International Journal of Pure and Applied Mathematics
Volume 93 No. 5 2014, 715-728
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v93i5.10

ijpam.eu

A STUDY OF SAIGO-MAEDA FRACTIONAL OPERATORS WITH GENERALIZED K-WRIGHT FUNCTION

Kantesh Gupta[1 §], Meena Kumari Gurjar[2], Jyotindra C. Prajapati[3]
[1,2]Department of Mathematics
Malaviya National Institute of Technology
Jaipur, 302017, Rajasthan, INDIA
[3]Department of Mathematical Sciences
Faculty of Applied Sciences, Charotar
University of Science and Technology
CHARUSAT, Changa, Anand, 388421, Gujarat, INDIA

Abstract: In this paper, we further study the generalized fractional integral and differential operators involving Appell's function $F_3\left(\,.\, \right)$ due to Saigo-Maeda [11]. During the course of our study, we obtain the images of the generalized K-Wright function in our operators. On account of the most general nature of our results, a large number of results obtained earlier by several authors such as Gehlot and Prajapati [3], Purohit et al. [10], Gupta and Gupta [4], Kilbas and Sebestian [7,8,9], Gupta and Gurjar [5], Kilbas [6] follow as special cases of our main findings.

AMS Subject Classification: 26A33, 33B15, 33C10, 33C20
Key Words: Saigo-Maeda fractional operators, Generalized K-Wright function, K-Gamma function, Bessel function

1. Introduction

Generalized K-Gamma function $\Gamma_k\left(x\right)$ defined as (Diaz and Pariguan [1])

Received: March 29, 2014

§Correspondence author

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!\, k^n (nk)^{\frac{x}{k}} - 1}{(x)_{n,k}},\ k>0,\ x \in C\backslash kZ^- \tag{1}$$

where $(x)_{n,k}$ is the k-Pochhammer symbol and is given by

$$(x)_{n,k} = x(x+k)(x+2k)\ldots\ldots\ldots\ldots(x+(n-1)k), \quad x\in C,\ k\in R,\ n\in N^+ \tag{2}$$

For $Re(x)>0,\ \Gamma_k(x)$ is defined as the integral

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\, dt \tag{3}$$

From equation (3) it follows that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right) \tag{4}$$

The generalized Wright function [13] for $z,\ a_i,\ b_j \in C$ and $\alpha_i, \beta_j \in R \quad (\alpha_i, \beta_j \neq 0;\ i=1,2,...,p; j=1,2,...,q)$ will be represented in the following manner:

$${}_p\psi_q(z) = {}_p\psi_q\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\middle|\, z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\frac{z^n}{n!} \tag{5}$$

The generalized K-Wright function ${}_p\psi\begin{smallmatrix}k\\q\end{smallmatrix}(z)$ is defined by Gehlot and Prajapati [2] for $a_i,\ b_j,\ z\in C,\ k\in R^+,\ \alpha_i,\beta_j\in R\ (\alpha_i,\beta_j\neq 0;\ i=1,2,...,p;\ j=1,2,...,q)$ and $(a_i+\alpha_i n), (b_j+\beta_j n) \in C\backslash kZ^-$

$${}_p\psi\begin{matrix}k\\q\end{matrix}(z) = {}_p\psi\begin{matrix}k\\q\end{matrix}\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\middle|\, z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma_k(b_j+\beta_j n)}\frac{z^n}{n!} \tag{6}$$

For convergence, we use the following notations

$$\delta = \sum_{j=1}^{q}\left(\frac{\beta_j}{k}\right) - \sum_{i=1}^{p}\left(\frac{\alpha_i}{k}\right); \quad D^{-1} = \prod_{i=1}^{p}\left|\frac{\alpha_i}{k}\right|^{-\frac{\alpha_i}{k}}\prod_{j=1}^{q}\left|\frac{\beta_j}{k}\right|^{\frac{\beta_j}{k}}$$

$$\mu = \sum_{j=1}^{q}\left(\frac{b_j}{k}\right) - \sum_{i=1}^{p}\left(\frac{a_i}{k}\right) + \frac{p-q}{2}$$

1. If $\delta > -1$,then the series (6) is absolutely convergent for all $z \in C$ and the generalized K-Wright function ${}_p\psi \begin{matrix} k \\ q \end{matrix} (z)$ is an entire function of z.
2. If $\delta = -1$,then the series (6) is absolutely convergent for all $|z| < D^{-1}$.
3. If $\delta = -1$,then the series (6) is absolutely convergent for all $|z| = D^{-1}$, $Re(\mu) > \frac{1}{2}$.

Special case of equation (6) becomes in the form by taking$k = 1, p = 0, q = 1, b_1 = 1 + \upsilon, \beta_1 = 1$ and $z = -\frac{z^2}{4}$

$$ {}_0\psi_1 \left[\begin{matrix} - \\ (\upsilon + 1, 1) \end{matrix} \middle| -\frac{z^2}{4} \right] = \sum_{n=0}^{\infty} \frac{1}{\Gamma(1 + \upsilon + n)} \frac{\left(-z^2/4\right)^n}{n!} = \left(\frac{z}{2}\right)^{-\upsilon} J_\upsilon(z) \quad (7) $$

with complex $z, \upsilon \in C$ known as Bessel function of first kind [12, p. 245, Eq. (A.29)].

