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COMMON FIXED POINT THEOREM ON FUZZY GROUPS

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Abstract: In this paper, we give some new definitions and concepts of $d-$ metric on fuzzy groups and some properties on these $d-$ metric are obtained.

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1. Introduction and Preliminaries

In this paper, G always denotes an arbitrary group with a multiplicative binary operation and identity e. Applying the concept of fuzzy sets of Zadeh[5] to group theory, Rosenfeld [3] introduced the notion of a fuzzy group as early as 1971. Though it was known for a long time that Rosenfeld's fuzzy subgroups of a given group form a complete lattice, the technique of generating a fuzzy group (the smallest fuzzy group) containing an arbitrarily chosen fuzzy set was

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developed only in 1993 by Ray [2].

In order to define the notion of fuzzy subgroups and to examine their properties, we introduce some operations on fuzzy subsets of a group G in terms of the group operation. In what follows I will denote $[0,1]$.

We first recall some basic definitions for sake of completeness.

Definition 1.1. Let X be a nonempty set. By a fuzzy subset of X, we mean a function from X into I. The set of all fuzzy subsets of X is called the I-power set of X and is denoted by I^X.

Definition 1.2. We define the binary operation $\circ$ on I^G and the unary $operation^{-1}$ on I^G as follows: $\forall \mu, \nu \in I^G, \forall x \in G$,

$$(\mu o \nu)(x) = \vee\{\mu(y) \wedge \nu(z) \ \mid \ y, z \in G, yz = x\},$$

$$\mu^{-1}(x) = \mu(x^{-1}).$$

We call $\mu o \nu$ the product of μ and ν, and μ^{-1} the inverse of μ.

Definition 1.3. A fuzzy subset μ of G is called a fuzzy subgroup of G if
(G1) $\mu(xy) \geq \mu(x) \wedge \mu(y) \ \ \forall x, y \in G$, and
(G2) $\mu(x^{-1}) \geq \mu(x) \ \ \forall x \in G$.

Example 1.4. Let $\mu : G \longrightarrow [0,1]$ defined by

1. $$\mu(x) = \begin{cases} 1 & \text{if } x > 0, \\ \frac{1}{2} & \text{if } x < 0, \end{cases}$$

   for $x \in \mathbb{R}^*$.

2. $$\mu(x) = \begin{cases} \frac{1}{2} & \text{if } x \ \ is \ \ even, \\ \frac{1}{3} & \text{if } x \ \ is \ \ odd,, \end{cases}$$

   for $x \in \mathbb{Z} - \{0\}$ and $\mu(0) = 1$.

3. $$\mu(x) = \begin{cases} \frac{1}{3} & \text{if } x \ \ is \ \ even, \\ \frac{1}{4} & \text{if } x \ \ is \ \ odd,, \end{cases}$$

   for $x \in \mathbb{Z}$.

If μ is a fuzzy subgroup of a group G, it follows that:

$$\mu(e) \geq \mu(x), \forall x \in G,$$

$$\mu(x^{-1}) = \mu(x), \forall x \in G.$$

Denote by $I(G)$, the set of all fuzzy subgroups of G. If $\mu \in I(G)$, we let

$$\mu_* = \{x \in G \mid \mu(x) = \mu(e)\}.$$

Definition 1.5. Let $\mu \in I^G$. Then μ is called an Abelian subset of G, if

$$\mu(xy) = \mu(yx) \quad \forall x, y \in G.$$

Definition 1.6. (Wu [4]). Let G be a group. A fuzzy subgroup μ of G is called to be normal if

$$\mu(xyx^{-1}) \geq \mu(y), \forall x, y \in G.$$

Also we denote by $NI(G)$ the set of all normal subgroups of G.

Definition 1.7. (Wu [4]). Let μ be a fuzzy subgroup of a group G. For any $a \in G$, $a\mu$ and μa are fuzzy subsets of G, defined by

$$\begin{aligned}(a\mu)(x) &= \mu(a^{-1}x), \forall x \in G,\\ (\mu a)(x) &= \mu(xa^{-1}), \forall x \in G.\end{aligned}$$

Then, $a\mu$ is said to be fuzzy left coset, and μa a fuzzy right coset.

