A STUDY OF SAIGO-MAEDA FRACTIONAL OPERATORS
WITH GENERALIZED K-WRIGHT FUNCTION

Kantesh Gupta\textsuperscript{1,}\textsuperscript{§}, Meena Kumari Gurjar\textsuperscript{2}, Jyotindra C. Prajapati\textsuperscript{3}

\textsuperscript{1,2}Department of Mathematics
Malaviya National Institute of Technology
Jaipur, 302017, Rajasthan, INDIA

\textsuperscript{3}Department of Mathematical Sciences
Faculty of Applied Sciences, Charotar
University of Science and Technology
CHARUSAT, Changa, Anand, 388421, Gujarat, INDIA

Abstract: In this paper, we further study the generalized fractional integral and differential operators involving Appell’s function $F_3 (.)$ due to Saigo-Maeda [11]. During the course of our study, we obtain the images of the generalized K-Wright function in our operators. On account of the most general nature of our results, a large number of results obtained earlier by several authors such as Gehlot and Prajapati [3], Purohit et al. [10], Gupta and Gupta [4], Kilbas and Sebastian [7,8,9], Gupta and Gurjar [5], Kilbas [6] follow as special cases of our main findings.

AMS Subject Classification: 26A33, 33B15, 33C10, 33C20
Key Words: Saigo-Maeda fractional operators, Generalized K-Wright function, K-Gamma function, Bessel function

1. Introduction

Generalized K-Gamma function $\Gamma_k (x)$ defined as (Diaz and Pariguan [1])
\[ \Gamma_k(x) = \lim_{n \to \infty} \frac{n!}{(x)_{n,k}^k} \left( x^{k-1} - 1 \right), \quad k > 0, \ x \in \mathbb{C} \setminus k\mathbb{Z}^- \]  

where \( (x)_{n,k} \) is the k-Pochhammer symbol and is given by

\[ (x)_{n,k} = x(x+k)(x+2k) \ldots \ldots(x+(n-1)k), \quad x \in \mathbb{C}, \ k \in \mathbb{R}, \ n \in \mathbb{N}^+ \]  

For \( \Re(x) > 0 \), \( \Gamma_k(x) \) is defined as the integral

\[ \Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{x}} dt \]  

From equation (3) it follows that

\[ \Gamma_k(x) = k^{\frac{x}{k}} \Gamma \left( \frac{x}{k} \right) \]  

The generalized Wright function [13] for \( z, \ a_i, \ b_j \in \mathbb{C} \) and \( \alpha_i, \beta_j \in \mathbb{R} \) \( (\alpha_i, \beta_j \neq 0; \ i = 1, 2, \ldots, p; \ j = 1, 2, \ldots, q) \) will be represented in the following manner:

\[ p^\psi_q(z) = p^\psi_q \left[ \frac{(a_i, \alpha_i)_{1,p}}{(b_j, \beta_j)_{1,q}} \bigg| z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i n) z^n}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j n) n!} \]  

The generalized K-Wright function \( p^\psi_q k^\gamma \mathbb{Z}^- \) is defined by Gehlot and Prajapati [2] for \( a_i, \ b_j, \ z \in \mathbb{C}, \ k \in \mathbb{R}^+, \alpha_i, \beta_j \in \mathbb{R} \) \( (\alpha_i, \beta_j \neq 0; \ i = 1, 2, \ldots, p; \ j = 1, 2, \ldots, q) \) and \( (a_i + \alpha_i n), (b_j + \beta_j n) \in \mathbb{C} \setminus k\mathbb{Z}^- \)

\[ p^\psi_q k^\gamma \mathbb{Z}^- = p^\psi_q \left[ \frac{(a_i, \alpha_i)_{1,p}}{(b_j, \beta_j)_{1,q}} \bigg| z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma k(a_i + \alpha_i n) z^n}{\prod_{j=1}^{q} \Gamma k(b_j + \beta_j n) n!} \]  

For convergence, we use the following notations

\[ \delta = \sum_{j=1}^{q} \left( \frac{\beta_j}{k} \right) - \sum_{i=1}^{p} \left( \frac{\alpha_i}{k} \right); \quad D^{-1} = \prod_{i=1}^{p} \left| \frac{\alpha_i}{k} \right|^{-\frac{x_i}{k}} \prod_{j=1}^{q} \left| \frac{\beta_j}{k} \right|^{-\frac{x_j}{k}} \]

\[ \mu = \sum_{j=1}^{q} \left( \frac{b_j}{k} \right) - \sum_{i=1}^{p} \left( \frac{a_i}{k} \right) + \frac{p-q}{2} \]
1. If $\delta > -1$, then the series (6) is absolutely convergent for all $z \in C$ and the generalized K-Wright function $\psi^k_q(z)$ is an entire function of $z$.

