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POISSON APPROXIMATION FOR THE NUMBER OF ISOLATED CYCLES IN A RANDOM INTERSECTION GRAPH

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Abstract: Let $W_{n,k}$ be the number of isolated cycles of order k in a random intersection graph $\mathbb{G}(n, m, p)$. In this paper, we demonstrate that $W_{n,k}$ can be approximated by Poisson distribution and give the bound of this approximation by using the Stein-Chen method.

Key Words: random intersection graph, isolated cycles, Stein-Chen method

1. Introduction

Given a set V with n vertices and another universal set U with m elements, define a bipartite graph $B(n, m, p)$ with independent vertex sets V and U and edges between $v \in V$ and $u \in U$ existing independently with probability p. The random intersection graph $\mathbb{G}(n, m, p)$, derived from $B(n, m, p)$, is defined on the vertex set V with vertices $v_1, v_2 \in V$ adjacent if and only if there exist some $u \in U$ such that both v_1 and v_2 are adjacent to u in $B(n, m, p)$. Also define S_i be a random subset of U such that each element of S_i is adjacent to $i \in V$, in which case two vertices $i, j \in V$ are adjacent if and only if $S_i \cap S_j \neq \phi$, and edge set $\mathrm{E}(\mathbb{G})$ is define as

$$\mathrm{E}(\mathbb{G}) = \{\{i, j\} : i, j \in V, S_i \cap S_j \neq \phi\}.$$

The properties of $\mathbb{G}(n, m, p)$ were studied in [2,3] contrasted with the well

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known random graph model $\mathbb{G}(n,p)$, in which vertices are made adjacent to each other independently and with probability p, and showed that for a fixed $\alpha > 0$, the number of elements m is taken to be $m = \lfloor n^{\alpha} \rfloor$. In 1999, Karonski, Scheinerman and Singer-Cohen[2] showed that the total variation distance between the distribution of $\mathbb{G}(n,m,p)$ and $\mathbb{G}(n,p)$ converges to 0 when $\alpha > 6$ and p is defined appropriately. Without loss of generality we consider the independent set V. For $i = 1, 2, 3, ..., n$, let

$$X_i = \begin{cases} 1 & \text{if vertex } i \text{ is an isolated in } \mathbb{G}(n,m,p) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$X = \sum_{i \in \Gamma} X_i.$$

Clearly, X is the number of isolated vertices in $\mathbb{G}(n,m,p)$.

In 2011, Yilum Shang[6] proved that the distribution function of X can be approximated by Poisson distribution with parameter

$$\lambda := \mathrm{E}W = nP(X_i = 1) = n \sum_{s=0}^{m} \binom{m}{s} p^s (1-p)^{m-s} (1-p)^{(n-1)s}$$
$$= n[1 - p + p(1-p)^{(n-1)}]^m.$$

In 2013, M. Donganont[9] showed the another proof of Poisson approximation for the number of isolated vertices in $\mathbb{G}(n,m,p)$ by Stein-Chen and coupling method. The results as the following,

Theorem 1.1 (9). *Let W be the number of isolated vertices in a random intersection graph $\mathbb{G}(n,m,p)$.*

For $A \subseteq \{0, 1, 2, ...n\}$ and $m = \lfloor n^{\alpha} \rfloor$ for some $\alpha > 0$, we have

$$|P(W \in A) - Poi_{\lambda}(A)| \leq \frac{nC_{\lambda,A}}{e^{pm(1-(1-p)^{n-2})}}$$

where $C_{\lambda,A} = \min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$,

$$\triangle(\lambda) = \begin{cases} e^{\lambda} + \lambda - 1 & \text{if } \lambda^{-1}(e^{\lambda} - 1) \leq M_A, \\ 2(e^{\lambda} - 1) & \text{if } \lambda^{-1}(e^{\lambda} - 1) > M_A, \end{cases}$$

and

$$M_A = \begin{cases} \max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\ \min\{w \mid w \in A\} & \text{if } 0 \notin A. \end{cases}$$

When $C_w = \{0, 1, ..., w\}$

Corollary 1.1. *[9] Let W be the number of isolated vertices in a random intersection graph $\mathbb{G}(n,m,p)$. Let $A \subseteq \{0,1,2,...n\}$, $m = \lfloor n^{\alpha} \rfloor$ for some $\alpha > 0$, $q = 1-p$, and $p = \frac{1}{n^{\gamma}}$ for any $\gamma \in \mathbb{R}^{+} \setminus \{1\}$,then*

$$|P(W \in A) - Poi_{\lambda}(A)| \leq \frac{C(\lambda, A)}{n^{2(\alpha-\gamma)(1-(q)^{n-2})-1}},$$

where $C(\lambda, A) = \min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$.

