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ON COMMUTATIVITY IN PRIME Γ-NEAR-RINGS WITH SYMMETRIC BI-DERIVATIONS

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Abstract: In this paper, we investigate the conditions of a prime Γ- near-ring to be commutative by means of symmetric bi-derivations. We prove that under certain conditions, prime Γ-near-rings will be the commutative Γ rings.

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Key Words: Γ-near-ring, prime Γ-near-ring, symmetric bi-derivation

1. Introduction

The derivations in Γ-near-rings have been introduced by Bell and Mason [2]. They investigated some basic properties of derivations in Γ-near-rings. Then Mustafa [11] obtained some commutativity conditions for a Γ-near-ring with derivations. Some characterizations of Γ-near-rings and some regularity conditions were obtained by Cho [5]. M. Kazaz and Akin Alkan [12] introduced the notion of two-sided Γ-α-derivation of a Γ-near-ring and investigated the com-

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mutativity of a prime and semiprime Γ-near-rings. M Uckun, M. A. Ozturk and Y. B. Jun [13] worked on prime Γ-near-rings with derivations and they investigated the conditions for a Γ-near-ring to be commutative.

In this paper, the commutativity conditions are investigated for Γ-near-rings with symmetric bi-derivations. We prove that every prime Γ-near-ring is a commutative Γ-ring by means of the trace of symmetric bi-derivations.

2. Preliminaries

A Γ-near-ring is a triple $(N, +, G)$, where:

(i) $(N, +)$ is a group (not necessarily Abelian),

(ii) G is a non-empty set of binary operations on N such that for each $\alpha \in G, (N, +, \alpha)$ is a left near-ring.

(iii) $x\alpha(y\beta z) = (x\alpha y)\beta z, \forall x, y, z \in N$ and $\alpha, \beta \in G$.

Exactly speaking, it is a *left* Γ*-near-ring* because it satisfies the left distributive law. We will use the word Γ*-near-ring* to mean *left* G*-near-ring.* For a near-ring N, the set $N_0 = \{x \in N : 0\alpha x = 0, \alpha \in G\}$ is called the *zero-symmetric part* of N. A Γ-near-ring N is said to be *zero-symmetric* if $N = N_0$. N will be prime, that is, will have the property that $xGNGy = 0$ for $x, y \in N$ implies $x = 0$ or $y = 0$. The symbol Z will denote the multiplicative center of N. For $x, y \in N$ and $\alpha \in G$, the symbol $[x, y]_\alpha$ will denote the commutator $x\alpha y - y\alpha x$, while the symbol (x, y) will denote the additive-group commutator $x + y - x - y$. A mapping $D : N \times N$®N is said to be symmetric if $D(x, y) = D(y, x), \forall x, y \in N$. A mapping $d : N$®N defined by $d(x) = D(x, x)$ is called the trace of D, where $D : N \times N$®N is a symmetric mapping. It is obvious that, if $D : N \times N$®N is a symmetric mapping which is also bi-additive (i.e., additive in both arguments), then the trace of D satisfies the relation $d(x + y) = d(x) + 2D(x, y) + d(y), \forall x, y \in N$. A symmetric bi-additive mapping $D : N \times N$®N is called a symmetric bi-derivation if $D(x\alpha y, z) = D(x, z)\alpha y + x\alpha D(y, z)$ is fulfilled $\forall x, y, z \in N, \alpha \in G$. Then, for any $y \in N$, a mapping x®$D(x, y)$ is a derivation. Let D be a symmetric bi-additive mapping of N. Then, $D(0, y) = 0$ is fulfilled for all $y \in N$ and so $D(-x, y) = -D(x, y), \forall x, y \in N$. Therefore, the mapping $d : N$®N defined by $d(x) = D(x, x)$ is an even function. As usual, an element $c \in N$ for which $d(c) = D(c, c) = 0$ is called a constant.