In the present paper, we first study the Saigo-Maeda fractional integral operator [11, p. 393, Eqs. (4.12) and (4.13)] defined and represented in the following manner:

$$ \left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) $$
$$ = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}; 1 - \frac{x}{t}\right) f(t)\, dt, \quad (8) $$

$$ \left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) $$
$$ = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{x}{t}; 1 - \frac{t}{x}\right) f(t)\, dt, \quad (9) $$

where $\alpha, \alpha', \beta, \beta', \gamma \in C, Re(\gamma) > 0, x > 0$ and Appell function or Horn's F_3-function is kernel of our study. Also, the corresponding fractional differential operators [11] will be represented in the following manner:

$$ \left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) $$
$$ = \left(\frac{d}{dx}\right)^r \left(I_{0+}^{-\alpha',-\alpha,-\beta'+r,-\beta,-\gamma+r} f\right)(x), (Re(\gamma) > 0; r = [Re(\gamma)] + 1) $$

$$= \frac{1}{\Gamma(r-\gamma)} \left(\frac{d}{dx}\right)^r \left(x^{\alpha'}\right) \int_0^x (x-t)^{r-\gamma-1} t^{\alpha} F_3 \left(-\alpha', -\alpha, -\beta' + r, -\beta, r-\gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, dt \quad (10)$$

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x)$$
$$= \left(-\frac{d}{dx}\right)^r \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+r,-\gamma+r} f\right)(x), \left(Re(\gamma) > 0; r = [Re(\gamma)] + 1\right)$$
$$= \frac{1}{\Gamma(r-\gamma)} \left(-\frac{d}{dx}\right)^r (x^{\alpha}) \int_x^{\infty} (t-x)^{r-\gamma-1} t^{\alpha'} F_3 \left(-\alpha', -\alpha, -\beta', r-\beta, r-\gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\, dt, \quad (11)$$

respectively, where $[Re(\gamma)]$ is the integral part of $Re(\gamma)$.

The left hand sided and right hand sided generalized integration of the type (8) and (9) for power functions are given by Saigo and Maeda [11, p. 394, Eqs. (4.18) and (4.19)] as follows:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1}\right)(x)$$
$$= \Gamma \left[\begin{array}{c} \sigma, \sigma+\gamma-\alpha-\alpha'-\beta, \sigma+\beta'-\alpha' \\ \sigma+\beta', \sigma+\gamma-\alpha-\alpha', \sigma+\gamma-\alpha'-\beta \end{array} \right] x^{\sigma+\gamma-\alpha-\alpha'-1} \quad (12)$$

where $Re(\gamma) > 0$, $Re(\sigma) > \max\{0, Re(\alpha+\alpha'+\beta-\gamma), Re(\alpha'-\beta')\}$ and

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1}\right)(x)$$
$$= \Gamma \left[\begin{array}{c} 1-\sigma-\gamma+\alpha+\alpha', 1-\sigma+\alpha+\beta'-\gamma, 1-\sigma-\beta \\ 1-\sigma, 1-\sigma-\gamma+\alpha+\alpha'+\beta', 1-\sigma+\alpha-\beta \end{array} \right] x^{\sigma+\gamma-\alpha-\alpha'-1} \quad (13)$$

where $Re(\gamma) > 0$, $Re(\sigma) < 1 + \min\{Re(-\beta), Re(\alpha+\alpha'-\gamma), Re(\alpha+\beta'-\gamma)\}$.

2. Saigo-Maeda Fractional Integral Operators of the Generalized K-Wright Function

First, we consider the generalized left-hand sided fractional integral operator of the generalized K-Wright function.

Theorem 1. Let $a, \alpha, \alpha', \beta, \beta', \gamma \in C$ such that $\mu > 0, Re(\gamma) > 0$ and $Re(\lambda) > \max.[0, Re(\alpha + \alpha' + \beta - \gamma), Re(\alpha' - \beta')]$ then for $\delta > -1$ fractional integral operator $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of generalized K-Wright function ${}_p\psi_q^k(z)$ is given by

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\frac{\lambda}{k}-1}{}_p\psi_q^k\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\middle| at^{\frac{\mu}{k}}\right]\right)\right)(x)$$
$$= k^{\gamma}x^{\frac{\lambda}{k}+\gamma-\alpha-\alpha'-1} \times {}_{p+3}\psi_{q+3}^{k}$$
$$\left[\begin{array}{c}(a_i,\alpha_i)_{1,p},(\lambda,\mu),(\lambda+(\gamma-\alpha-\alpha'-\beta)k,\mu),(\lambda+(\beta'-\alpha')k,\mu)\\(b_j,\beta_j)_{1,q},(\lambda+\beta'k,\mu),(\lambda+(\gamma-\alpha-\alpha')k,\mu),(\lambda+(\gamma-\alpha'-\beta)k,\mu)\end{array}\middle| ax^{\frac{\mu}{k}}\right]. \tag{14}$$