Let

$$\Gamma_{n,k} = \{i =: \{i_1, i_2, \ldots, i_k\} \mid 1 \leq i_1 < \ldots < i_k \leq n\}$$

be the set of all possible combinations of k vertices. we note that T_k is a tree of order k in $\mathbb{G}(n,m,p)$ and say that T_k is isolated in $\mathbb{G}(n,m,p)$ if there is no edge in $\mathbb{G}(n,m,p)$ between a vertex in the tree and the other outside of the tree.

For each $i \in \Gamma_{n,k}$, we define the indicator random variable

$$X_i = \begin{cases} 1 & \text{if there is an isolated tree in } \mathbb{G}(n,m,p) \text{ that spans the vertices} \\ & i = (i_1, \ldots, i_k), \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$W_{n,k} = \sum_{i \in \Gamma_{n,k}} X_i.$$

Then $W_{n,k}$ is the number of isolated trees in $\mathbb{G}(n,m,p)$.

In 2013, Mana [10] shows that if $m = \lfloor n^{\alpha} \rfloor$; $\alpha > 0$, then $W_{n,k}$ can be approximated by Poisson approximation with parameter

$$\begin{aligned} \lambda := \mathbb{E}(W_{n,k}) = \binom{n}{k} P(X_i = 1) &= \binom{n}{k} \sum_{s=0}^{m} \binom{m}{s} p^{s+k-1}(1-p)^{m-s}(1-p)^{(n-k)s} \\ &= \binom{n}{k} p^{k-1} \left[1 - p + p(1-p)^{(n-k)}\right]^{m}. \qquad (1) \end{aligned}$$

By using Stein-Chen and Coupling Method. The result is following,

Theorem 1.2 (10). *Let W be the number of isolated trees in a random intersection graph $\mathbb{G}(n,m,p)$.*

For $A \subseteq \{0,1,2,...n\}$ and $m = \lfloor n^{\alpha} \rfloor$ for some $\alpha > 0$, we have

$$|P(W \in A) - Poi_{\lambda}(A)| \leq \frac{C_{(\lambda,A)}(1 + \frac{n^k}{k!}pk^2)p^{k-1}}{e^{mp(1-(1-p)^{n-2k})}}$$

where $C_{(\lambda,A)} = \min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$,

$$\triangle(\lambda) = \begin{cases} e^{\lambda} + \lambda - 1 & \text{if } \lambda^{-1}(e^{\lambda}-1) \le M_A, \\ 2(e^{\lambda}-1) & \text{if } \lambda^{-1}(e^{\lambda}-1) > M_A, \end{cases}$$

and

$$M_A = \begin{cases} \max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\ \min\{w \mid w \in A\} & \text{if } 0 \notin A. \end{cases}$$

When $C_w = \{0, 1, ..., w\}$

Corollary 1.2. *[10] Let W be the number of isolated trees in a random intersection graph $\mathbb{G}(n,m,p)$. Let $A \subseteq \{0,1,2,...n\}$, $m = \lfloor n^{\alpha} \rfloor$ for some $\alpha > 0$, and $p = \dfrac{1}{n^{\gamma}}$ for any $\gamma \in \mathbb{R}^{+}\{1\}$*

1. *If $\gamma > k > 0$, $q = 1 - p$,then*

$$|P(W \in A) - Poi_{\lambda}(A)| \le \frac{C(\lambda, A)}{n^{2(\alpha-\gamma)(1-(q)^{n-2k})+(k\gamma-\gamma)}},$$

where $C(\lambda, A) = 2\min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$.

2. *If $k > \gamma > 0$, $q = 1 - p$, then*

$$|P(W \in A) - Poi_{\lambda}(A)| \le \frac{C(\lambda, A)}{n^{2(\alpha-\gamma)(1-(q)^{n-2k})+(k\gamma-k)}},$$

where $C(\lambda, A) = 2\min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$.

This work, we use this idea in order to show that the number of isolated cycles can be approximated by Poisson distribution.