Theorem 1.8. *(Wu [4]). Let μ be a fuzzy subgroup of a group G. Then μ is a normal fuzzy subgroup of a group G if and only if any one of the following conditions is satisfied:*

1. $\mu(xyx^{-1}) = \mu(y), \forall x, y \in G$
2. $\mu(xy) = \mu(yx), \forall x, y \in G$
3. $x\mu = \mu x, \forall x \in G$

2. The Main Results

In this section, we give some new definitions and concepts of fuzzy subgroups and give some properties.

Lemma 2.1. *Let μ be a fuzzy subset of G, then*

1. *if $\mu \in I(G)$, then μ_* is a subgroup of G,*

2. *if $\mu \in NI(G)$, then μ_* is a normal subgroup of G,*

3. *if H is a arbitrary subgroup of G, then there exists a $\mu \in I(G)$ such that $\mu_* = H$.*

Proof. The proof (1) is easy and we can see this proof in [4]. For proof (2), since $\mu \in I(G)$, it follows from (1) that μ_* is a subgroup of G. Let $x \in G$ and $y \in \mu_*$. Since $\mu \in NI(G)$, hence we have $\mu(xyx^{-1}) = \mu(y) = \mu(e)$ and thus $xyx^{-1} \in \mu_*$. For proof (3) we define

$$\mu(x) = \begin{cases} \mu(e) & \text{if } x \in H, \\ a & \text{if } x \notin H, \end{cases}$$

for $a \in [0, \mu(e))$. Now, we prove that $\mu \in I(G)$. For $x \in G$

$$\begin{aligned} \mu^{-1}(x) &= \mu(x^{-1}) = \begin{cases} \mu(e) & \text{if } x \in H, \\ a & \text{if } x \notin H, \end{cases} \\ &= \mu(x) \end{aligned}$$

For every $x, y \in G$, we consider
Step1: if $x, y \in H$, then

$$\mu(xy) = \mu(e) = \mu(x) \wedge \mu(y).$$

Step2: if at most one of x and y belongs to H, then

$$\mu(xy) \geq \mu(x) \wedge \mu(y) = a.$$

Hence $\mu \in I(G)$. On the other hand

$$\mu_* = \{x \in G \mid \mu(x) = \mu(e)\} = H.$$

□

Let G be a group and $\mu \in I(G)$. For $0 < r < \mu(e)$, the ball $B_\mu(x, r)$ with center $x \in G$ and radius r is defined by

$$B_\mu(x, r) = \{y \in G : \mu(xy^{-1}) > r\}.$$

Lemma 2.2. *Let G be a group and $\mu \in I(G)$. If $0 < r < \mu(e)$, then ball $B_\mu(x, r)$ with center $x \in G$ and radius r is open ball.*

Proof. Let $y \in B_\mu(x,r)$, hence $\mu(xy^{-1}) > r$. If set $\mu(xy^{-1}) = \delta$ then we can find a δ' such that $\delta' > r$. Now, we prove that $B_\mu(y,\delta') \subseteq B_\mu(x,r)$. Let $z \in B_\mu(y,\delta')$, then we have $\mu(xz^{-1}) = \mu(xy^{-1}yz^{-1}) \geq \mu(xy^{-1}) \wedge \mu(yz^{-1}) > \delta \wedge \delta' > r$. Hence $B_\mu(y,\delta') \subseteq B_\mu(x,r)$. It induces that ball $B_\mu(x,r)$ is open ball. □

Example 2.3. In Example 1.4 for part (2), we have

$$B_\mu(2,\frac{1}{2}) = \{y \in \mathbb{Z} | \mu(2-y) > \frac{1}{2}\} = \{2\},$$

and

$$B_\mu(3,\frac{1}{3}) = \{y \in \mathbb{Z} | \mu(3-y) > \frac{1}{3}\} = \{1,3,5,\cdots\}.$$