Proof. Using equation (6) and (8), and then changing the order of integration and summation, which is justified under the conditions stated with Theorem 1, we get

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\frac{\lambda}{k}-1}{}_p\psi_q^k\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\middle| at^{\frac{\mu}{k}}\right]\right)\right)(x)$$
$$=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma_k(b_j+\beta_j n)}\frac{(a)^n}{n!}\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\frac{\lambda}{k}+\frac{\mu n}{k}-1}\right)\right)(x).$$

Now, applying the known result (12) with σ replaced by $\frac{\lambda}{k} + \frac{\mu n}{k}$, we obtain

$$k^{\gamma}x^{\frac{\lambda}{k}+\gamma-\alpha-\alpha'-1}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k(a_i+\alpha_i n)\Gamma_k(\lambda+\mu n)}{\prod_{j=1}^{q}\Gamma_k(b_j+\beta_j n)\Gamma_k(\lambda+\mu n+\beta' k)}$$
$$\times\frac{\Gamma_k(\lambda+\mu n+(\gamma-\alpha-\alpha'-\beta)k)\Gamma_k(\lambda+\mu n+(\beta'-\alpha')k)}{\Gamma_k(\lambda+\mu n+(\gamma-\alpha-\alpha')k)\Gamma_k(\lambda+\mu n+(\gamma-\alpha'-\beta)k)}\frac{\left(ax^{\mu/k}\right)^n}{n!}.$$

Finally, interpreting the above equation, in view of the definition (6), we arrive at the result (14).

Corollary 1. If we put $\alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha$ in equation (14), we get the following new and interesting result concerning Saigo fractional integral operator

$$\left(I_{0+}^{\alpha,\beta,\eta}\left(t^{\frac{\lambda}{k}-1}{}_{p}\psi\begin{matrix}k\\q\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle|at^{\frac{\mu}{k}}\right]\right)\right)(x)$$
$$=k^{\alpha}x^{\frac{\lambda}{k}-\beta-1}{}_{p+2}\psi\begin{matrix}k\\q+2\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p},(\lambda,\mu),(\lambda+(-\beta+\eta)k,\mu)\\(b_j,\beta_j)_{1,q},(\lambda-\beta k,\mu),(\lambda+(\alpha+\eta)k,\mu)\end{matrix}\middle|ax^{\frac{\mu}{k}}\right]. \quad (15)$$

Corollary 2. If we take $\beta=-\alpha$ and $\lambda=\gamma$ in above equation (15), we get following known result due to Gehlot and Prajapati [3, P. 285, Eq. (13)] concerning Riemann-Liouville fractional integral operator

$$\left(I_{0+}^{\alpha}\left(t^{\frac{\gamma}{k}-1}{}_{p}\psi\begin{matrix}k\\q\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle|at^{\frac{\mu}{k}}\right]\right)\right)(x)=k^{\alpha}x^{\frac{\gamma}{k}+\alpha-1}$$
$$\times{}_{p+1}\psi\begin{matrix}k\\q+1\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p},(\gamma,\mu)\\(b_j,\beta_j)_{1,q},(\gamma+\alpha k,\mu)\end{matrix}\middle|ax^{\frac{\mu}{k}}\right] \quad (16)$$

Further, taking $k=1$in equation (16), we get known result due to Kilbas [6, p. 117, Eq. (11)].

Now, we establish a theorem that gives the generalized right-hand sided fractional integral operator of the generalized K-Wright function.

Theorem 2. Let $a,\alpha,\alpha',\beta,\beta',\gamma\in C$ such that $\mu>0, Re(\gamma)>0$ and $Re(\lambda)>\max.[Re(\gamma-\alpha-\alpha'), Re(\gamma-\alpha-\beta'), Re(\beta)]$ then for $\delta>-1$ the fractional integral operator $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of generalized K-Wright function ${}_{p}\psi\begin{matrix}k\\q\end{matrix}(z)$ is given by

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\frac{\lambda}{k}}{}_{p}\psi\begin{matrix}k\\q\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle|at^{-\frac{\mu}{k}}\right]\right)\right)(x)=k^{\gamma}x^{-\frac{\lambda}{k}+\gamma-\alpha-\alpha'}$$
$$\times{}_{p+3}\psi\begin{matrix}k\\q+3\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p},(\lambda+(-\gamma+\alpha+\alpha')k,\mu),(\lambda+(\alpha+\beta'-\gamma)k,\mu),(\lambda-\beta k,\mu)\\(b_j,\beta_j)_{1,q},(\lambda,\mu),(\lambda+(-\gamma+\alpha+\alpha'+\beta')k,\mu),(\lambda+(\alpha-\beta)k,\mu)\end{matrix}\middle|ax^{-\frac{\mu}{k}}\right]. \quad (17)$$

Proof. Using equation (6) and (9), and then changing the order of integration and summation, which is justified under the conditions stated with Theorem 2, we get