Now, we define,

$$\Gamma_{n,k} = \{i =: \{i_1, i_2, \ldots, i_k\} \mid 1 \le i_1 < \ldots < i_k \le n\}$$

be the set of all possible combinations of k vertices. we note that C_k is a cycle of order k in $\mathbb{G}(n,m,p)$ and say that C_k is isolated in $\mathbb{G}(n,m,p)$ if there is no edge in $\mathbb{G}(n,m,p)$ between a vertex in the cycle and the other outside of the cycle.

For each $i \in \Gamma_{n,k}$, we define the indicator random variable

$$X_i = \begin{cases} 1 & \text{if there is an isolated cycle in } \mathbb{G}(n,m,p) \text{ that spans the vertices} \\ & i = (i_1, \ldots, i_k), \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$W_{n,k} = \sum_{i \in \Gamma_{n,k}} X_i.$$

Thus, $W_{n,k}$ is the number of isolated cycles in $\mathbb{G}(n,m,p)$.

If $m = \lfloor n^\alpha \rfloor$; $\alpha > 0$, we demonstrate that $W_{n,k}$ can be approximated by Poisson approximation with parameter

$$\begin{aligned}\lambda := \mathbb{E}(W_{n,k}) = \binom{n}{k} P(X_i = 1) &= \binom{n}{k} \sum_{s=0}^{m} \binom{m}{s} p^{s+k}(1-p)^{m-s}(1-p)^{(n-k)s} \\ &= \binom{n}{k} p^k \left[1 - p + p(1-p)^{(n-k)}\right]^m. \end{aligned} \tag{2}$$

By using Stein-Chen and Coupling Method. The main result is following,

Theorem 1.3. *Let W be the number of isolated cycles in a random intersection graph $\mathbb{G}(n,m,p)$. For $A \subseteq \{0,1,2,...,n\}$ and $m = \lfloor n^\alpha \rfloor$ for some $\alpha > 0$, then*

$$|P(W \in A) - Poi_\lambda(A)| \leq \frac{C_{(\lambda,A)}(1 + \frac{n^k}{k!}k^2)p^k}{e^{mp(1-(1-p)^{n-2k})}}$$

where $C_{(\lambda,A)} = \min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$,

$$\triangle(\lambda) = \begin{cases} e^\lambda + \lambda - 1 & \text{if } \lambda^{-1}(e^\lambda - 1) \leq M_A, \\ 2(e^\lambda - 1) & \text{if } \lambda^{-1}(e^\lambda - 1) > M_A, \end{cases}$$

and

$$M_A = \begin{cases} \max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\ \min\{w \mid w \in A\} & \text{if } 0 \notin A. \end{cases}$$

When $C_w = \{0,1,....,w\}$

Corollary 1.3. *Let W be the number of isolated cycles in a random intersection graph $\mathbb{G}(n,m,p)$. If $A \subseteq \{0,1,2,...,n\}$, $m = \lfloor n^\alpha \rfloor$ for some $\alpha > 0$, $q = 1-p$ and $p = \dfrac{1}{n^\gamma}$ for any $\gamma \in \mathbb{R}^+\{1\}$,then*

$$|P(W \in A) - Poi_\lambda(A)| \leq \frac{C(\lambda,A)}{n^{2(\alpha-\gamma)(1-(q)^{n-2k})+(k\gamma-k)}},$$

where $C(\lambda,A) = 2\min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$.

2. Stein-Chen and Coupling Method

In 1972, Stein [1] gave a new technique to find a bound in the normal approximation to a distribution of a sum of dependent random variables. His technique was relied instead on the elementary differential equation. In 1975, Chen [4, 5] applied Stein's idea to the Poisson case. The central idea of the Stein-Chen method is the difference equation

$$I_A(j) - \mathcal{P}_\lambda(A) = \lambda g_{\lambda,A}(j+1) - j g_{\lambda,A}(j), j \in \mathbb{N} \cup \{0\} \tag{3}$$

where $\lambda > 0$ and $A \subseteq \mathbb{N} \cup \{0\}$ and $I_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$ is defined by

$$I_A(w) = \begin{cases} 1 & ; w \in A, \\ 0 & ; w \notin A. \end{cases}$$

The equation (3) is called Stein's equation for Poisson distribution function and its solution is

$$g_{\lambda,A}(w) = \begin{cases} (w-1)!\lambda^{-w} e^{\lambda}[\mathcal{P}_\lambda(I_{A \cap C_{w-1}}) - \mathcal{P}_\lambda(I_A)\mathcal{P}_\lambda(I_{C_{w-1}})] & ; w \geq 1, \\ 0 & ; w = 0 \end{cases}$$

where

$C_{w-1} = \{0, 1, \ldots, w-1\}$ and $\mathcal{P}_\lambda(I_A) = e^{-\lambda} \sum_{l=0}^{\infty} I_A(l) \frac{\lambda^l}{l!}$. [7]

