4. Let $Q$ be a $XI$–lower incomplete net. If the semilattice $Q$ and $D' = \{ \tilde{T}_{01}, \tilde{T}_{10}, \ldots, \tilde{T}_{sk} \}$ (see Fig. 2.2) are $\alpha$–isomorphic and $|\Omega(Q)| = m_0$, then the following equalities are valid:

\[
x_{j} = m_0 \cdot \left( j \left| T_{02} \setminus T_{s1} \right| - 1 \right) \cdot \left( 3 \left| T_{03} \setminus T_{s2} \right| - 2 \left| T_{03} \setminus T_{s2} \right| \right) \\
\ldots \left( (k - 1) \left| T_{0k-1} \setminus T_{sk-2} \right| - (k - 2) \left| T_{0k-1} \setminus T_{sk-2} \right| \right) \\
\left( k \left| T_{0k} \setminus T_{sk-1} \right| - (k - 1) \left| T_{0k} \setminus T_{sk-1} \right| \right) \cdot \left( 2 \left| T_{20} \setminus T_{1k} \right| - 1 \right) \\
\left( 3 \left| T_{20} \setminus T_{2k} \right| - 2 \left| T_{20} \setminus T_{2k} \right| \right) \ldots \left( (s - 1) \left| T_{s-10} \setminus T_{s-2k} \right| - (i - 1) \left| T_{s-10} \setminus T_{s-2k} \right| \right)
\]
automorphisms (i.e., $\bar{f}$ or $f$)

if $s \neq k$ or $b$

\[
|R(D')| = 2 \cdot m_0 \cdot \left( j|\bar{T}_{02}| - 1 \right) \cdot \left( 3|\bar{T}_{03}| - 2|\bar{T}_{03}| \right) \cdot \ldots \cdot \left( (k - 1)^{|\bar{T}_{0k-1}|} - 1 \right) \cdot \left( 2|\bar{T}_{20}| - 1 \right) \cdot \left( 3|\bar{T}_{30}| - 2|\bar{T}_{2k}| \right) \ldots \left( (s - 1)^{|\bar{T}_{s-10}|} - 1 \right) \cdot \left( 2|\bar{T}_{03}| - 1 \right) \cdot \left( 3|\bar{T}_{03}| - 2|\bar{T}_{30}| \right) \ldots \left( (s - 1)^{|\bar{T}_{s-10}|} - 1 \right) \cdot |Q| \cdot |X|^{-\bar{T}_{sk}},
\]

if $s = k$.

\textbf{Proof.} In the first place, we note that the given semilattice $Q$ has one automorphism (i.e., $|\Phi(Q, Q)| = 1$) if $s \neq k$ and two automorphism (i.e., $|\Phi(Q, Q)| = 2$) if $s = k$ (see [4, Theorem 11.7.1]). Hence, we have $|\Phi(Q, D')| = 1$ or $|\Phi(Q, D)| = 2$ (see [4, Lemma 6.3.2]).

Next, assume that $\alpha \in \bar{R}(Q, D')$ and a quasinormal representation of a regular binary relation $\alpha$ has the form

\[
\alpha = \bigcup_{T_{ij} \in Q} (Y_{ij}^\alpha \times T_{ij}) \quad (2.2)
\]

Then, according to Theorem 2.1, we have

\[
Y_{10}^\alpha \supseteq \bar{T}_{10}, \; Y_{10}^\alpha \cup Y_{20}^\alpha \supseteq \bar{T}_{20}, \ldots, \; Y_{10}^\alpha \cup Y_{20}^\alpha \cup \ldots \cup Y_{s0}^\alpha \supseteq \bar{T}_{s0}, \quad (2.3)
\]

\[
Y_{01}^\alpha \supseteq \bar{T}_{01}, \; Y_{01}^\alpha \cup Y_{02}^\alpha \supseteq \bar{T}_{02}, \ldots, \; Y_{01}^\alpha \cup Y_{02}^\alpha \cup \ldots \cup Y_{ok}^\alpha \supseteq \bar{T}_{ok},
\]