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\frac{\lambda}{k}}{}_{p}\psi\begin{matrix}k\\q\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle|at^{-\frac{\mu}{k}}\right]\right)\right)(x)$$

$=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma_k(b_j+\beta_j n)}\frac{(a)^n}{n!}\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\frac{\lambda}{k}-\frac{\mu n}{k}}\right)\right)(x)$Now, applying the known result (13) with σ replaced by $-\frac{\lambda}{k} - \frac{\mu n}{k} + 1$, we obtain

$$= k^{\gamma} x^{-\frac{\lambda}{k} + \gamma - \alpha - \alpha'} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma_k \left(a_i + \alpha_i n\right) \Gamma_k \left(\lambda + \mu n + \left(\alpha + \alpha' - \gamma\right) k\right)}{\prod_{j=1}^{q} \Gamma_k \left(b_j + \beta_j n\right) \Gamma_k \left(\lambda + \mu n\right)}$$

$$\times \frac{\Gamma_k \left(\lambda + \mu n + \left(\alpha + \beta' - \gamma\right) k\right) \Gamma_k \left(\lambda - \beta k + \mu n\right)}{\Gamma_k \left(\lambda + \mu n + \left(\alpha + \alpha' + \beta' - \gamma\right) k\right) \Gamma_k \left(\lambda + \mu n + \left(\alpha - \beta\right) k\right)} \frac{\left(a x^{-\mu/k}\right)^n}{n!}$$

Finally, interpreting the above equation, in view of the definition (6), we arrive at the result (17).

Corollary 1. If we put $\alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha$ in equation (17), we get the following new and interesting result concerning Saigo fractional integral operator

$$\left(I_{-}^{\alpha,\beta,\eta} \left(t^{-\frac{\lambda}{k}} {}_{p}\psi_{q}^{k} \left[\begin{array}{c} \left(a_i, \alpha_i\right)_{1,p} \\ \left(b_j, \beta_j\right)_{1,q} \end{array} \middle| a t^{-\frac{\mu}{k}} \right] \right) \right)(x)$$

$$= k^{\alpha} x^{-\frac{\lambda}{k} - \beta} {}_{p+2}\psi_{q+2}^{k} \left[\begin{array}{c} \left(a_i, \alpha_i\right)_{1,p}, \left(\lambda + \beta k, \mu\right), \left(\lambda + \eta k, \mu\right) \\ \left(b_j, \beta_j\right)_{1,q}, \left(\lambda, \mu\right), \left(\lambda + \left(\alpha + \beta + \eta\right) k, \mu\right) \end{array} \middle| a x^{-\frac{\mu}{k}} \right] \tag{18}$$

Corollary 2. If we take $\beta = -\alpha$ and $\lambda = \gamma$ in above equation (18), we get following known result due to Gehlot and Prajapati [3, P. 286, Eq. (14)] concerning Riemann-Liouville fractional integral operator

$$\left(I_{-}^{\alpha} \left(t^{-\frac{\gamma}{k}} {}_{p}\psi_{q}^{k} \left[\begin{array}{c} \left(a_i, \alpha_i\right)_{1,p} \\ \left(b_j, \beta_j\right)_{1,q} \end{array} \middle| a t^{-\frac{\mu}{k}} \right] \right) \right)(x)$$

$$= k^{\alpha} x^{\alpha - \frac{\gamma}{k}} {}_{p+1}\psi_{q+1}^{k} \left[\begin{array}{c} \left(a_i, \alpha_i\right)_{1,p}, \left(\gamma - \alpha k, \mu\right) \\ \left(b_j, \beta_j\right)_{1,q}, \left(\gamma, \mu\right) \end{array} \middle| a x^{-\frac{\mu}{k}} \right] \tag{19}$$

Further, taking $k = 1$ in equation (19), we get known result due to Kilbas [6, p. 118, Eq. (13)].

3. Saigo-Maeda Fractional Differential Operators of the Generalized K-Wright Function

First, we consider the generalized left-hand sided fractional differential operator of the generalized K-Wright function.

Theorem 3. Let $a, \alpha, \alpha', \beta, \beta', \gamma \in C$ such that $\mu > 0, Re(\gamma) > 0$ and $Re(\lambda) > -\min.[0, Re(\alpha + \alpha' + \beta' - \gamma), Re(\alpha - \beta)]$ then for $\delta > -1$ fractional differential operator $D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of generalized K-Wright function ${}_p\psi\begin{smallmatrix}k\\q\end{smallmatrix}(z)$ is given by

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\frac{\lambda}{k}-1}{}_p\psi\begin{matrix}k\\q\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle| at^{\frac{\mu}{k}}\right]\right)\right)(x) = k^{-\gamma}x^{\frac{\lambda}{k}-\gamma+\alpha+\alpha'-1}$$
$$\times {}_{p+3}\psi\begin{matrix}k\\q+3\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p},(\lambda,\mu),(\lambda+(\alpha+\alpha'+\beta'-\gamma)k,\mu),(\lambda+(\alpha-\beta)k,\mu)\\(b_j,\beta_j)_{1,q},(\lambda-\beta k,\mu),(\lambda+(\alpha+\alpha'-\gamma)k,\mu),(\lambda+(\alpha+\beta'-\gamma)k,\mu)\end{matrix}\middle| ax^{\frac{\mu}{k}}\right]. \tag{20}$$