By substituting j and λ in (3) by any integer-valued random variable W and $\lambda = \mathrm{E}(W)$, we have

$$P(W \in A) - Poi_\lambda(A) = \mathrm{E}(\lambda g_{\lambda,A}(W+1)) - \mathrm{E}(W g_{\lambda,A}(W)). \tag{4}$$

So far W could be $\sum_{i \in \Gamma} X_i$ and $\lambda = \mathbb{E}(W) = \sum_{i \in \Gamma} p_i$ where $p_i = \mathbb{E}(X_i) = P(X_i = 1)$.

In 1992, Barbour, Holst and Janson[7] constructed coupling random variable W_i and used Stein-Chen method to find the bound in Poisson approximation of W. They assumed that for each i the distribution $L(W_i)$ of W_i equals to the conditional distribution $L(W - X_i | X_i = 1)$ and gave the fundamental theorem as follows:

Theorem 2.1. *If W and W_i are defined as above, then*

$$|P(W \in A) - Poi_\lambda(A)| \leq \| g_{\lambda,A} \| \sum_{i \in \Gamma} p_i \mathbb{E}(|W - W_i|) \tag{5}$$

where $\| g_{\lambda,A} \| := \sup_w [g_{\lambda,A}(w+1) - g_{\lambda,A}(w)]$.

In 2006, Santiwipanont and Teerapabolarn [8] proved that for any subset A of $\{0, 1, \ldots, n\}$,

$$\| g_{\lambda,A} \| \leq \lambda^{-1} \min \left\{ 1, \lambda, \frac{\triangle(\lambda)}{M_A + 1} \right\} \tag{6}$$

where

$$\triangle(\lambda) = \begin{cases} e^{\lambda} + \lambda - 1 & \text{if } \lambda^{-1}(e^{\lambda} - 1) \leq M_A, \\ 2(e^{\lambda} - 1) & \text{if } \lambda^{-1}(e^{\lambda} - 1) > M_A, \end{cases}$$

and

$$M_A = \begin{cases} \max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\ \min\{w \mid w \in A\} & \text{if } 0 \notin A. \end{cases}$$

In next section, we will use Theorem 2.1 and (6) to prove our main result by constructing the random variable W_i.

3. Proof of Main Result

Let $A \subseteq \mathbb{N}$. By (5), it is enough to bound $\mathbb{E}|W - W_i|$ for $i \in \Gamma_{n,k}$ where the distribution of W_i equals to the conditional distribution of $W - X_i$ given $X_i = 1$.

Let W_i be the number of isolated cycles of order k in a random intersection graph $\mathbb{G}(n, m, p) - i$, $\mathbb{G}(n, m, p) - i$ obtained from $\mathbb{G}(n, m, p)$ by dropping the set $i \subseteq V$ and all the edges containing any of these vertices. Then for $w_0 \in \{0, 1, \ldots, \lfloor \frac{n-k}{k} \rfloor\}$, we have

$$\begin{aligned} P(W_i = w_0) &= \binom{\binom{n-k}{k}}{w_0} \left[\sum_{s=0}^{m} \binom{m}{s} p^{s+k} (1-p)^{m-s} (1-p)^{(n-2k)s} \right]^{w_0} \\ &= \binom{\binom{n-k}{k}}{w_0} p^{kw_0} \left[1 - p + p(1-p)^{(n-2k)} \right]^{mw_0} \end{aligned} \tag{7}$$