$Y_{20}^\alpha \cap \bar{T}_{20} \neq \varnothing$, ..., $Y_{s0}^\alpha \cap \bar{T}_{s0} \neq \varnothing$, $Y_{02}^\alpha \cap \bar{T}_{02} \neq \varnothing$, ..., $Y_{ok}^\alpha \cap \bar{T}_{ok} \neq \varnothing$,

for any $\bar{T}_{ij} \in \{ \bar{T}_{01}, \bar{T}_{02}, \ldots, \bar{T}_{0k}, \bar{T}_{10}, \bar{T}_{20}, \ldots, \bar{T}_{s0} \}$.

Further, let $f_{\alpha}$ be a mapping the set $X$ in the semilattice $D$ satisfying the conditions $f_{\alpha}(t) = t\alpha$ for all $t \in X$.

\[
f_{01\alpha}, f_{02\alpha}, \ldots, f_{0k-1\alpha}, f_{0k\alpha}, \quad \bar{T}_{01}, \bar{T}_{02} \setminus \bar{T}_{s1}, \ldots, \bar{T}_{0k-1} \setminus \bar{T}_{sk-2}, \bar{T}_{0k} \setminus \bar{T}_{sk-1},
\]

and $f_{sk\alpha}$, are the restrictions of the mapping $f_{\alpha}$, on the sets
\[ \bar{T}_{01}, \bar{T}_{20} \backslash \bar{T}_{1k}, \ldots, \bar{T}_{s-10} \backslash \bar{T}_{s-2k}, \bar{T}_{s0} \backslash \bar{T}_{s-1k}, X \backslash \bar{T}_{sk} \]

respectively. It is clear that the intersection disjoint elements of the set

\[
\{ \bar{T}_{01}, \bar{T}_{02} \backslash \bar{T}_{s1}, \ldots, \bar{T}_{0k-1} \backslash \bar{T}_{sk-2}, \bar{T}_{0k} \backslash \bar{T}_{sk-1}, \bar{T}_{10}, \bar{T}_{20} \backslash \bar{T}_{1k}, \ldots, \bar{T}_{s-10} \backslash \bar{T}_{s-2k}, \bar{T}_{s0} \backslash \bar{T}_{s-1k}, X \backslash \bar{T}_{sk} \}
\]

are empty set, and

\[
\bar{T}_{01} \cup (\bar{T}_{02} \backslash \bar{T}_{s1}) \cup \ldots \cup (\bar{T}_{0k-1} \backslash \bar{T}_{sk-2}) \cup (\bar{T}_{0k} \backslash \bar{T}_{sk-1}) \cup \bar{T}_{10} \cup (\bar{T}_{20} \backslash \bar{T}_{1k}) \cup \ldots \cup (\bar{T}_{s-10} \backslash \bar{T}_{s-2k}) \cup (\bar{T}_{s0} \backslash \bar{T}_{s-1k}) \cup (X \backslash \bar{T}_{sk}) = X.
\]

We are going to find properties of the maps \( f_{01\alpha}, f_{02\alpha}, \ldots, f_{0k-1\alpha}, f_{0k\alpha}, f_{10\alpha}, f_{20\alpha}, \ldots, f_{s-10\alpha}, f_{s0\alpha} \) and \( f_{sk\alpha} \).

1) \( t \in \bar{T}_{01} \). Then by properties (2.3) we have \( Y_{01}^\alpha \supseteq \bar{T}_{01} \), i.e., \( t \in Y_{01}^\alpha \) and \( t\alpha = T_{01} \) by definition of the set \( Y_{01}^\alpha \). Therefore \( f_{01\alpha}(t) = T_{01} \) for all \( t \in \bar{T}_{01} \).