2. If $\delta = -1$, then the series (6) is absolutely convergent for all $|z| < D^{-1}$.

3. If $\delta = -1$, then the series (6) is absolutely convergent for all $|z| = D^{-1}$, $\Re(\mu) > \frac{1}{2}$.

Special case of equation (6) becomes in the form by taking $k = 1, p = 0, q = 1, b_1 = 1 + \nu, \beta_1 = 1$ and $z = -\frac{z^2}{4}$

$$0\psi^1 \left[ \frac{-z^2}{4} \right] = \sum_{n=0}^{\infty} \frac{1}{\Gamma(1 + \nu + n)} \frac{(-z^2/4)^n}{n!} = (\frac{z}{2})^{-\nu} J_\nu(z) \quad (7)$$

with complex $z, \nu \in C$ known as Bessel function of first kind [12, p. 245, Eq. (A.29)].

In the present paper, we first study the Saigo-Maeda fractional integral operator [11, p. 393, Eqs. (4.12) and (4.13)] defined and represented in the following manner:

$$\left( I_{0+}^{\alpha,\beta,\gamma} f \right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha} F_3 \left( \alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) f(t) \, dt, \quad (8)$$

$$\left( I_{-\alpha,\beta,\gamma} f \right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3 \left( \alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{x}{t}, 1 - \frac{t}{x} \right) f(t) \, dt, \quad (9)$$

where $\alpha, \alpha', \beta, \beta', \gamma \in C, \Re(\gamma) > 0, x > 0$ and Appell function or Horn’s $F_3$-function is kernel of our study. Also, the corresponding fractional differential operators [11] will be represented in the following manner:

$$\left( D_{0+}^{\alpha,\beta,\gamma} f \right)(x) = \left( I_{0+}^{-\alpha',-\alpha,-\beta,-\gamma} f \right)(x)$$

$$= \left( \frac{d}{dx} \right)^r \left( I_{0+}^{-\alpha',-\alpha,-\beta+r,-\gamma+r} f \right)(x), \Re(\gamma) > 0; r = [\Re(\gamma)] + 1$$
\begin{align*}
\frac{1}{\Gamma(r - \gamma)} \left(\frac{d}{dx}\right)^{r} (x^{\alpha}) \int_{0}^{x} (x - t)^{r-\gamma-1} t^{\alpha} F_{3} (-\alpha', -\alpha, -\beta' + r, -\beta, r - \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t}) f(t) dt \tag{10}
\end{align*}

\begin{align*}
\left(D^{-\alpha',\beta,\gamma}_{-}\right)(x) &= \left(I_{-\alpha',\beta,\gamma}^{-\alpha,\beta,-\gamma} f\right)(x) \\
&= \left(-\frac{d}{dx}\right)^{r} \left(I_{-\alpha',\beta,-\gamma}^{-\alpha,-\beta,-\gamma+r} f\right)(x), (Re(\gamma) > 0; r = [Re(\gamma)] + 1) \\
&= \frac{1}{\Gamma(r - \gamma)} \left(-\frac{d}{dx}\right)^{r} (x^{\alpha}) \int_{x}^{\infty} (t - x)^{r-\gamma-1} t^{\alpha} F_{3} (-\alpha', -\alpha, -\beta', r - \beta, r - \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t}) f(t) dt, \tag{11}
\end{align*}

respectively, where \([Re(\gamma)]\) is the integral part of \(Re(\gamma)\).

The left hand sided and right hand sided generalized integration of the type (8) and (9) for power functions are given by Saigo and Maeda [11, p. 394, Eqs. (4.18) and (4.19)] as follows:

\begin{align*}
\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1} \right)(x) &= \Gamma \left[\frac{\sigma + \gamma - \alpha - \alpha' - \beta, \sigma + \beta' - \alpha'}{\sigma + \beta', \sigma + \gamma - \alpha - \alpha', \sigma + \gamma - \alpha' - \beta} \right] x^{\sigma + \gamma - \alpha - \alpha' - 1} \tag{12}
\end{align*}

where \(Re(\gamma) > 0, Re(\sigma) > \max\{0, Re(\alpha + \alpha' + \beta - \gamma), Re(\alpha' - \beta')\}\) and