3. Main Results

We need the following lemmas to obtain our main results

Lemma 3.1. *Let N be a prime Γ-near-ring.*

(i) If $z \in Z - \{0\}$, then z is not a zero divisor.

(ii) If $Z - \{0\}$ contains an element z for which $z + z \in Z$, then $(N, +)$ is Abelian.

Proof. (i) If $z \in C - \{0\}$ and $z\alpha x = 0, x \in N, \alpha \in G$, then $z\alpha r\beta x = 0, x, r \in N, \alpha \in G$. Thus we get $zGNGx = 0$, by primeness of $N, x = 0$.

(ii) Let $z \in C - \{0\}$ be an element such that $z+z \in C$, and let $x, y \in N, \alpha \in G$. Since $z + z$ is distributive we get

$$(x+y)\alpha(z+z) = x\alpha(z+z)+y\alpha(z+z) = x\alpha z+x\alpha z+y\alpha z+y\alpha z = z\alpha(x+x+y+y).$$

On the other hand

$$(x + y)\alpha(z + z) = (x + y)\alpha z + (x + y)\alpha z = z\alpha((x + y + x + y).$$

Thus, $x + x + y + y = x + y + x + y$ and therefore $x + y = y + x$. Hence $(N, +)$ is Abelian. □

Lemma 3.2. *Let N be a 2-torsion free G-near-ring, D a symmetric bi-additive mapping of N and d the trace of D. If d(x) = 0 for all x ∈ N, then D = 0.*

Proof. For any $x, y \in N, d(x+y) = d(x)+2D(x, y)+d(y)$ and so, from the hypothesis and since N is 2-torsion free, we get $D(x, y) = 0 \forall x, y \in N$. Thus we get that $D = 0$. □

Lemma 3.3. *Let N be a 2 -torsion free prime G-near-ring, D a symmetric bi-derivation of N and d the trace of D. If $x\alpha d(y) = 0, \forall x, y \in N, \alpha \in G$, then $x = 0$ or $D = 0$.*

Proof. For any $y, z \in N, d(y+z) = d(y)+2D(y, z)+d(z)$. Now $x\alpha d(y+z) = x\alpha d(y) + 2x\alpha D(y, z) + x\alpha d(z), \forall x, y, z \in N, \alpha \in G$, so, from the hypothesis and since N is 2-torsion free, we get for all $x, y, z \in N, \alpha \in G$,

$$x\alpha D(y, z) = 0. \tag{1}$$

Taking $y\beta w, \beta \in G$, instead of y in (1), $0 = x\alpha D(y\beta w, z) = x\alpha D(y, z)\beta w + x\alpha y\beta D(w, z)$ and so from (1) we get $x\alpha y\beta D(w, z) = 0, \forall x, y, z, w \in N, \alpha, \beta \in G$. Since N is a prime Γ-near-ring, we get that $x = 0$ or $D = 0$. □

Lemma 3.4. *Let N be a Γ-near-ring, D a symmetric bi-additive mapping of N. Then the following are equivalent:*

(i) $D(x\alpha y, z) = D(x, z)\alpha y + x\alpha D(y, z), \forall x, y, z \in N, \alpha \in G$.

(ii) $D(x\alpha y, z) = x\alpha D(y, z) + D(x, z)\alpha y, \forall x, y, z \in N, \alpha \in G$.

Proof. $(ii) \Rightarrow (i)$ Suppose $D(x\alpha y, z) = D(x, z)\alpha y + x\alpha D(y, z), \forall x, y, z \in N, \alpha \in G$. Then

$$\begin{aligned} D(x\alpha(y + y), z) &= D(x, z)\alpha(y + y) + x\alpha D(y + y, z) \\ &= D(x, z)\alpha y + D(x, z)\alpha y + x\alpha D(y, z) + x\alpha D(y, z), \end{aligned}$$

and

$$\begin{aligned} D(x\alpha(y + y), z) &= D(x\alpha y + x\alpha y, z) = D(x\alpha y, z) + D(x\alpha y, z) \\ &= D(x, z)\alpha y + x\alpha D(y, z) + D(x, z)\alpha y + x\alpha D(y, z). \end{aligned}$$

Therefore we get $D(x, z)\alpha y + x\alpha D(y, z) = x\alpha D(y, z) + D(x, z)\alpha y, \forall x, y, z \in N, \alpha \in G$ and so we get that $D(x\alpha y, z) = x\alpha D(y, z) + D(x, z)\alpha y, \forall x, y, z \in N, \alpha \in G$.