2) \( t \in \bar{T}_{0j} \backslash \bar{T}_{sj-1} \) (\( j = 2, \ldots, k - 1, k \)). Then by properties (2.3) we have \( \bar{T}_{0j} \backslash \bar{T}_{sj-1} \subseteq \bar{T}_{0j} \subseteq Y_{01}^\alpha \cup Y_{02}^\alpha \cup \ldots \cup Y_{0j}^\alpha \) i.e., \( t \in Y_{01}^\alpha \cup Y_{02}^\alpha \cup \ldots \cup Y_{0j}^\alpha \) and \( t\alpha \in \{T_{01}, T_{02}, \ldots, T_{0j}\} \) by definition of the sets \( Y_{01}^\alpha, Y_{02}^\alpha, \ldots, Y_{0j}^\alpha \). Therefore \( f_{0j\alpha}(t) \in \{T_{01}, T_{02}, \ldots, T_{0j}\} \) for all \( t \in \bar{T}_{0j} \backslash \bar{T}_{sj-1} \).

By suppose we have that \( Y_{0j}^\alpha \cap \bar{T}_{0j} \neq \emptyset \), i.e., \( t_{0j} \alpha = T_{0j} \) for some \( t_{0j} \in \bar{T}_{0j} \). If \( t_{0j} \in \bar{T}_{sj-1} \), then \( T_{sj-1} = \bar{T}_{s0} \cup \bar{T}_{0j-1} \) by definition of the non complete net, and

\[
t_{0j} \in \bar{T}_{sj-1} = \bar{T}_{s0} \cup \bar{T}_{0j-1} \subseteq (Y_{10}^\alpha \cup Y_{20}^\alpha \cup \ldots \cup Y_{s0}^\alpha) \cup (Y_{01}^\alpha \cup \ldots \cup Y_{0j-1}^\alpha).
\]
So,

\[ t_{0j}^\alpha \in \{ T_{10}, T_{20}, \ldots, T_{s0}, T_{01}, T_{02}, \ldots, T_{0j-1} \} \]

by definition of the sets \( Y_{10}^\alpha, Y_{20}^\alpha, \ldots, Y_{s0}^\alpha, Y_{01}^\alpha, Y_{02}^\alpha, \ldots, Y_{0j-1}^\alpha \). The condition \( t_{0j}^\alpha \in \{ T_{10}, T_{20}, \ldots, T_{s0}, T_{01}, T_{02}, \ldots, T_{0j-1} \} \) contradict of the equality \( t_{0j}^\alpha \in T_{0j} \), while \( T_{0j}^\neq \{ T_{10}, T_{20}, \ldots, T_{s0}, T_{01}, T_{02}, \ldots, T_{0j-1} \} \). Therefore, \( f_{0j}\alpha(t_{0j}) = T_{0j}^\alpha \) for some \( t_{0j} \in T_{0j}\backslash T_{sj-1} \).

3) \( t \in \bar{T}_{10} \). Then by properties (2.3) we have \( Y_{10}^\alpha \subseteq \bar{T}_{10} \), i.e., \( t \in Y_{10}^\alpha \) and \( t\alpha = T_{10} \) by definition of the set \( Y_{10}^\alpha \). Therefore \( f_{10}\alpha(t) = T_{10} \) for all \( t \in \bar{T}_{10} \).

4) \( t \in \bar{T}_{i0}\backslash \bar{T}_{i-1k} \) (\( j = 2, \ldots, s-1, s \)). Then by properties (2.3) we have \( \bar{T}_{i0}\backslash \bar{T}_{i-10} \subseteq \bar{T}_{i0} \subseteq Y_{10}^\alpha \cup Y_{20}^\alpha \cup \cdots \cup Y_{si}^\alpha \), i.e., \( t \in Y_{10}^\alpha \cup Y_{20}^\alpha \cup \cdots \cup Y_{si}^\alpha \) and \( t\alpha \in \{ T_{10}, T_{20}, \ldots, T_{i0} \} \) by definition of the sets \( Y_{10}^\alpha, Y_{20}^\alpha, \ldots, Y_{si}^\alpha \). Therefore \( f_{i0}\alpha(t) \in \{ T_{10}, T_{20}, \ldots, T_{i0} \} \) for all \( t \in \bar{T}_{i0}\backslash \bar{T}_{i-1k} \).