\begin{align*}
\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1} \right)(x) &= \Gamma \left[\frac{1 - \sigma - \gamma + \alpha + \alpha', 1 - \sigma + \alpha + \beta' - \gamma, 1 - \sigma - \beta}{1 - \sigma, 1 - \sigma - \gamma + \alpha + \alpha' + \beta', 1 - \sigma + \alpha - \beta} \right] x^{\sigma + \gamma - \alpha - \alpha' - 1} \tag{13}
\end{align*}

where \(Re(\gamma) > 0, Re(\sigma) < 1 + \min\{Re(-\beta), Re(\alpha + \alpha' - \gamma), Re(\alpha + \beta' - \gamma)\}\).

2. Saigo-Maeda Fractional Integral Operators of the Generalized K-Wright Function

First, we consider the generalized left-hand sided fractional integral operator of the generalized K-Wright function.
Theorem 1. Let \( a, \alpha, \alpha', \beta, \beta', \gamma \in C \) such that \( \mu > 0, \Re(\gamma) > 0 \) and \( \Re(\lambda) > \max\{0, \Re(\alpha + \alpha' + \beta - \gamma), \Re(\alpha' - \beta')\} \) then for \( \delta > -1 \) fractional integral operator \( I_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \) of generalized K-Wright function \( p\psi_k q \) of \( z \) is given by

\[
(I_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} (t^{\frac{k-1}{p}\psi} q \left[ \frac{(a_i, \alpha_i)_{1,p} (b_j, \beta_j)_{1,q}}{at^{\frac{k}{p}}} \right])) (x)
\]

\[
= k^{\gamma} x^{\lambda + \gamma - \alpha - \alpha' - 1} \times \psi_k q \[ (a_i, \alpha_i)_{1,p} (b_j, \beta_j)_{1,q} \mid a x^{\frac{k}{p}} \].
\]

Proof. Using equation (6) and (8), and then changing the order of integration and summation, which is justified under the conditions stated with Theorem 1, we get

\[
(I_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} (t^{\frac{k-1}{p}\psi} q \left[ \frac{(a_i, \alpha_i)_{1,p} (b_j, \beta_j)_{1,q}}{at^{\frac{k}{p}}} \right])) (x)
\]

\[
= \sum_{n=0}^{\infty} \prod_{i=1}^{p} \Gamma_k (a_i + \alpha_i n) \frac{(a)^n}{n!} \left( I_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} (t^{\frac{k-1}{p}\psi} q \left[ \frac{(a_i, \alpha_i)_{1,p} (b_j, \beta_j)_{1,q}}{at^{\frac{k}{p}}} \right]) \right) (x).
\]

Now, applying the known result (12) with \( \sigma \) replaced by \( \frac{\lambda}{k} + \frac{\mu n}{k} \), we obtain

\[
k^{\gamma} x^{\lambda + \gamma - \alpha - \alpha' - 1} \sum_{n=0}^{\infty} \prod_{i=1}^{p} \Gamma_k (a_i + \alpha_i n) \Gamma_k (\lambda + \mu n + \beta k)
\]

\[
\times \frac{\prod_{j=1}^{q} \Gamma_k (b_j + \beta j n) \Gamma_k (\lambda + \mu n + \beta' k)}{\prod_{j=1}^{q} \Gamma_k (\lambda + \mu n + (\gamma - \alpha - \alpha') k) \Gamma_k (\lambda + \mu n + (\gamma - \alpha' - \beta) k)} \left( a x^{\mu/k} \right)^n n!.
\]

Finally, interpreting the above equation, in view of the definition (6), we arrive at the result (14).

Corollary 1. If we put \( \alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha \) in equation (14), we get the following new and interesting result concerning Saigo fractional integral operator
\[
\left( I_{0+}^{\alpha,\beta,\eta} \left( t^{-\beta} p \psi_k \frac{k}{q} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \text{at}^\frac{t}{k} \right) \right)(x) \\
= k^{\alpha} x^{\frac{\gamma}{k} - \beta - 1} \left( p+2 \psi_k \frac{k}{q} + 2 \left( a_i, \alpha_i \right)_{1,p}, (\lambda, \mu), (\lambda + (\beta + \eta) k, \mu) \\
\left( b_j, \beta_j \right)_{1,q}, (\lambda - \beta k, \mu), (\lambda + (\alpha + \eta) k, \mu) \right| ax^\frac{t}{k} \right) \text{.} (15)
\]