Let G be a group. Let τ be the set of all $A \subset G$ with $x \in A$ if and only if there exist $0 < r < \mu(e)$ such that $B_\mu(x,r) \subset A$. Then τ is a topology on G (induced by the fuzzy subgroup μ). Let $\mu \in I(G)$, a sequence $\{x_n\}$ in G converges to x if and only if $\mu(xx_n^{-1}) \to \mu(e)$ as $n \to \infty$. On other hand,for each $0 < \varepsilon < \mu(e)$, there exits $n_0 \in \mathbb{N}$ such that $|\mu(xx_n^{-1}) - \mu(e)| < \varepsilon$ for each $n \geq n_0$. It follows that, for each $0 < \varepsilon < \mu(e)$, there exits $n_0 \in \mathbb{N}$ such that $\mu(xx_n^{-1}) > \mu(e) - \varepsilon$ for each $n \geq n_0$. It is called a Cauchy sequence if for each $0 < \varepsilon < \mu(e)$, there exits $n_0 \in \mathbb{N}$ such that $|\mu(x_m {x_n}^{-1}) - \mu(e)| < \varepsilon$ for each $n, m \geq n_0$.

Definition 2.4. Let X be a nonempty set and $\mu \in I(G)$. Then the (X,μ) is called μ-topology space induced by fuzzy subgroup μ .

A μ-topology space (X,μ) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges. Suppose that $\{x_n\}$ is a sequence in μ-topology space (X,μ), then we define $L(x_n) = \{x | x_n \longrightarrow x\}$.

The following example shows that the limit of a convergent sequence is not unique.

Example 2.5. Let $G = \mathbb{Z}$ and

$$\mu(x) = \begin{cases} \frac{1}{3} & \text{if } x \ \ is \ \ even, \\ \frac{1}{4} & \text{if } x \ \ is \ \ odd,, \end{cases}$$

for $x \in \mathbf{Z}$. Let $x_n = \{2,4,6,\cdots\}$, then we have $\lim_{n\to\infty} \mu(x - x_n) = \mu(0) = \frac{1}{3}$, therefore $L(x_n) = \{2,4,6,\cdots\}$.

The following Lemma shows that under certain conditions the limit is unique.

Lemma 2.6. *Let (G, μ) be a μ-topology space such that for every $x \in G$ from $\mu(x) = \mu(e)$ implies that $x = e$. If sequence $\{x_n\}$ in G converges to x, then x is unique.*

Proof. Let $x_n \longrightarrow y$ and $y \neq x$. Since $\{x_n\}$ converges to x and y, for each $0 < \varepsilon < \mu(e)$ there exist $n_1, n_2 \in \mathbb{N}$ such that
$\mu(x{x_n}^{-1}) > \mu(e) - \varepsilon$ for every $n \geq n_1$ and $\mu(x_n y^{-1}) > \mu(e) - \varepsilon$ for every $n \geq n_2$.
If set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ we have

$$\mu(xy^{-1}) = \mu(x{x_n}^{-1}.x_n y^{-1}) \geq \mu(x{x_n}^{-1}) \wedge \mu(x_n y^{-1}) > \mu(e) - \varepsilon.$$

Hence, for every $0 < \varepsilon < \mu(e)$ we get $\mu(xy^{-1}) > \mu(e) - \varepsilon$ and so $\mu(xy^{-1}) \geq \mu(e)$. Since $\mu \in I(G)$ and $\mu(e) \geq \mu(xy^{-1})$ we get $\mu(xy^{-1}) = \mu(e)$. Therefore $x = y$. □

Henceforth, we assume that (G, μ) be a μ-topology space such that for every $x \in G$ from $\mu(x) = \mu(e)$ implies that $x = e$.

Lemma 2.7. *Let (G, μ) be a μ-topology space. If sequence $\{x_n\}$ in G is converges to x, then sequence $\{x_n\}$ is a Cauchy sequence.*

Proof. Since $x_n \longrightarrow x$ for each $0 < \varepsilon < \mu(e)$ there exist $n_1, n_2 \in \mathbb{N}$ such that $\mu(x_n x^{-1}) > \mu(e) - \varepsilon$ for every $n \geq n_1$ and $\mu(x{x_m}^{-1}) > \mu(e) - \varepsilon$ for every $m \geq n_2$.
If we set $n_0 = \max\{n_1, n_2\}$, then for every $n, m \geq n_0$ we have $\mu(x_n {x_m}^{-1}) = \mu(x_n x^{-1} x {x_m}^{-1}) \geq \mu(x{x_m}^{-1}) \wedge \mu(x{x_n}^{-1}) > \mu(e) - \varepsilon$. Hence sequence $\{x_n\}$ is a Cauchy sequence. □