By suppose we have that \( Y_{i0}^\alpha \cap \bar{T}_{i0} \neq \varnothing \), i.e., \( t_{i0}\alpha = T_{i0} \) for some \( t_{i0} \in \bar{T}_{i0} \).

If \( t_{i0} \in \bar{T}_{i-1k} \), then \( \bar{T}_{i-1k} = \bar{T}_{i-10} \cup \bar{T}_{0k} \) by definition of the non complete net, and

\[ t_{i0} \in \bar{T}_{i-1k} = \bar{T}_{i-10} \cup \bar{T}_{0k} \subseteq (Y_{10}^\alpha \cup Y_{20}^\alpha \cup \cdots \cup Y_{i-10}^\alpha) \cup (Y_{01}^\alpha \cup \cdots \cup Y_{0k}^\alpha) \]

So,

\[ t_{i0}\alpha \in \{ T_{10}, T_{20}, \ldots, T_{i-10}, T_{01}, T_{02}, \ldots, T_{0k} \} \]

by definition of the sets \( Y_{10}^\alpha, Y_{20}^\alpha, \ldots, Y_{i-10}^\alpha, Y_{01}^\alpha, Y_{02}^\alpha, \ldots, Y_{0k}^\alpha \). The condition \( t_{i0}\alpha \in \{ T_{10}, T_{20}, \ldots, T_{i-10}, T_{01}, T_{02}, \ldots, T_{0k} \} \) contradict of the equality \( t_{i0}\alpha = T_{i0} \), while \( T_{i0} \notin \{ T_{10}, T_{20}, \ldots, T_{i-10}, T_{01}, \ldots, T_{0k} \} \). Therefore, \( f_{i0}\alpha(t_{i0}) = T_{i0} \) for some \( t_{i0} \in \bar{T}_{i0}\backslash \bar{T}_{i-1k} \).

5) \( t \in X\backslash \bar{T}_{sk} \). Then by definition quasinormal representation binary relation \( \alpha \) and by property (2.3) we have \( t \in X\backslash \bar{T}_{sk} \subseteq X = \bigcup_{i \in N_s, j \in N_k} Y_{ij} \) \( ((i, j) \notin \{ (0, 0) \}) \), i.e., \( t\alpha \in Q \) by definition of the sets \( Y_{ij}^\alpha \). Therefore \( f_{sk}\alpha(t) \in Q \) for all \( t \in X\backslash \bar{T}_{sk} \).

Therefore, for every binary relation \( \alpha \in R(Q, D') \) there exists an ordered system

\[ (f_{01}\alpha, f_{02}\alpha, \ldots, f_{0k-1}\alpha, f_{0k}\alpha, f_{10}\alpha, f_{20}\alpha, \ldots, f_{s-10}\alpha, f_{s0}\alpha, f_{sk}\alpha) \quad (2.4) \]

Further, let

\[ f_{01} : \bar{T}_{01} \to \{ T_{01} \}, \quad f_{10} : \bar{T}_{10} \to \{ T_{10} \}, \]
\[ f_{0j} : \bar{T}_{0j}\backslash \bar{T}_{sj-1} \to \{ T_{00}, T_{01}, T_{02}, \ldots, T_{0j} \}, \quad (j = 2, \ldots, k-1, k), \]
\[ f_{i0} : \bar{T}_{i0}\backslash \bar{T}_{i-1k} \to \{ T_{00}, T_{10}, T_{20}, \ldots, T_{i0} \}, \quad (i = 2, \ldots, s-1, s), \]
\[ f_{sk} : X\backslash \bar{T}_{sk} \to \{ T_{ij} : i \in N_s, j \in N_k \}, \]
are such mappings, which satisfying the conditions:

6) $f_{01}(t) = T_{01}$ for all $t \in T_{01}$;
7) $f_{10}(t) = T_{10}$ for all $t \in T_{10}$;
8) $f_{0j}(t) \in \{T_{01}, T_{02}, \ldots, T_{0j}\}$ for all $t \in T_{0j} \setminus T_{sj-1}$ and $f_{0j}(t_{0j}) = T_{0j}$ for some $t_{0j} \in T_{0j} \setminus T_{sj-1}$;
9) $f_{10}(t) \in \{T_{10}, T_{20}, \ldots, T_{i0}\}$ for all $t \in T_{i0} \setminus T_{i-1k}$ and $f_{10}(t_{i0}) = T_{i0}$ for some $t_{i0} \in T_{i0} \setminus T_{i-1k}$;
10) $f_{sk}(t) \in Q$ for all $t \in X \setminus T_{sk}$.

Now, we define a map $f$ of a set $X$ in the semilattice $D$, which satisfies the condition:

$$f(t) = \begin{cases} 
  f_{01}(t), & \text{if } t \in T_{01} \\
  f_{0j}(t), & \text{if } t \in T_{0j} \setminus T_{sj-1} \ (j = 2, \ldots, k-1, k) \\
  f_{10}(t), & \text{if } t \in T_{10} \\
  f_{10}(t), & \text{if } t \in T_{i0} \setminus T_{i-1k} \ (i = 2, \ldots, s-1, s) \\
  f_{sk}(t), & \text{if } t \in X \setminus T_{sk}.
\end{cases}$$

Further, let $\beta = \bigcup_{x \in X} (\{x\} \times f(x))$, and $Y_{ij}^\beta = \{t: t\beta = T_{ij}\} \ (T_{ij} \in Q)$. Then binary relation $\beta$ may be representation by form $\beta = \bigcup_{T_{ij} \in Q} \left( Y_{ij}^\beta \times T_{ij} \right)$ and satisfying the conditions:

$$Y^\alpha_{01} \supset T_{01}, \ Y^\alpha_{01} \cup Y^\alpha_{02} \supset T_{02}, \ldots, \ Y^\alpha_{01} \cup Y^\alpha_{02} \cup \cdots \cup Y^\alpha_{0s} \supset T_{0s},$$

$$Y^\alpha_{01} \supset T_{01}, \ Y^\alpha_{01} \cup Y^\alpha_{02} \supset T_{02}, \ldots, \ Y^\alpha_{01} \cup Y^\alpha_{02} \cup \cdots \cup Y^\alpha_{0k} \supset T_{0k},$$

$$Y^\alpha_{20} \cap T_{20} \neq \emptyset, \ldots, \ Y^\alpha_{20} \cap T_{s0} \neq \emptyset, \ Y^\alpha_{20} \cap T_{20} \neq \emptyset, \ Y^\alpha_{20} \cap T_{sk} \neq \emptyset.$$

(By suppose $f_{i0}(t_{i0}) = T_{i0}$ for some $t_{i0} \in T_{i0} \setminus T_{i-1k}$ and $f_{0j}(t_{0j}) = T_{0j}$ for some $t_{0j} \in T_{0j} \setminus T_{sj-1}$). From this and by Theorem 2.1 we have that $\beta \in \bar{R}(Q, D')$. Therefore for every binary relation $\alpha \in \bar{R}(Q, D')$ and ordered system (2.4) exist one to one mapping.