**Corollary 2.** If we take \( \beta = -\alpha \) and \( \lambda = \gamma \) in above equation (15), we get the following known result due to Gehlot and Prajapati [3, P. 285, Eq. (13)] concerning Riemann-Liouville fractional integral operator.

\[
\left( I_{0+}^{\alpha} \left( t^{-\beta} p \psi_k \frac{k}{q} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \text{at}^\frac{t}{k} \right) \right)(x) = k^{\alpha} x^{\frac{\gamma}{k} + \alpha - 1} \\
\times p+1 \psi_k \frac{k}{q} + 1 \left( a_i, \alpha_i \right)_{1,p}, (\gamma, \mu) \\
\left( b_j, \beta_j \right)_{1,q}, (\gamma + \alpha k, \mu) \right| ax^\frac{t}{k} \right) \text{.} (16)
\]

Further, taking \( k = 1 \) in equation (16), we get the known result due to Kilbas [6, p. 117, Eq. (11)].

Now, we establish a theorem that gives the generalized right-hand sided fractional integral operator of the generalized K-Wright function.

**Theorem 2.** Let \( a, \alpha, \alpha', \beta, \beta', \gamma \in C \) such that \( \mu > 0, \text{Re}(\gamma) > 0 \) and \( \text{Re}(\lambda) > \text{max} \left[ \text{Re}(\gamma - \alpha - \alpha'), \text{Re}(\gamma - \alpha - \beta'), \text{Re}(\beta) \right] \) then for \( \delta > -1 \) the fractional integral operator \( I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} \) of generalized K-Wright function \( \underbrace{p \psi \frac{k}{q} }_{z} \) is given by

\[
\left( I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} \left( t^{-\beta} p \psi_k \frac{k}{q} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q} \end{array} \right] \text{at}^\frac{t}{k} \right) \right)(x) = k^{\gamma} x^{\frac{-\gamma}{k} + \gamma - \alpha - \alpha'} \\
\times p+3 \psi_k \frac{k}{q} + 3 \left( a_i, \alpha_i \right)_{1,p}, (\lambda + (-\gamma + \alpha + \alpha') k, \mu), (\lambda + (\alpha + \beta' - \gamma) k, \mu), (\lambda - \beta k, \mu) \\
\left( b_j, \beta_j \right)_{1,q}, (\lambda + (-\gamma + \alpha + \alpha' + \beta') k, \mu), (\lambda + (\alpha - \beta) k, \mu) \right| ax^\frac{t}{k} \right) \text{.} (17)
\]

**Proof.** Using equation (6) and (9), and then changing the order of integration and summation, which is justified under the conditions stated with Theorem 2, we get

\[
\left( I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} \left( t^{-\beta} p \psi_k \frac{k}{q} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q} \end{array} \right] \text{at}^\frac{t}{k} \right) \right)(x) \\
= \sum_{n=0}^{\infty} \prod_{i=1}^{p} \Gamma_k(a_i + \alpha_i n) \left( a \right)^n \prod_{j=1}^{q} \Gamma_k(b_j + \beta_j n) \left( a \right)^n \left( I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} \left( t^{-\beta} \frac{a^n}{k} \right) \right)(x) \text{.} (18)
\]
sult (13) with $\sigma$ replaced by $-\frac{\lambda}{k} - \frac{\mu n}{k} + 1$, we obtain

$$= k^{\gamma} x^{-\frac{\lambda}{k}} + \gamma - \alpha - \alpha' \sum_{n=0}^{\infty} \prod_{i=1}^{p} \Gamma_k (a_i + \alpha_i n) \Gamma_k (\lambda + \mu n + (\alpha + \alpha' - \gamma) k)$$

$$\times \frac{\Gamma_k (\lambda + \mu n + (\alpha + \beta' - \gamma) k) \Gamma_k (\lambda - \beta k + \mu n)}{\Gamma_k (\lambda + \mu n + (\alpha + \alpha' + \beta' - \gamma) k) \Gamma_k (\lambda + \mu n + (\alpha - \beta) k)} (ax^{-\mu/k})^n \frac{n!}{n}$$

Finally, interpreting the above equation, in view of the defi nition (6), we arrive at the result (17).