Definition 2.8. Let (G, μ) be a μ-topology space. $\mu : G \longrightarrow [0, 1]$ is said to be continuous function on G if

$$\lim_{n\to\infty} \mu(x_n y_n) = \mu(xy)$$

whenever sequences $\{x_n\}$ and $\{y_n\}$ in G converges to a point x and $y \in G$ respectively , i.e.

$$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y.$$

Lemma 2.9. *Let (G, μ) be a μ-topology space such that $\mu(xy) = \mu(yx)$ for every $x, y \in G$. Then μ is continuous function on G.*

Proof. As $\{x_n\}$ and $\{y_n\}$ converge to a $x \in X$ and $y \in X$ respectively, for each $0 < \varepsilon < \mu(e)$ there exist $n_0 \in \mathbb{N}$ such that

$$\mu(xx_n^{-1}) > \mu(e) - \varepsilon \text{ and } \mu(yy_n^{-1}) > \mu(e) - \varepsilon \text{ for } n \geq n_0.$$

We have that

$$\begin{aligned} \mu(xy) &\geq \mu(xx_n^{-1}) \wedge \mu(x_n y) \\ &\geq \mu(xx_n^{-1}) \wedge \mu(x_n y_n) \wedge \mu(y_n^{-1} y) \\ &\geq (\mu(e) - \varepsilon) \wedge \mu(x_n y_n) \wedge (\mu(e) - \varepsilon) \\ &= \mu(e) \wedge \mu(x_n y_n) \wedge \mu(e) \\ &\geq \mu(x_n y_n). \end{aligned}$$

Taking the upper limit as $n \to \infty$ we obtain

$$\mu(xy) \geq \limsup_{n \longrightarrow \infty} \mu(x_n y_n).$$

On the other hand

$$\begin{aligned} \mu(x_n y_n) &\geq \mu(x_n x^{-1}) \wedge \mu(xy_n)) \\ &\geq \mu(x_n x^{-1}) \wedge \mu(xy) \wedge \mu(y^{-1} y_n) \\ &\geq (\mu(e) - \varepsilon) \wedge \mu(xy) \wedge (\mu(e) - \varepsilon) \\ &= \mu(e) \wedge \mu(xy) \wedge \mu(e) \\ &\geq \mu(xy). \end{aligned}$$

And taking the lower limit as $n \to \infty$ we obtain

$$\liminf_{n \longrightarrow \infty} \mu(x_n y_n) \geq \mu(xy).$$

So we have

$$\mu(xy) \geq \limsup_{n \longrightarrow \infty} \mu(x_n y_n) \geq \liminf_{n \longrightarrow \infty} \mu(x_n y_n) \geq \mu(xy),$$

that is, $\lim_{n \longrightarrow \infty} \mu(x_n y_n) = \mu(xy)$. □

Lemma 2.10. *Let (G, μ) be a μ-topology space . If there exists sequence $\{x_n\}$ in G such that*

$$\lim_{n \to \infty} \mu(x_n x_{n+1}^{-1}) = \mu(e),$$

then sequence $\{x_n\}$ is a Cauchy sequence.

Proof. Since $\mu(x_n {x_{n+1}}^{-1}) \longrightarrow \mu(e)$ for each $0 < \varepsilon < \mu(e)$, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n {x_{n+1}}^{-1}) > \mu(e) - \varepsilon$ for every $n \geq n_0$. For every $m \geq n \geq n_0$ we have

$$\begin{aligned} \mu(x_n {x_m}^{-1}) &= \mu(x_n {x_{n+1}}^{-1}.x_{n+1} {x_{n+2}}^{-1} x_{n+2} ... x_{m-1} {x_m}^{-1}) \\ &\geq \mu(x_n {x_{n+1}}^{-1}) \wedge \mu(x_{n+1} {x_{n+2}}^{-1}) \wedge ... \wedge \mu(x_{m-1} {x_m}^{-1}) \\ &> \mu(e) - \varepsilon. \end{aligned}$$

Hence sequence $\{x_n\}$ is a Cauchy sequence. □

Lemma 2.11. *Let (G, μ) be a μ-topology space. If we define $d : X^2 \longrightarrow [0, \infty)$ by $d(x, y) = \log_a \mu(xy^{-1}) - \log_a \mu(e)$ for all $a \in (0, 1)$, then d is a metric on X.*

Proof. It is clear from the definition that $d(x, y)$ is well defined for each $x, y \in G$.