By the Theorem 1.1 the number of the mappings $f_{01\alpha}, f_{02\alpha}, \ldots, f_{0k-1\alpha}, f_{0k\alpha}, f_{10\alpha}, f_{20\alpha}, \ldots, f_{s-10\alpha}, f_{s0\alpha}, f_{sk\alpha}$ are respectively:

$$1, j |T_{02} \setminus T_{s1}| - 1, 3 |T_{03} \setminus T_{s2}| - 2 |T_{03} \setminus T_{s2}|, \ldots, (k-1) |T_{0k-1} \setminus T_{sk-2}| - (k-2) |T_{0k-1} \setminus T_{sk-2}|, \ k |T_{0k} \setminus T_{sk-1}| - (k-1) |T_{0k} \setminus T_{sk-1}|, 2 |T_{20} \setminus T_{1k}| - 1, 3 |T_{30} \setminus T_{2k}| - 2 |T_{30} \setminus T_{2k}|, \ldots,$$

$$(s-1) \|T_{s-10} \setminus T_{s-2k}| - (i-1) \|T_{s-10} \setminus T_{s-2k}|, s |T_{s0} \setminus T_{s-1k}| - (s-1) |T_{s0} \setminus T_{s-1k}|, |D| X \setminus T_{sk}.$$

Therefore the equality

$$|\bar{R}(Q, D')| = \left( 2 |T_{02} \setminus T_{s1}| - 1 \right) \cdot \left( 3 |T_{03} \setminus T_{s2}| - 2 |T_{03} \setminus T_{s2}| \right) \ldots$$
\[
\left( (k - 1)^{k|T_{0k} \setminus T_{sk-2}|} - (k - 2)^{k|T_0| - 1}\right) \cdot \left( 2|T_0| - 1\right) \\
\cdot \left( 3|T_0| - 2|T_{sk-2}|\right) \\
\cdot \left( s - 1\right)^{s|T_0| - 1}\right) \\
\cdot \left( (s - 1)^{s|T_0| - 1}\right) \\
\cdot \left( 2|T_0| - 1\right) \\
\cdot \left( 3|T_0| - 2|T_{sk-2}|\right) \\
\cdot \left( s - 1\right)^{s|T_0| - 1}\right).
\]

is valid. Now, using the equalities $|\Omega(Q)| = m_0$ we obtain

\[
|R(D')| = m_0 \cdot \left( 2s|T_0| - 1\right) \\
\cdot \left( 3|T_0| - 2|T_{sk-2}|\right) \\
\cdot \left( s - 1\right)^{s|T_0| - 1}\right).
\]

if $s \neq k$ or

\[
|R(D')| = 2 \cdot m_0 \cdot \left( 2s|T_0| - 1\right) \\
\cdot \left( 3|T_0| - 2|T_{sk-2}|\right) \\
\cdot \left( s - 1\right)^{s|T_0| - 1}\right).
\]

if $s = k$ (see Theorem 1.6).

**Corollary 2.1.** Let $Q$ be a $\Omega$-lower incomplete net and $E_X^{(r)}(Q)$ be the set of all right units of the semigroup $B_x(Q)$. If $X$ is a finite set, then the following formula is true

\[
|E_X^{(r)}(Q)| = \left( 2s|T_0| - 1\right) \\
\cdot \left( 3|T_0| - 2|T_{sk-2}|\right) \\
\cdot \left( s - 1\right)^{s|T_0| - 1}\right).
\]
\begin{equation}
\left( (k - 1)^{|T_{0k-1}\setminus T_{sk-2}}| - (k - 2)^{|T_{0k-1}\setminus T_{sk-2}}| \right) .
\end{equation}

\begin{equation}
\left( k^{|T_{0k}\setminus T_{sk-1}}| - (k - 1)^{|T_{0k}\setminus T_{sk-1}}| \right) . \left( 2^{|T_{20}\setminus T_{1k}}| - 1 \right) .
\end{equation}

\begin{equation}
\left( 3^{|T_{30}\setminus T_{2k}}| - 2^{|T_{30}\setminus T_{2k}}| \right) . \left( (s - 1)^{|T_{s-10}\setminus T_{s-2k}}| - (i - 1)^{|T_{s-10}\setminus T_{s-2k}}| \right) . \left( |Q| \cdot X \setminus \bar{T}_{sk} \right) .
\end{equation}

**Proof.** By virtue of Theorem [4, Theorem 6.3.11] (see [4]) we have 
\[ E_{X}^{(r)}(Q) = R_{\varepsilon Q}(Q, Q) \], where \( \varepsilon Q \) is the identity mapping of the net \( Q \). Now, taking into account Theorem 1.5 and Theorem 2.4, we obtain the validity of corollary.