**Corollary 1.** If we put $\alpha = \alpha + \beta, \alpha' = \beta, \gamma = \alpha$ in equation (17), we get the following new and interesting result concerning Saigo fractional integral operator

$$(I_{\alpha, \beta, \eta}^{\alpha, \beta, \eta} (t^{-\lambda} p_{\psi} k q \left[ \left( a_i, \alpha_i \right)_{1, p}, (b_j, \beta_j)_{1, q}, t^{-\mu/k} \right]) (x))$$

$$= k^\alpha x^{-\frac{\lambda - \beta}{k} + 2} q + 2 \left[ (a_i, \alpha_i)_{1, p}, (b_j, \beta_j)_{1, q}, (\lambda + (\alpha + \beta + \eta) k, \mu) | ax^{-\mu/k} \right]$$

**Corollary 2.** If we take $\beta = -\alpha$ and $\lambda = \gamma$ in above equation (18), we get following known result due to Gehlot and Prajapati [3, P. 286, Eq. (14)] concerning Riemann-Liouville fractional integral operator

$$(I_\alpha^{\alpha} (t^{-\gamma} p_{\psi} k q \left[ \left( a_i, \alpha_i \right)_{1, p}, (b_j, \beta_j)_{1, q}, t^{-\mu/k} \right]) (x))$$

$$= k^\alpha x^{-\frac{\gamma - \alpha}{k} + 1} q + 1 \left[ (a_i, \alpha_i)_{1, p}, (\gamma - \alpha k, \mu) | ax^{-\mu/k} \right]$$

Further, taking $k = 1$ in equation (19), we get known result due to Kilbas [6, p. 118, Eq. (13)].


First, we consider the generalized left-hand sided fractional differential operator of the generalized K-Wright function.
Theorem 3. Let \( a, \alpha, \alpha', \beta, \beta', \gamma \in C \) such that \( \mu > 0, Re(\gamma) > 0 \) and \( Re(\lambda) > -\min[0, Re(\alpha + \alpha' + \beta' - \gamma), Re(\alpha - \beta)] \) then for \( \delta > -1 \) fractional differential operator \( D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \) of generalized K-Wright function \( p\psi^k q (z) \) is given by

\[
(D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} (t_k^{-1} p\psi^k q \left[ (a_i, \alpha_i)_{1,p} \left( b_j, \beta_j \right)_{1,q} \right]) (x) = k^{\gamma - \gamma + \alpha + \alpha' - 1} (x)
\]

\[
\times p + 3^k q + 3 \left[ (a_i, \alpha_i)_{1,p}, (b_j, \beta_j)_{1,q}, (\lambda, \mu), (\lambda + (\alpha + \alpha' + \beta' - \gamma) k, \mu), (\lambda + (\alpha - \beta) k, \mu), (\lambda - \beta k, \mu), (\lambda + (\alpha + \alpha' - \gamma + r) k, \mu), (\lambda + (\alpha + \beta' - \gamma) k, \mu) \right] \left( x^k \right).
\]

Proof. Let \( r = Re (\gamma) + 1 \). Using (10) and applying (14), with \( \alpha, \alpha', \beta, \beta', \gamma \) replaced by \(-\alpha', -\alpha, -\beta' + r, -\beta, -\gamma + r \), we have

\[
(D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} (t_k^{-1} p\psi^k q \left[ (a_i, \alpha_i)_{1,p} \left( b_j, \beta_j \right)_{1,q} \right]) (x)
\]

\[
= \left( \frac{d}{dx} \right)^r \left\{ L_{r+}^{-\alpha, -\alpha, -\beta + r, -\beta, -\gamma + r} (t_k^{-1} p\psi^k q \left[ (a_i, \alpha_i)_{1,p} \left( b_j, \beta_j \right)_{1,q} \right]) (x)
\]

\[
= \left( \frac{d}{dx} \right)^r \left\{ \Gamma_k (\lambda + (\alpha + \alpha' + \beta' - \gamma) k, \mu), (\lambda + (\alpha - \beta) k, \mu), (\lambda - \beta k, \mu), (\lambda + (\alpha + \alpha' - \gamma + r) k, \mu), (\lambda + (\alpha + \beta' - \gamma) k, \mu) \right\} \left( x^k \right).
\]

Now, using equation (6) and changing the order of differentiation and summation, we get

\[
k^{\gamma + r} \sum_{n=0}^{\infty} \prod_{i=1}^{p} \Gamma_k (a_i + \alpha_i n) \Gamma_k (\lambda + \mu n) \Gamma_k (\lambda + \mu n + (-\gamma + \alpha + \alpha' + \beta') k)
\]

\[
\prod_{j=1}^{q} \Gamma_k (b_j + \beta_j n) \Gamma_k (\lambda - \beta k + \mu n) \Gamma_k (\lambda + \mu n + (-\gamma + r + \alpha + \alpha') k)
\]

\[
\frac{\Gamma_k (\lambda + \mu n + (\alpha - \beta) k)}{\Gamma_k (\lambda + \mu n + (-\gamma + \alpha + \beta') k)} \frac{(a)^n}{n!} \left( \frac{d}{dx} \right)^r \left( x^k \right) - \gamma + r + \alpha + \alpha' - 1 + \frac{\mu n}{k}.
\]