(i) $d(x, y) \geq 0$ for all $x, y \in G$ is trivial.

(ii)

$$d(x, y) = 0 \Longleftrightarrow \log_a \mu(xy^{-1}) - \log_a \mu(e) = 0 \Longleftrightarrow \mu(xy^{-1}) = \mu(e) \Longleftrightarrow x = y.$$

(iii)

$$d(x, y) = \log_a \mu(xy^{-1}) - \log_a \mu(e) = \log_a \mu(yx^{-1}) - \log_a \mu(e) = d(y, x).$$

(iv)

Since $\mu(xy^{-1}) \geq \mu(xz^{-1}) \wedge \mu(zy^{-1})$,

and also since $\log_a$ is decreasing, it follows that,

$$\log_a \mu(xy^{-1}) \leq \max\{\log_a \mu(xz^{-1}), \log_a \mu(zy^{-1})\}.$$

Therefore,

$$\begin{aligned} d(x, y) &= \log_a \mu(xy^{-1}) - \log_a \mu(e) \\ &\leq \max\{\log_a \mu(xz^{-1}) - \log_a \mu(e), \log_a \mu(zy^{-1}) - \log_a \mu(e)\} \\ &\leq \log_a \mu(xz^{-1}) - \log_a \mu(e) + \log_a \mu(zy^{-1}) - \log_a \mu(e) \\ &= d(x, z) + d(z, y) \end{aligned}$$

This proves that d is a metric on X. □

Lemma 2.12. *Let (G,μ) be a μ-topology space. If we define $d : X^2 \longrightarrow [0,\infty)$ by $d(x,y) = a^{\mu(xy^{-1})} - a^{\mu(e)}$ for all $a \in (0,1)$, then d is a metric on X.*

Proof. It is clear from the definition that $d(x,y)$ is well defined for each $x,y \in G$.

(i) $d(x,y) \geq 0$ for all $x,y \in G$ is trivial.

(ii)

$$d(x,y) = 0 \Longleftrightarrow a^{\mu(xy^{-1})} - a^{\mu(e)} = 0 \Longleftrightarrow \mu(xy^{-1}) = \mu(e) \Longleftrightarrow x = y.$$

(iii)

$$d(x,y) = a^{\mu(xy^{-1})} - a^{\mu(e)} = a^{\mu(yx^{-1})} - a^{\mu(e)} = d(y,x).$$

(iv)

$$\textit{Since } \mu(xy^{-1}) \quad \geq \quad \mu(xz^{-1}) \wedge \mu(zy^{-1}),$$

it follows that, $a^{\mu(xy^{-1})} \leq a^{\mu(xz^{-1})\wedge\mu(zy^{-1})} = \max\{a^{\mu(xz^{-1})}, a^{\mu(zy^{-1})}\}$. Therefore,

$$\begin{aligned} d(x,y) &= a^{\mu(xy^{-1})} - a^{\mu(e)} \\ &= \max\{a^{\mu(xz^{-1})}, a^{\mu(zy^{-1})}\} - a^{\mu(e)} \\ &= \max\{a^{\mu(xz^{-1})} - a^{\mu(e)}, a^{\mu(zy^{-1})} - a^{\mu(e)}\} \\ &\leq a^{\mu(xz^{-1})} - a^{\mu(e)} + a^{\mu(zy^{-1})} - a^{\mu(e)} \\ &= d(x,z) + d(z,y) \end{aligned}$$

This proves that d is a metric on X. □

Lemma 2.13. *Let (G,μ) be a μ-topology space. If define $d : G^2 \longrightarrow [0,\infty)$ by $d(x,y) = \mu(e) - \mu(xy^{-1})$, then d is a metric on G.*