**Corollary 2.2.** Let \( Q = \{T_{01}, T_{10}, T_{11}\} \) be a \( XI \)-subsemilattice of the semilattice \( D \) (see Fig. 2.3). If the semilattices \( Q \) and \( D' = \{\bar{T}_{01}, \bar{T}_{10}, \bar{T}_{11}\} \) are \( \alpha \)-isomorphic, \( |\Omega(Q)| = m_{0} \), then the following equality is valid: 
\[ |R(D')| = 2 \cdot m_{0} \cdot 3^{|X\setminus \bar{T}_{11}|} \] .

**Proof.** It is obvious that in this case \( |\Phi(Q, D')| = 2 \). Therefore the corollary immediately follows from Theorem 2.4.

**Corollary 2.3.** Let \( Q = \{T_{01}, T_{10}, T_{02}, T_{11}, T_{20}, T_{12}, T_{21}, T_{22}\} \) be a \( XI \)-subsemilattice of the semilattice \( D \) (see Fig. 2.4). If the semilattices \( Q \) and 
\[ D' = \{\bar{T}_{01}, \bar{T}_{10}, \bar{T}_{02}, \bar{T}_{11}, \bar{T}_{20}, \bar{T}_{12}, \bar{T}_{21}, \bar{T}_{22}\} \] are \( \alpha \)-isomorphic, \( |\Omega(Q)| = m_{0} \), then the following equality is valid: 
\[ |R(D')| = 2 \cdot m_{0} \cdot \left( 2^{|T_{20}\setminus T_{12}|} - 1 \right) \cdot \left( 2^{|T_{02}\setminus T_{21}|} - 1 \right) \cdot 8^{|X\setminus \bar{T}_{22}|} \] .

**Proof.** It is obvious that in this case \( |\Phi(Q, D')| = 2 \). Therefore the corollary immediately follows from Theorem 2.4.
Corollary 2.4. Let \( Q = \{T_{01}, T_{10}, T_{02}, T_{11}, T_{20}, \ldots, T_{33}\} \) be a \( XI-\)sub-semilattice of the semilattice \( D \) (see Fig. 2.5). If the semilattices \( Q \) and
\[
D' = \{\bar{T}_{01}, \bar{T}_{10}, \bar{T}_{02}, \bar{T}_{11}, \bar{T}_{20}, \ldots, \bar{T}_{33}\}
\]
are \( \alpha-\)isomorphic, \( |\Omega(Q)| = m_0 \), then the following equality is valid:
\[
|R(D')| = 2 \cdot m_0 \cdot \left( 2|\bar{T}_{20}\setminus\bar{T}_{12}| - 1 \right) \cdot \left( 3|\bar{T}_{30}\setminus\bar{T}_{23}| - 2|\bar{T}_{30}\setminus\bar{T}_{21}| \right) 
\]
\[
\cdot \left( 2|\bar{T}_{02}\setminus\bar{T}_{21}| - 1 \right) \cdot \left( 3|\bar{T}_{03}\setminus\bar{T}_{32}| - 2|\bar{T}_{03}\setminus\bar{T}_{31}| \right) 
\]
\[
\cdot \left( s|\bar{T}_{s0}\setminus\bar{T}_{s1}| - (s - 1)|\bar{T}_{s0}\setminus\bar{T}_{s1-1}| \right) \cdot (2 \cdot (s + 1) - 1)|X\setminus\bar{T}_{33}|.
\]