Finally, interpreting the above equation, in view of the definition (6), we arrive at the result (20) after a little simplification.
Corollary 1. If we put $\alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha$ in equation (20), we get the following new and interesting result concerning Saigo fractional differential operator

$$\left( D_{0+}^{\alpha,\beta,\eta} \left( t^{\frac{\lambda}{k}} - p \psi_q^k \left( \begin{array}{c} (a_i, \alpha)_{1,p} \mathcal{at}^\frac{\mu}{\lambda} \\ \frac{b_j, \beta}{1,q} \end{array} \right) \right) \right)(x)$$

$$= k^{-\alpha} x^{\frac{\lambda}{k} + \beta - 1} p + 2 \psi_q^k \left[ \begin{array}{c} (a_i, \alpha)_{1,p}, (\lambda, \mu), (\lambda + (\alpha + \beta + \eta) k, \mu) \\ (b_j, \beta)_{1,q}, (\lambda + \eta k, \mu), (\lambda + \beta k, \mu) \end{array} \right] a x^\frac{\mu}{\lambda}.$$

(21)

Further, taking $k = 1$ in equation (21), we get known result due to Gupta and Gupta [4, p. 52, Eq. (25)].

Corollary 2. If we take $\beta = -\alpha$ and $\lambda = \gamma$ in equation (21), we get following known result due to Gehlot and Prajapati [3, p. 286, Eq. (15)] concerning Riemann-Liouville fractional differential operator

$$\left( D_{0+}^\alpha \left( t^{\frac{\lambda}{k}} - p \psi_q^k \left( \begin{array}{c} (a_i, \alpha)_{1,p} \mathcal{at}^\frac{\mu}{\lambda} \\ \frac{b_j, \beta}{1,q} \end{array} \right) \right) \right)(x) = k^{-\alpha} x^{\frac{\lambda}{k} - \alpha - 1}$$

$$\times p + 1 \psi_q^k \left[ \begin{array}{c} (a_i, \alpha)_{1,p}, (\gamma, \mu) \\ (b_j, \beta)_{1,q}, (\gamma - \alpha k, \mu) \end{array} \right] a x^\frac{\mu}{\lambda}.$$

(22)

Further, taking $k = 1$ in equation (22), we get known result due to Kilbas [6, p. 119, Eq. (14)].

Now, we establish a theorem that gives the generalized right-hand sided fractional differential operator of the generalized K-Wright function.

Theorem 4. Let $a, \alpha, \alpha', \beta, \beta', \gamma \in C$ such that $\mu > 0, Re(\gamma) > 0$ and $Re(\lambda) > -\min \{Re(\gamma - \alpha - \alpha' - r), Re(\gamma - \alpha' - \beta), Re(\beta')\}$ then for $\delta > -1$ fractional differential operator $D_{-}^{\alpha,\beta,\gamma}$ of generalized K-Wright function $p \psi_q^k (z)$ is given by

$$\left( D_{-}^{\alpha,\alpha',\beta,\gamma} \left( t^{-\frac{\lambda}{k}} p \psi_q^k \left( \begin{array}{c} (a_i, \alpha)_{1,p} \mathcal{at}^\frac{\mu}{\lambda} \\ \frac{b_j, \beta}{1,q} \end{array} \right) \right) \right)(x) = k^{-\gamma} x^{\alpha + \alpha' - \gamma - \frac{\lambda}{\delta}}$$

$$\times p + 3 \psi_q^k \left[ \begin{array}{c} (a_i, \alpha)_{1,p}, (\lambda + (-\alpha - \beta + \gamma) k, \mu), (\lambda + \beta k, \mu), (\lambda + (\gamma - \alpha - \beta) k, \mu) \end{array} \right] a x^\frac{\mu}{\lambda}.$$