$(i) \Rightarrow (ii)$. This is proved in a similar way. □

Lemma 3.5. *Let N be a Γ-near-ring, D a symmetric bi-additive mapping of N and d the trace of D. Then we have, $\forall x, y, z, w \in N, \alpha, \beta \in G$:*

(i) $(D(x, z)\alpha y + x\alpha D(y, z))\beta w = D(x, z)\alpha y\beta w + x\alpha D(y, z)\beta w$.

(ii) $(d(x)\alpha y + x\alpha D(x, y))\beta w = d(x)\alpha y\beta w + x\alpha D(x, y)\beta w$.

(iii) $(x\alpha D(y, z) + D(x, z)\alpha y)\beta w = x\alpha D(y, z)\beta w + D(x, z)\alpha y\beta w$.

(iv) $(x\alpha D(x, z) + d(x)\alpha y)\beta w = x\alpha D(x, y)\beta w + d(x)\alpha y\beta w$.

Proof. (i) From the associative law, we have

$$\begin{aligned} &D(x\alpha y, z) = D(x, z)\alpha y + x\alpha D(y, z), \forall x, y, z \in N, \alpha \in G, D((x\alpha y)\beta w, z) \\ &= D(x\alpha y, z)\beta w + x\alpha y\beta D(w, z) = (D(x, z)\alpha y + x\alpha D(y, z))\beta w + x\alpha y\beta D(w, z), \end{aligned}$$

and

$$\begin{aligned} D(x\alpha(y\beta w), z) &= D(x, z)\alpha y\beta w + x\alpha D(y\beta w, z) \\ &= D(x, z)\alpha y\beta w + x\alpha D(y, z)\beta w + x\alpha y\beta D(w, z). \end{aligned}$$

Comparing the two expression we get, $\forall x, y, z, w \in N, \alpha, \beta \in G, (D(x,z)\alpha y + x\alpha D(y,z))\beta w = D(x,z)\alpha y\beta w + x\alpha D(y,z)\beta w$.

(ii) Taking x instead of z in (i) and since D is a symmetric we get, $\forall x, y, w \in N, \alpha, \beta \in G, (d(x)\alpha y + x\alpha D(x,y))\beta w = d(x)\alpha y\beta w + x\alpha D(x,y)\beta w$.

The proof of (iii) (by Lemma 3.4) and (iv) are straightforward. □

Theorem 3.1. *Let N be a prime Γ-near-ring, D a non-zero symmetric bi-derivation of N and d the trace of D. If N is 2-torsion free and $d(N)Z$, then N is a commutative Γ-ring.*

Proof. Let c be an arbitrary constant and x a non-constant. Then $d(x+c) = d(x) + 2D(x,c) + d(c) in Z$ and so, since $d(x) \in Z - \{0\}$ and N is 2-torsion free, we get,

$$D(x,c) \in Z - \{0\}. \tag{2}$$

Taking $c\alpha x, \alpha \in G$, instead of x in (2) we get, $D(c\alpha x, c) = c\alpha D(x,c) + D(c,c)\alpha x = c\alpha D(x,c) \in Z - \{0\}$. From (2), it follows easily that $c \in Z - \{0\}$. Since $d(c+c) = 0$, for all constant c, so from Lemma 2.1(ii), we get that $(N,+)$ is Abelian. In this case, suppose 0 is the only constant, since $d(x) \in Z, \forall x \in N$, we get for all $x, y \in N$,