Proof. Let $r = Re[(\gamma)]+1$. Using (10) and applying (14), with $\alpha, \alpha', \beta, \beta', \gamma$ replaced

by $-\alpha', -\alpha, -\beta' + r, -\beta, -\gamma + r$, we have

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\frac{\lambda}{k}-1}{}_p\psi\begin{matrix}k\\q\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle| at^{\frac{\mu}{k}}\right]\right)\right)(x)$$

$$= \left(\frac{d}{dx}\right)^r\left\{I_{0+}^{-\alpha',-\alpha,-\beta'+r,-\beta,-\gamma+r}\left(t^{\frac{\lambda}{k}-1}{}_p\psi\begin{matrix}k\\q\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle| at^{\frac{\mu}{k}}\right]\right)\right\}(x)$$

$$= \left(\frac{d}{dx}\right)^r\left\{k^{-\gamma+r}x^{\frac{\lambda}{k}-\gamma+r+\alpha+\alpha'-1}\;{}_{p+3}\psi\begin{matrix}k\\q+3\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p},\\(b_j,\beta_j)_{1,q},\end{matrix}\right.\right.$$

$$\left.\left.\begin{matrix}(\lambda,\mu),(\lambda+(\alpha+\alpha'+\beta'-\gamma)k,\mu),(\lambda+(\alpha-\beta)k,\mu)\\(\lambda-\beta k,\mu),(\lambda+(\alpha+\alpha'-\gamma+r)k,\mu),(\lambda+(\alpha+\beta'-\gamma)k,\mu)\end{matrix}\middle| ax^{\frac{\mu}{k}}\right]\right\}(x).$$

Now, using equation (6) and changing the order of differentiation and summation, we get

$$k^{-\gamma+r}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k(a_i+\alpha_i n)\Gamma_k(\lambda+\mu n)\Gamma_k(\lambda+\mu n+(-\gamma+\alpha+\alpha'+\beta')k)}{\prod_{j=1}^{q}\Gamma_k(b_j+\beta_j n)\Gamma_k(\lambda-\beta k+\mu n)\Gamma_k(\lambda+\mu n+(-\gamma+r+\alpha+\alpha')k)}$$
$$\frac{\Gamma_k(\lambda+\mu n+(\alpha-\beta)k)}{\Gamma_k(\lambda+\mu n+(-\gamma+\alpha+\beta')k)}\frac{(a)^n}{n!}\left(\frac{d}{dx}\right)^r x^{\frac{\lambda}{k}-\gamma+r+\alpha+\alpha'-1+\frac{\mu n}{k}}.$$

Finally, interpreting the above equation, in view of the definition (6), we arrive at the result (20) after a little simplification.

Corollary 1. If we put $\alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha$ in equation (20), we get the following new and interesting result concerning Saigo fractional differential operator

$$\left(D_{0+}^{\alpha,\beta,\eta}\left(t^{\frac{\lambda}{k}-1}{}_p\psi\begin{matrix}k\\q\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle| at^{\frac{\mu}{k}}\right]\right)\right)(x)$$
$$= k^{-\alpha}x^{\frac{\lambda}{k}+\beta-1}{}_{p+2}\psi\begin{matrix}k\\q+2\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p},(\lambda,\mu),(\lambda+(\alpha+\beta+\eta)k,\mu)\\(b_j,\beta_j)_{1,q},(\lambda+\eta k,\mu),(\lambda+\beta k,\mu)\end{matrix}\middle| ax^{\frac{\mu}{k}}\right]. \tag{21}$$

Further, taking $k = 1$ in equation (21), we get known result due to Gupta and Gupta [4, p. 52, Eq. (25)].

Corollary 2. If we take $\beta = -\alpha$ and $\lambda = \gamma$ in equation (21), we get following known result due to Gehlot and Prajapati [3, p. 286, Eq. (15)] concerning Riemann-Liouville fractional differential operator

$$\left(D_{0+}^{\alpha}\left(t^{\frac{\gamma}{k}-1}{}_p\psi\begin{matrix}k\\q\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle| at^{\frac{\mu}{k}}\right]\right)\right)(x) = k^{-\alpha}x^{\frac{\gamma}{k}-\alpha-1}$$
$$\times {}_{p+1}\psi\begin{matrix}k\\q+1\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p},(\gamma,\mu)\\(b_j,\beta_j)_{1,q},(\gamma-\alpha k,\mu)\end{matrix}\middle| ax^{\frac{\mu}{k}}\right]. \tag{22}$$

Further, taking $k = 1$ in equation (22), we get known result due to Kilbas [6, p. 119, Eq. (14)].

Now, we establish a theorem that gives the generalized right-hand sided fractional differential operator of the generalized K-Wright function.