(23)
Proof. Let \( r = \text{Re} [(\gamma)] + 1 \). Using (11) and applying (17), with \( \alpha, \alpha', \beta, \beta', \gamma \) replaced by \(-\alpha', -\alpha, -\beta', -\beta + r, -\gamma + r, \) we have

\[
\left( D_{-\alpha, \alpha', \beta, \beta', \gamma} (t^{-\frac{\lambda}{k} p_\psi} k q \left[ (a_i, \alpha_i)_{1,p} \mid at^{-\frac{\lambda}{k}} \right]) \right) (x) = \left( -\frac{d}{dx} \right)^r \left\{ \Lambda_{-\alpha, -\beta, -\beta + r, -\gamma + r} (t^{-\frac{\lambda}{k} p_\psi} k q \left[ (a_i, \alpha_i)_{1,p} \mid at^{-\frac{\lambda}{k}} \right]) \right\} (x)
\]

Now, using equation (6) and changing the order of differentiation and summation, we get

\[
k^{-\gamma + r} \sum_{n=0}^{\infty} \prod_{i=1}^{p} \Gamma_k (a_i + \alpha n) \prod_{j=1}^{q} \Gamma_k (b_j + \beta n) \Gamma_k (\lambda + \mu n + (\gamma - \alpha - \alpha' - r) k) \Gamma_k (\lambda + \mu n + (\gamma - \alpha' - \beta) k) \times \frac{\Gamma_k (\lambda + \mu n + \beta k)}{\Gamma_k (\lambda + \mu n + (-\alpha' - \beta) k)} (a)^n (\frac{\lambda}{k}) (x) = k^{-\alpha} x^{-\frac{\lambda}{k} + \beta + p + 2} \psi q + 2 \left[ (a_i, \alpha_i)_{1,p} (\lambda + (\alpha + \eta) k, \mu), (\lambda + \beta k, \mu), (\lambda, \mu), (\lambda + (\beta' - \alpha') k, \mu) \mid ax^{-\frac{\lambda}{k}} \right]
\]

Finally, interpreting the above equation, in view of the definition (6), we arrive at the result (23) after a little simplification.

**Corollary 1.** If we put \( \alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha \) in equation (23), we get the following new and interesting result concerning Saigo fractional differential operator

\[
\left( D_{-\alpha, \alpha, \beta, \beta, \gamma} (t^{-\frac{\lambda}{k} p_\psi} k q \left[ (a_i, \alpha_i)_{1,p} \mid at^{-\frac{\lambda}{k}} \right]) \right) (x) = k^{-\alpha} x^{-\frac{\lambda}{k} - \frac{\mu}{k}} \psi q + 2 \left[ (a_i, \alpha_i)_{1,p} (\lambda + (\alpha + \eta) k, \mu), (\lambda + \beta k, \mu), (\lambda, \mu), (\lambda + (\beta' - \alpha') k, \mu) \mid ax^{-\frac{\lambda}{k}} \right]
\]

**Corollary 2.** If we take \( \beta = -\alpha \) and \( \lambda = \gamma \) in equation (24), we get following known result due to Gehlot and Prajapati [3, P. 287, Eq. (16)] concerning Riemann-Liouville fractional differential operator

\[
\left( D_{-\alpha} (t^{-\frac{\lambda}{k} p_\psi} k q \left[ (a_i, \alpha_i)_{1,p} \mid at^{-\frac{\lambda}{k}} \right]) \right) (x) = k^{-\alpha} x^{-\alpha - \frac{\lambda}{k}}
\]
Further, taking \( k = 1 \) in equation (25), we get known result due to Kilbas [6, p. 120, Eq. (16)].

4. Special Cases

(i) If we reduce the generalized K-Wright function to the Bessel function of first kind [12] by taking \( k = 1, p = 0, q = 1, b_1 = 1 + v, \beta_1 = 1, a = -z^2/4 \) and \( \mu = 2 \) in equation (14), we get an interesting result

\[
\times_{p+1} p^k q + 1 \left[ (a_i, \alpha_i)_{1,p}, (\gamma + \alpha_k, \mu) \bigg| ax - \mu \right] \cdot (25)
\]

Further, setting \( \lambda - v = \rho \) and \( z = 1 \) in (26), we get known result due to Purohit et al. [10, p. 24, Eq. (10)].

Again, setting \( \lambda - v = \sigma, z = 1 \) and \( \alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha \) in (26), we get known result due to Kilbas and Sebestian [8, p. 873, Eq. (26)].

(ii) On reducing the generalized Wright function in R.H.S. of equation (26) to the generalized hypergeometric function and taking \( \lambda - v = \sigma, z = 1 \) and \( \alpha = \alpha + \beta \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha \) in (26), we get known result due to Kilbas and Sebestian [8, p. 875, Eq. (37); see also 7, p. 166, Eq. (2.9)].