Proof. It is obvious that in this case \( |\Phi(Q, D')| = 2 \). Therefore the corollary immediately follows from Theorem 2.4. \( \square \)

Corollary 2.5. Let \( Q = \{T_{01}, T_{10}, T_{02}, T_{11}, T_{20}, \ldots, T_{s-11}, T_{s0}, T_{s1}\} \) be a \( XI-\)sub-semilattice of the semilattice \( D \) (see Fig. 2.6). If the semilattices \( Q \) and
\[
D' = \{\bar{T}_{01}, \bar{T}_{10}, \bar{T}_{02}, \bar{T}_{11}, \bar{T}_{20}, \ldots, \bar{T}_{s-11}, \bar{T}_{s0}, \bar{T}_{s1}\}
\]
are \( \alpha-\)isomorphic, \( |\Omega(Q)| = m_0 \), then the following equality is valid:
\[
|R(D')| = m_0 \cdot \left( 2|\bar{T}_{20}\setminus\bar{T}_{11}| - 1 \right) \cdot \left( 3|\bar{T}_{30}\setminus\bar{T}_{23}| - 2|\bar{T}_{30}\setminus\bar{T}_{21}| \right) \ldots 
\]
\[
\left( (s - 1)|\bar{T}_{s-10}\setminus\bar{T}_{s-21}| - (s - 2)|\bar{T}_{s-10}\setminus\bar{T}_{s-21}| \right) 
\]
\[
\cdot \left( s|\bar{T}_{s0}\setminus\bar{T}_{s-11}| - (s - 1)|\bar{T}_{s0}\setminus\bar{T}_{s-11}| \right) \cdot (2 \cdot (s + 1) - 1)|X\setminus\bar{T}_{s1}|.
\]

Proof. It is obvious that in this case \( |\Phi(Q, D')| = 1 \). Therefore the corollary immediately follows from Theorem 2.4. \( \square \)

Corollary 2.6. Let \( Q = \{T_{01}, T_{10}, T_{11}, T_{20}, T_{21}\} \) be a \( XI-\)subsemilattice of the semilattice \( D \) (see Fig. 2.7). If the semilattices \( Q \) and
\[
D' = \{\bar{T}_{01}, \bar{T}_{10}, \bar{T}_{11}, \bar{T}_{20}, \bar{T}_{21}\}
\]
are α−isomorphic, \( |\Omega(Q)| = m_0 \), then the following equality is valid: \( |R(D')| = m_0 \cdot \left( 2^{|\bar{T}_{20}\setminus\bar{T}_{11}|} - 1 \right) \cdot 5^{|X\setminus\bar{T}_{21}|} \).

Proof. It is obvious that in this case \( |\Phi(Q, D')| = 1 \). Therefore the corollary immediately follows from Theorem 2.4. \( \square \)

**Corollary 2.7.** Let \( Q = \{T_{01}, T_{10}, T_{11}, T_{20}, T_{21}, T_{30}, T_{31}\} \) be a XI−sub-semilattice of the semilattice \( D \) (see Fig. 2.8). If the semilattices \( Q \) and

\[
D' = \{\bar{T}_{01}, \bar{T}_{10}, \bar{T}_{11}, \bar{T}_{20}, \bar{T}_{21}, \bar{T}_{30}, \bar{T}_{31}\}
\]

are α−isomorphic, \( |\Omega(Q)| = m_0 \), then the following equality is valid: \( |R(D')| = m_0 \cdot \left( 2^{|\bar{T}_{20}\setminus\bar{T}_{11}|} - 1 \right) \cdot \left( 3^{|\bar{T}_{30}\setminus\bar{T}_{21}|} - 2^{|\bar{T}_{30}\setminus\bar{T}_{21}|} \right) \cdot 7^{|X\setminus\bar{T}_{31}|} \).

Proof. It is obvious that in this case \( |\Phi(Q, D')| = 1 \). Therefore the corollary immediately follows from Theorem 2.4. \( \square \)

**References**