and

$$\begin{aligned} & P(W - X_i = w_0 \mid X_i = 1) \\ &= \frac{P(W - X_i = w_0, X_i = 1)}{P(X_i = 1)} \\ &= \frac{P(W = w_0 + 1, X_i = 1)}{P(X_i = 1)} \\ &= \frac{\binom{\binom{n-k}{k}}{w_0} \left[\sum_{s=0}^{m} \binom{m}{s} p^{s+k} (1-p)^{m-s} (1-p)^{(n-2k)s} \right]^{w_0} \sum_{s=0}^{m} \binom{m}{s} p^{s+k} (1-p)^{m-s} (1-p)^{(n-2k)s}}{\sum_{s=0}^{m} \binom{m}{s} p^{s+k} (1-p)^{m-s} (1-p)^{(n-2k)s}} \\ &= \binom{\binom{n-k}{k}}{w_0} \left[\sum_{s=0}^{m} \binom{m}{s} p^{s+k} (1-p)^{m-s} (1-p)^{(n-2k)s} \right]^{w_0} \end{aligned}$$

$$= \binom{\binom{n-k}{k}}{w_0} p^{kw_0} \left[1 - p + p(1-p)^{(n-2k)}\right]^{mw_0}. \tag{8}$$

From (7) and (8), the distribution of W_i equals to the conditional distribution of $(W - X_i | X_i = 1)$.

For $i, j \in \Gamma_{n,k}$ such that $i \neq j$, we define the indicator random variable $X_j^{(i)}$ and E_{ij}, as follow

$$X_j^{(i)} = \begin{cases} 1 & \text{if here is an isolated cycle in } \mathbb{G}(n,m,p) - i \text{ that} \\ & \text{spans the vertices } i = (i_1, \ldots, i_k), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$E_{ij} = \begin{cases} 1 & \text{if there exists adjacent between } i_k \in i \text{ and } j_l \in j, \\ 0 & \text{otherwise.} \end{cases}$$

We observe that in case $X_i = 1$, that is we have an isolated cycle in $\mathbb{G}(n,m,p)$ that span vertices $i = \{i_1, \ldots, i_k\}$. Thus the number of isolated cycles in a random intersection graph $\mathbb{G}(n,m,p) - i$ equals to the number of isolated cycles in a random intersection graph $\mathbb{G}(n,m,p)$ minus 1, that is

$$W_i = W_{n,k} - 1. \tag{9}$$

In case $X_i = 0$. For $j \in \Gamma_{n,k}$ such that $j \cap i = \emptyset$ and $j \neq i$, then

$$W_i = W_{n,k} + \sum_{j \in \Gamma_{n,k}, j \cap i = \emptyset} E_{ij} X_j^{(i)} \tag{10}$$

that is the number of isolated cycles in $\mathbb{G}(n,m,p) - i$ equals to the sum of the number of isolated cycles in $\mathbb{G}(n,m,p)$ and the number of isolated cycles in $\mathbb{G}(n,m,p) - i$ which are connected to i.

We know that

$$|W_{n,k} - W_i| = (W_{n,k} - W_i)^+ + (W_{n,k} - W_i)^-$$

where $(W_{n,k} - W_i)^+ = \max\{W_{n,k} - W_i, 0\}$ and $(W_{n,k} - W_i)^- = -\min\{W_{n,k} - W_i, 0\}$. Since $-\min\{W_{n,k} - W_i, 0\} = \max\{W_i - W_{n,k}, 0\} = (W_i - W_{n,k})^+$, we have

$$\mathbb{E}|W_{n,k} - W_i| = \mathrm{E}(W_{n,k} - W_i)^+ + \mathrm{E}(W_i - W_{n,k})^+.$$

Form (9) and (10), we have

$$\begin{aligned}(W_{n,k} - W_i)^+ &\leq X_i \\ (W_i - W_{n,k})^+ &\leq \sum_{j \in \Gamma_{n,k}, j \cap i = \emptyset} E_{ij} X_j^{(i)}.\end{aligned}$$

We note that,

$$\begin{aligned}\mathbb{E}(X_i) &= \frac{\mathbb{E}(W_{n,k})}{\binom{n}{k}} \\ &= \frac{\binom{n}{k} \sum_{s=0}^{m} \binom{m}{s} p^{s+k} (1-p)^{m-s} (1-p)^{(n-k)s}}{\binom{n}{k}} \\ &= \sum_{s=0}^{m} \binom{m}{s} p^{s+k} (1-p)^{m-s} (1-p)^{(n-k)s} \\ &= p^k \left[1 - p + p(1-p)^{(n-k)}\right]^m. \end{aligned} \tag{11}$$

and

$$\begin{aligned}\sum_{j \in \Gamma_{n,k}, j \cap i = \emptyset} \mathrm{E}[E_{ij} X_j^{(i)}] &= \sum_{j \in D, j \cap i = \emptyset} P(E_{ij} = 1, X_j^{(i)} = 1) \\ &= \binom{n-k}{k} (1 - q^{k^2}) \sum_{s=0}^{m} \binom{m}{s} p^{s+k} (1-p)^{m-s} (1-p)^{(n-2k)s} \\ &\leq \frac{n^k}{k!} p^k k^2 \left[1 - p + p(1-p)^{(n-2k)}\right]^m. \end{aligned} \tag{12}$$