(iii) If we reduce the generalized K-Wright function to the Bessel function of first kind [12] by taking \( k = 1, p = 0, q = 1, b_1 = 1 + v, \beta_1 = 1, a = -z^2/4 \) and \( \mu = 2 \) in equation (17), we get an interesting result

\[
\times_{3} \psi_4 \left[ \frac{I_0^{\alpha, \alpha', \beta, \gamma} (t^{\lambda - v - 1} J_v (zt))}{(\lambda, \lambda + \gamma - \alpha' - \beta, 2, (\lambda + \beta - \gamma, 2), (\lambda + \gamma - \alpha' - \beta, 2), (\lambda + \gamma - \beta, 2)} \right] \cdot (27)
\]

Further, setting \( v - \lambda = \rho - 1 \) and \( z = 1 \) in (27), we get known result due to Purohit et al. [10, p.24, Eq. (12)].

Again, taking \( v - \lambda = \sigma - 1, z = 1, \alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta \) and \( \gamma = \alpha \) in (27), we get known result due to Kilbas and Sebestian [8, p.874, Eq. (31)].
(iv) On reducing the generalized Wright function in R.H.S. of equation (27) to the generalized hypergeometric function and taking \( v - \lambda = \sigma - 1, z = 1, \alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta \) and \( \gamma = \alpha \) in (27), we get known result due to Kilbas and Sebestian [8, p. 876-877, Eq. (41); see also 7, p. 170, Eq. (3.6)].

(v) If we reduce the generalized K-Wright function to the Bessel function of first kind [12] by taking \( k = 1, p = 0, q = 1, b_1 = 1 + v, \beta_1 = 1, a = -\frac{z^2}{4} \) and \( \mu = 2 \) in equation (20), we get an interesting result

\[
\begin{align*}
\left\{ D_{0+}^{\alpha, \alpha', \beta, \beta'} \left( t^{\lambda - v - 1} J_v \left( zt \right) \right) \right\}(x) &= \left( \frac{z}{2} \right)^v x^{\lambda - \gamma + \alpha + \alpha' - 1} \\
\times 3\psi_4 \left[ \begin{array}{c}
(\lambda, 2), (\lambda + \alpha + \alpha' + \beta' - \gamma, 2), (\lambda + \alpha - \beta, 2) \\
(1 + v, 1), (\lambda - \beta, 2), (\lambda + \alpha + \alpha' - \gamma, 2), (\lambda + \alpha + \beta' - \gamma, 2)
\end{array} \right] - \left( \frac{zx^2}{2} \right)^2 \end{align*}
\]

Further, setting \( \lambda - v = \sigma \) and \( z = 1 \) in (28), we get known result due to Gupta and Gurjar [5].

Again, taking \( \lambda - v = \sigma, z = \lambda \) and \( \alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha \) in (28), we get known result due to Kilbas and Sebestian [9, p.330, Eq. (32)].

(vi) If we reduce the generalized K-Wright function to the Bessel function of first kind [12] by taking \( k = 1, p = 0, q = 1, b_1 = 1 + v, \beta_1 = 1, a = -\frac{z^2}{4} \) and \( \mu = 2 \) in equation (23), we get an interesting result

\[
\begin{align*}
\left\{ D_{-}^{\alpha, \alpha', \beta, \beta'} \left( t^{-\lambda + v} J_v \left( \frac{z}{t} \right) \right) \right\}(x) &= \left( \frac{z}{2} \right)^v x^{-\lambda - \gamma + \alpha + \alpha'} \\
\times 3\psi_4 \left[ \begin{array}{c}
(\lambda - \alpha' - \beta + \gamma, 2), (\lambda + \beta', 2), (\lambda + \gamma - \alpha - \alpha', 2) \\
(1 + v, 1), (\lambda, 2), (\lambda + \gamma - \alpha - \alpha' - \beta, 2), (\lambda + \beta' - \alpha', 2)
\end{array} \right] - \left( \frac{zx^2}{2x} \right)^2 \end{align*}
\]

Further, setting \( v - \lambda = \sigma - 1, z = \lambda, \alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta \) and \( \gamma = \alpha \) in (29), we get known result due to Kilbas and Sebestian [9, p.331, Eq. (37)].

5. Acknowledgments

The authors are highly thankful to Prof. S. L. Kalla for his valuable comments and suggestions to improve the quality of this paper.
References