Theorem 4. Let $a, \alpha, \alpha', \beta, \beta', \gamma \in C$ such that $\mu > 0, Re(\gamma) > 0$ and $Re(\lambda) > -\min.[Re(\gamma-\alpha-\alpha'-r), Re(\gamma-\alpha'-\beta), Re(\beta')]$ then for $\delta > -1$ fractional differential operator $D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of generalized K-Wright function ${}_p\psi\begin{matrix}k\\q\end{matrix}(z)$ is given by

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\frac{\lambda}{k}}{}_p\psi\begin{matrix}k\\q\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle| at^{-\frac{\mu}{k}}\right]\right)\right)(x) = k^{-\gamma}x^{\alpha+\alpha'-\gamma-\frac{\lambda}{k}}$$
$$\times{}_{p+3}\psi\begin{matrix}k\\q+3\end{matrix}\left[\begin{matrix}(a_i,\alpha_i)_{1,p},(\lambda+(-\alpha'-\beta+\gamma)k,\mu),(\lambda+\beta' k,\mu),(\lambda+(\gamma-\alpha-\alpha')k,\mu)\\(b_j,\beta_j)_{1,q},(\lambda,\mu),(\lambda+(\gamma-\alpha-\alpha'-\beta)k,\mu),(\lambda+(\beta'-\alpha')k,\mu)\end{matrix}\middle| ax^{-\frac{\mu}{k}}\right]. \tag{23}$$

Proof. Let $r = Re\,[(\gamma)]+1$. Using (11) and applying (17), with $\alpha, \alpha', \beta, \beta', \gamma$ replaced by $-\alpha', -\alpha, -\beta', -\beta + r, -\gamma + r$, we have

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\frac{\lambda}{k}}{}_{p}\psi\begin{array}{c}k\\q\end{array}\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\middle| at^{-\frac{\mu}{k}}\right]\right)\right)(x)$$

$$= \left(-\frac{d}{dx}\right)^{r}\left\{I_{-}^{-\alpha',-\alpha,-\beta',-\beta+r,-\gamma+r}\left(t^{-\frac{\lambda}{k}}{}_{p}\psi\begin{array}{c}k\\q\end{array}\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\middle| at^{-\frac{\mu}{k}}\right]\right)\right\}(x)$$

$$= \left(-\frac{d}{dx}\right)^{r}\left\{k^{-\gamma+r}x^{-\frac{\lambda}{k}-\gamma+r+\alpha+\alpha'}{}_{p+3}\psi\begin{array}{c}k\\q+3\end{array}\left[\begin{array}{l}(a_i,\alpha_i)_{1,p}\,,\\(b_j,\beta_j)_{1,q}\,,\end{array}\right.\right.$$

$$\left.\left.\begin{array}{l}(\lambda+(\gamma-r-\alpha-\alpha')\,k,\mu)\,,(\lambda+(\gamma-\alpha'-\beta)\,k,\mu)\,,(\lambda+\beta'k,\mu)\\(\lambda,\mu)\,,(\lambda+(\gamma-\alpha-\alpha'-\beta)\,k,\mu)\,,(\lambda+(\beta'-\alpha')\,k,\mu)\end{array}\middle| ax^{-\frac{\mu}{k}}\right]\right\}(x)\,.$$

Now, using equation (6) and changing the order of differentiation and summation, we get

$$k^{-\gamma+r}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k\left(a_i+\alpha_i n\right)\Gamma_k\left(\lambda+\mu n+(\gamma-\alpha-\alpha'-r)\,k\right)\Gamma_k\left(\lambda+\mu n+(\gamma-\alpha'-\beta)\,k\right)}{\prod_{j=1}^{q}\Gamma_k\left(b_j+\beta_j n\right)\Gamma_k\left(\lambda+\mu n\right)\Gamma_k\left(\lambda+\mu n+(\gamma-\alpha-\alpha'-\beta)\,k\right)}$$

$$\times\frac{\Gamma_k\left(\lambda+\mu n+\beta' k\right)}{\Gamma_k\left(\lambda+\mu n+(-\alpha'+\beta')\,k\right)}\frac{(a)^n}{n!}\left(-\frac{d}{dx}\right)^{r}x^{-\frac{\lambda}{k}-\frac{\mu n}{k}-\gamma+r+\alpha+\alpha'}.$$

Finally, interpreting the above equation, in view of the definition (6), we arrive at the result (23) after a little simplification.

Corollary 1. If we put $\alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha$ in equation (23), we get the following new and interesting result concerning Saigo fractional differential operator

$$\left(D_{-}^{\alpha,\beta,\eta}\left(t^{-\frac{\lambda}{k}}{}_{p}\psi\begin{array}{c}k\\q\end{array}\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\middle| at^{-\frac{\mu}{k}}\right]\right)\right)(x)$$

$$= k^{-\alpha}x^{-\frac{\lambda}{k}+\beta}{}_{p+2}\psi\begin{array}{c}k\\q+2\end{array}\left[\begin{array}{l}(a_i,\alpha_i)_{1,p}\,,(\lambda+(\alpha+\eta)\,k,\mu)\,,(\lambda-\beta k,\mu)\,,\\(b_j,\beta_j)_{1,q}\,,(\lambda,\mu)\,,(\lambda+(-\beta+\eta)\,k,\mu)\end{array}\middle| ax^{-\frac{\mu}{k}}\right] \tag{24}$$