$$D(x,y) \in Z. \tag{3}$$

Now, suppose that u is not a zero divisor for $u \in N$ and let $x, y \in N, \alpha \in G$. Then

$$\begin{aligned} D(u\alpha(x+u), y) &= D(u,y)\alpha(x+u) + u\alpha D(x+u, y) \\ &= D(u,y)\alpha x + D(u,y)\alpha u + u\alpha D(x,y) + u\alpha D(u,y) \end{aligned}$$

and

$$\begin{aligned} D(u\alpha(x+u), y) &= D(u\alpha x + u\alpha u, y) = D(u\alpha x, y) + D(u\alpha u, y) \\ &= D(u,y)\alpha x + u\alpha D(x,y) + D(u,y)\alpha u + u\alpha D(u,y). \end{aligned}$$

Comparing the two expression we get, for all $x, y \in N, \alpha \in G$,

$$D(u,y)\alpha u + u\alpha D(x,y) = u\alpha D(x,y) + D(u,y)\alpha u.$$

From (3) and this equation we get, for all $x, y \in N, \alpha \in G$,

$$0 = u\alpha(D(u,y) + D(x,y) - D(x,y) - D(u,y)) = u\alpha D(u + x - u - x, y).$$

Thus, since u is not zero divisor, we get, for all $x, y \in N$.

$$D((u, x), y) = 0. \tag{4}$$

Taking (u, x) instead of y in (4) we get $d((u, x)) = 0, \forall x \in N$, and so (u, x) is constant, i.e., $(u, x) = 0, \forall x \in N$. Thus $u \in C(N)$ which is center of $(N, +)$. Now, let x be a non-zero element of N. From the hypothesis and Lemma 3.3 $d(x)$ is not a zero divisor, we get $d(x) \in C(N), \forall 0x \in N$. Since $d(x + z) = d(x) + 2D(x, z) + d(z) \in C(N)$, where $0z \in N$, we get $D(x, z) \in C(N), \forall 0x$ and $0y \in N$. Thus $0 = D(x, z) + D(y, z) - D(x, z) - D(y, z) = D((x, y), z)$ and so, we get that $(x, y) = 0, \forall x, y \in N$, i.e., $(N, +)$ is Abelian. Taking $x\beta w, \beta \in G$, instead of x in (3), we have, $z\alpha(D(x, y)\beta w + x\alpha D(w, y)) = (D(x, y)\beta w + x\alpha\beta D(w, y))z, \forall x, y, z, w \in N$. Thus from (3), Lemma 3.4 and Lemma 3.5 we get, $\forall x, y, z, w \in N, \alpha, \beta \in G$,

$$D(x, y)\beta[z, w]\alpha = D(w, y)\beta[z, x]\alpha. \tag{5}$$

Taking $d(w)$ instead of w in (5). From the hypothesis we get, for all $x, y, z, w \in N, \alpha, \beta \in G$,

$$D(d(w), y)\beta[z, x]\alpha = 0. \tag{6}$$

Substituting ydr for y, where $r \in N$, in (6) and from (6) we get, for all $x, y, z, w \in N, \alpha, \beta, d \in G$,

$$D(d(w), y)dr\beta[z, x]\alpha = 0. \tag{7}$$

Now, suppose that N is not commutative. In this case, since N is prime and from (6) we get, for all $y, w \in N$

$$D(d(w), y) = 0. \tag{8}$$

Substituting $x + w$ for w in (8), since N is a 2-torsion free and from (8) we get, for all $x, y, w \in N$

$$D(D(x, w), y) = 0. \tag{9}$$

Substituting xlw for x in (9) and so, from (8) and (9) we get, for all $x, y, w \in N, l \in G$,

$$D(x, w)lD(w, y) + D(x, y)ld(w) = 0. \tag{10}$$

Substituting x for w in (10) and since N is 2-torsion free, we get, for all $x, y \in N, l \in G$,

$$D(x, y)ld(x) = 0. \tag{11}$$

Taking xgy instead of y in (11), from (11) we get, for all $x, y \in N, l, g \in G$,

$$d(x)lygd(x) = 0. \tag{12}$$

From (12) and since N is 3-prime we get that $d(x) = 0$. By Lemma 3.2 we have $D = 0$. But this is a contradiction. □