From (11), (12) and use the fact that $1 - p \leq \frac{1}{e^p}$, we have

$$\begin{aligned}\mathbb{E}(|W - W_i|) &= \mathbb{E}(W - W_i)^+ + \mathbb{E}(W_i - W)^+ \\ &\leq \mathbb{E}(X_i) + \sum_{j \in \Gamma_{n,k}, j \cap i = \emptyset} \mathbb{E}(E_{ij} X_j^{(i)}) \\ &\leq p^k \left[1 - p + p(1-p)^{(n-k)}\right]^m + \frac{n^k}{k!} p^k k^2 \left[1 - p + p(1-p)^{(n-2k)}\right]^m \\ &\leq \left[1 - p + p(1-p)^{(n-2k)}\right]^m (1 + \frac{n^k}{k!} k^2) p^k \\ &= \left[1 - p(1 - (1-p)^{(n-2k)}\right]^m (1 + \frac{n^k}{k!} k^2) p^k\end{aligned}$$

$$\leq \frac{(1+\frac{n^k}{k!}k^2)p^k}{e^{mp(1-(1-p)^{n-2k})}} \tag{13}$$

Hence, by (5), (6) and (13), we have

$$|P(W \in A) - Poi_\lambda(A)| \leq \frac{C_{(\lambda,A)}(1+\frac{n^k}{k!}k^2)p^k}{e^{mp(1-(1-p)^{n-2k})}}$$

where $C_{(\lambda,A)} = \min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$.

This complete the proof of Theorem 1.3.

4. Proof of Corollary 1.3.

From (13), we have

$$\begin{aligned}\mathbb{E}(|W - W_i|) &\leq \frac{(1+\frac{n^k}{k!}k^2)p^k}{e^{mp(1-(1-p)^{n-2k})}} \\ &= \frac{p^k}{e^{mp(1-(1-p)^{n-2k})}} + \frac{k(np)^k}{(k-1)!e^{mp(1-(1-p)^{n-2k})}} \end{aligned} \tag{14}$$

Suppose that $q = 1-p$ and $m = \lfloor n^\alpha \rfloor$ for some $\alpha > 0$ and $p = \dfrac{1}{n^\gamma}$ for any $\gamma \in \mathbb{R}^+ \setminus \{1\}$, then we obtain

$$\begin{aligned}\mathbb{E}(|W - W_i|) &\leq \frac{1}{n^{(k)\gamma}e^{mp(1-(1-p)^{n-2k})}} + \frac{k}{(k-1)!n^{(\gamma-1)k}e^{mp(1-(1-p)^{n-2k})}} \\ &\leq \frac{1}{n^{k\gamma}e^{mp(1-(1-p)^{n-2k})}} + \frac{n^k k}{(k-1)!n^{k\gamma}e^{mp(1-(1-p)^{n-2k})}} \\ &\leq \frac{(1+n^k)}{n^{k\gamma}e^{mp(1-(1-p)^{n-2k})}} \\ &\leq \frac{2}{n^{(k\gamma-k)}e^{mp(1-(1-p)^{n-2k})}} \\ &\leq \frac{2}{n^{(k\gamma-k)}e^{n^{\alpha-\gamma}(1-(1-p)^{n-2k})}} \\ &\leq \frac{2}{n^{(k\gamma-k)}n^{2(\alpha-\gamma)(1-(q)^{n-2k})}} \\ &= \frac{2}{n^{2(\alpha-\gamma)(1-(q)^{n-2k})+(k\gamma-k)}} \end{aligned} \tag{15}$$

By (5), (6) and (15), we have

$$|P(W \in A) - Poi_\lambda(A)| \leq \frac{C(\lambda, A)}{n^{2(\alpha-\gamma)(1-(q)^{n-2k})+(k\gamma-k)}},$$

where $C(\lambda, A) = 2\min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$.

This complete the proof of Corollary 1.3.

Acknowledgments

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