Corollary 2. If we take$\beta = -\alpha$ and $\lambda = \gamma$ in equation (24), we get following known result due to Gehlot and Prajapati [3, P. 287, Eq. (16)] concerning Riemann-Liouville fractional differential operator

$$\left(D_{-}^{\alpha}\left(t^{-\frac{\gamma}{k}}{}_{p}\psi\begin{array}{c}k\\q\end{array}\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\middle| at^{-\frac{\mu}{k}}\right]\right)\right)(x) = k^{-\alpha}x^{-\alpha-\frac{\gamma}{k}}$$

$$\times_{p+1}\psi\begin{matrix} k \\ q+1 \end{matrix}\left[\begin{matrix} (a_i,\alpha_i)_{1,p}, (\gamma+\alpha k,\mu) \\ (b_j,\beta_j)_{1,q}, (\gamma,\mu) \end{matrix}\middle| ax^{-\frac{\mu}{k}}\right]. \tag{25}$$

Further, taking $k = 1$ in equation (25), we get known result due to Kilbas [6, p. 120, Eq. (16)].

4. Special Cases

(i) If we reduce the generalized K-Wright function to the Bessel function of first kind [12] by taking $k = 1, p = 0, q = 1, b_1 = 1 + \upsilon, \beta_1 = 1, a = -\frac{z^2}{4}$ and $\mu = 2$ in equation (14), we get an interesting result

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\lambda-\upsilon-1}J_\upsilon(zt)\right)\right\}(x) = \left(\frac{z}{2}\right)^\upsilon x^{\lambda+\gamma-\alpha-\alpha'-1}$$
$$\times\, {}_3\psi_4\left[\begin{matrix}(\lambda,2),(\lambda+\gamma-\alpha-\alpha'-\beta,2),(\lambda+\beta'-\alpha',2)\\(1+\upsilon,1),(\lambda+\beta',2),(\lambda+\gamma-\alpha-\alpha',2),(\lambda+\gamma-\alpha'-\beta,2)\end{matrix}\middle| -\left(\frac{zx}{2}\right)^2\right]. \tag{26}$$

Further, setting $\lambda - \upsilon = \rho$ and $z = 1$ in (26), we get known result due to Purohit et al. [10, p. 24, Eq. (10)].

Again, setting $\lambda - \upsilon = \sigma, z = 1$ and $\alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha$ in (26), we get known result due to Kilbas and Sebestian [8, p.873, Eq. (26)].

(ii) On reducing the generalized Wright function in R.H.S. of equation (26) to the generalized hypergeometric function and taking $\lambda - \upsilon = \sigma, z = 1$ and $\alpha = \alpha + \beta\alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha$ in (26), we get known result due to Kilbas and Sebestian [8, p. 875, Eq. (37); see also 7, p. 166, Eq. (2.9)].

(iii) If we reduce the generalized K-Wright function to the Bessel function of first kind [12] by taking $k = 1, p = 0, q = 1, b_1 = 1 + \upsilon, \beta_1 = 1, a = -\frac{z^2}{4}$ and $\mu = 2$ in equation (17), we get an interesting result

$$\left\{I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\lambda+\upsilon}J_\upsilon\left(\frac{z}{t}\right)\right)\right\}(x) = \left(\frac{z}{2}\right)^\upsilon x^{-\lambda+\gamma-\alpha-\alpha'}$$
$$\times_3\psi_4\left[\begin{matrix}(\lambda-\gamma+\alpha+\alpha',2),(\lambda+\alpha+\beta'-\gamma,2),(\lambda-\beta,2)\\(1+\upsilon,1),(\lambda,2),(\lambda-\gamma+\alpha+\alpha'+\beta',2),(\lambda+\alpha-\beta,2)\end{matrix}\middle| -\left(\frac{z}{2x}\right)^2\right]. \tag{27}$$

Further, setting $\upsilon - \lambda = \rho - 1$ and $z = 1$ in (27), we get known result due to Purohit et al. [10, p.24, Eq. (12)].

Again, taking $\upsilon - \lambda = \sigma - 1, z = 1, \alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta$ and $\gamma = \alpha$ in (27), we get known result due to Kilbas and Sebestian [8, p.874, Eq. (31)].

(iv) On reducing the generalized Wright function in R.H.S. of equation (27) to the generalized hypergeometric function and taking $\upsilon - \lambda = \sigma - 1, z = 1, \quad \alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta$ and $\gamma = \alpha$ in (27), we get known result due to Kilbas and Sebestian [8, p. 876-877, Eq. (41); see also 7, p. 170, Eq. (3.6)].