Theorem 3.2. *Let N be a prime Γ-near-ring, D a non-zero symmetric bi-derivation of N and d the trace of D. If N is 2-torsion free and $d(y), d(y)+d(y) \in C(D(x,y)), forall x, y, z \in N$, then N is a commutative G-ring.*

Proof. From the hypothesis, if both w and $w + w$ commute element-wise with $D(x, z), \forall x, y \in N, \alpha \in G$, then $(D(x, z) + D(y, z))\alpha(w + w) = (D(x, z) + D(x, z) + D(y, z) + D(y, z))\alpha w$.

On the other hand

$$(D(x, z) + D(y, z))\alpha(w + w) = (D(x, z) + D(y, z) + D(x, z) + D(y, z))\alpha w.$$

Comparing the two expression we get, for all $x, y \in N, \alpha \in G, D(x + y - x - y)\alpha w = 0$. This implies that

$$D((x, y), z)\alpha w = 0. \tag{13}$$

Thus, let $w = d(r)$ in (13), we get $D((x, y), z)\alpha d(r) = 0, \forall x, y, z, r \in N$ and so, $D((x, y), z) = 0, \forall x, y, z, \in N$ by Lemma 3.3. Since $z\alpha(x, y)$ is also an additive commutator for any $z \in N, \alpha \in G$, we have $0 = D(z\alpha(x, y), z) = d(z)\alpha(x, y), \alpha \in G$, and so $(x, y) = 0$ by Lemma 3.3. That is, $(N, +)$ is Abelian. Now, since $d(y) \in C(D(x, y))\forall x, y, z \in N, , \alpha \in G$, we have

$$[D(x, z), d(y)]\alpha = 0. \tag{14}$$

Thus, replacing z by $z\beta w$ in (14), and from (14), we get, for all $x, y, z, w \in N, \alpha, \beta \in G$,

$$\begin{aligned} 0 &= D(x, z\beta w)\alpha d(y) - d(y)\alpha D(x, z\beta w) \\ &= (D(x, z)\beta w + z\beta D(x, w))\alpha d(y) - d(y)\alpha(D(x, z)\beta w + z\beta D(x, w)), \end{aligned}$$

and so, from (14) and Lemma 3.5(i), since $(N, +)$ is Abelian, we get for all $x, y, z, w \in N, \alpha, \beta \in G$,
$0 = D(x, z)\beta w\alpha d(y) - D(x, z)\alpha d(y)\beta w + z\beta d(y)\alpha D(x, w) - d(y)\alpha z\beta D(x, w)$.

Taking $d(z)$ for z in the previous equation, and the hypothesis, we get,

$$\forall x, y, z, w \in N, \alpha, \beta \in G, D(x, d(z))\beta[w, d(y)]\alpha = 0. \tag{15}$$

Replacing x by xdr in (15), and from (15), we get, $\forall x, y, z, w \in N, \alpha, \beta, d \in G$,

$$D(x, d(z))dr\beta[w, d(y)]\alpha = 0. \tag{16}$$

From (16) and since N is prime Γ-near-ring we get that $[w, d(y)]\alpha = 0, \forall y, w \in N, \alpha \in G$, or $D(x, d(z)) = 0, \forall x, z \in N$. If $d(N)Z$, then N is commutative Γ-ring by Theorem 3.6. If $D(d(z), x) = 0, \forall x, z \in N$, then the argument used in the proof of Theorem 3.6 (see, equation (8)) shows that $D = 0$. But this is a contradiction. □

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