(v) If we reduce the generalized K-Wright function to the Bessel function of first kind [12] by taking $k = 1, p = 0, q = 1, b_1 = 1 + \upsilon, \beta_1 = 1, a = -\frac{z^2}{4}$ and $\mu = 2$ in equation (20), we get an interesting result

$$\left\{ D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\lambda-\upsilon-1} J_\upsilon \left(zt \right) \right) \right\} (x) = \left(\frac{z}{2} \right)^\upsilon x^{\lambda-\gamma+\alpha+\alpha'-1}$$
$$\times {}_3\psi_4 \left[\begin{array}{l} (\lambda, 2), (\lambda + \alpha + \alpha' + \beta' - \gamma, 2), (\lambda + \alpha - \beta, 2) \\ (1 + \upsilon, 1), (\lambda - \beta, 2), (\lambda + \alpha + \alpha' - \gamma, 2), (\lambda + \alpha + \beta' - \gamma, 2) \end{array} \right| - \left(\frac{zx}{2} \right)^2 \right] \tag{28}$$

Further, setting $\lambda - \upsilon = \sigma$ and $z = 1$ in (28), we get known result due to Gupta and Gurjar [5].

Again, taking $\lambda - \upsilon = \sigma$, $z = \lambda$ and $\alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha$ in (28), we get known result due to Kilbas and Sebestian [9, p.330, Eq. (32)].

(vi) If we reduce the generalized K-Wright function to the Bessel function of first kind [12] by taking $k = 1, p = 0, q = 1, b_1 = 1 + \upsilon, \beta_1 = 1, a = -\frac{z^2}{4}$ and $\mu = 2$ in equation (23), we get an interesting result

$$\left\{ D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{-\lambda+\upsilon} J_\upsilon \left(\frac{z}{t} \right) \right) \right\} (x) = \left(\frac{z}{2} \right)^\upsilon x^{-\lambda-\gamma+\alpha+\alpha'}$$
$$\times {}_3\psi_4 \left[\begin{array}{l} (\lambda - \alpha' - \beta + \gamma, 2), (\lambda + \beta', 2), (\lambda + \gamma - \alpha - \alpha', 2) \\ (1 + \upsilon, 1), (\lambda, 2), (\lambda + \gamma - \alpha - \alpha' - \beta, 2), (\lambda + \beta' - \alpha', 2) \end{array} \right| - \left(\frac{z}{2x} \right)^2 \right] \tag{29}$$

Further, setting $\upsilon - \lambda = \sigma - 1, z = \lambda, \alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta$ and $\gamma = \alpha$ in (29), we get known result due to Kilbas and Sebestian [9, p.331, Eq. (37)].

5. Acknowledgments

The authors are highly thankful to Prof. S. L. Kalla for his valuable comments and suggestions to improve the quality of this paper.

References

[1] R. Diaz, E. Pariguan, On hypergeometric functions and Pochhamer k-symbol, *Divulgaciones Mathematics*, **15**, No. 2 (2007), 179-192.

[2] K. S. Gehlot, J. C. Prajapati, On generalization of K-Wright function and its properties, *Communicated for Publication.*

[3] K. S. Gehlot, J. C. Prajapati, Fractional calculus of generalized K-Wright function, *J. Fract. Calc. and Applications*, **4**, No. 2 (2013), 283-289.

[4] K. Gupta, A. Gupta, Generalized fractional differentiation of the multivariable H-function, *J. of the Applied Mathematics, Statistics and Informatics (JAMSI)*, **7**, No. 2 (2011), 45-54.

[5] K. Gupta, M. K. Gurjar, Saigo-Maeda fractional differential operators of the multivariable H-function, *Int. J. of Computational Sci. & Mathematics,* **5**, No. 2 (2013), 43-54.

[6] A. A. Kilbas, Fractional calculus of the generalized Wright function, *Fract. Calc. and Appl. Analysis,* **8**, No. 2 (2005), 113-126.

[7] A. A. Kilbas, N. Sebastian, Fractional integration of the product of Bessel functions of the first kind, *Fract. Calc. Appl. Anal.,* **13**, No. 2 (2010), 159-175.

[8] A. A. Kilbas, N. Sebastian, Generalized fractional integration of Bessel functions of the first kind, *Integral Transform and Special Functions*, **19**, No. 12 (2008), 869-883, **doi:** 10.1080/10652460802295978.

[9] A. A. Kilbas, N. Sebastian, Generalized fractional differentiation of Bessel function of the first kind, *Mathematica Balkanica,* New Series Vol. 22 (2008), 323-346.

[10] S. D. Purohit, D. L. Suthar, S. L. Kalla, Marichev Saigo-Maeda fractional integration operators of the Bessel functions, *Le Matematiche,* Vol. LXVII (2012), 21-32.

[11] M. Saigo, N. Maeda, More generalization of fractional calculus, *Transform Methods and Special Functions*, Varna, Bulgaria, (1996), 386-400.

[12] H. M. Srivastava, K. C. Gupta, S. P. Goyal, *The H-Functions of One and Two Variables With Applications*, South Asian Publications, New Delhi, Madras (1982).

[13] E. M. Wright, The asymptotic expansion of generalized hypergeometric function, *J. London Math. Soc.*, 10 (1935), 286-293, **doi:** 10.1112/jlms/s1-10.40.286.