Proof. (i) It is easy to see that $d(x,y) \geq 0$,
(ii)

$$d(x,y) = 0 \Longleftrightarrow \mu(e) - \mu(xy^{-1}) = 0 \Longleftrightarrow \mu(xy^{-1}) = \mu(e) \Longleftrightarrow x = y$$

(iii)

$$d(x,y) = \mu(e) - \mu(xy^{-1}) = \mu(e) - \mu(yx^{-1}) = d(y,x)$$

(iv) Since

$$\begin{aligned} d(x,y) &= \mu(e) - \mu(xy^{-1}) = \mu(e) - \mu(xz^{-1}zy^{-1}) \\ &\leq \mu(e) - \min\{\mu(xz^{-1}), \mu(zy^{-1})\} \\ &= \max\{\mu(e) - \mu(xz^{-1}), \mu(e) - \mu(zy^{-1})\} \\ &= \max\{d(x,z), d(z,y)\} \\ &\leq d(x,z) + d(z,y) \end{aligned}$$

That is d is a metric on G. □

Lemma 2.14. *If define*

$$d(x,y) = \int_{\mu(xy^{-1})}^{\mu(e)} \varphi(s)ds,$$

for all $x, y \in G$, where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a Lesbegue integrable mapping, then d is a metric on G.

Proof. (i) It is easy to see that $d(x,y) \geq 0$,

(ii)

$$d(x,y) = 0 \Longleftrightarrow \mu(xy^{-1}) = \mu(e) \Longleftrightarrow xy^{-1} = e \Longleftrightarrow x = y$$

(iii)

$$d(x,y) = d(y,x)$$

(iv) Since

$$\begin{aligned} d(x,y) &= \int_{\mu(xy^{-1})}^{\mu(e)} \varphi(s)ds \\ &\leq \int_{\min\{\mu(xz^{-1}),\mu(zy^{-1})}^{\mu(e)} \varphi(s)ds \\ &= \max\{\int_{\mu(xz^{-1})}^{\mu(e)} \varphi(s)ds, \int_{\mu(zy^{-1})}^{\mu(e)} \varphi(s)ds\} \\ &= \max\{d(x,z), d(z,y)\} \\ &\leq d(x,z) + d(z,y) \end{aligned}$$

That is d is a metric on G. □

Definition 2.15. Let (G, μ) be a μ-topology space. $f : G \longrightarrow G$ is said to be continuous function on G if

$$\lim_{n\to\infty} f(x_n) = f(x),$$

whenever sequence $\{x_n\}$ in G converges to a point $x \in G$.

Theorem 2.16. *[1] Let A, B, S and T are four self maps on a complete metric space (X, d) satisfying:*

(i) $T(X) \subseteq A(X)$ *and* $S(X) \subseteq B(X)$,

(ii) *pairs* (S, A) *and* (T, B) *are commuting,*

(iii) *there exists* $k \in [0, 1)$ *such that for each* $x, y \in X$,
$d(Sx, Ty) \leq k.N(x, y)$,
where

$$N(x, y) = \max\{d(Ax, By), d(Ax, Sx), d(By, Ty)\};$$

(iv)*one of* S, A, T *and* B *is continuous,*
then A, B, S *and* T *have a unique common fixed point in* X.

We next apply Theorem 2.16 to establish the following theorem in μ-topology space (G, μ).

Theorem 2.17. *Let* (G, μ) *be a* μ*-topology space and* A, B, S *and* T *be self maps on* G *satisfying:*

(i) $T(X) \subseteq A(X)$ *and* $S(X) \subseteq B(X)$,

(ii) *pairs* (S, A) *and* (T, B) *are commuting,*

(iii) *there exists* $k \in [0, 1)$ *such that for each* $x, y \in X$,
$\mu(Sx(Ty)^{-1}) \geq N(x, y)^k$,
where

$$N(x, y) = \min\{\mu(Ax(By)^{-1}), \mu(Ax(Sx)^{-1}), \mu(By(Ty)^{-1})\};$$

(iv)*one of* S, A, T *and* B *is continuous,*
then A, B, S *and* T *have a unique common fixed point in* X.

Proof. We define $d(x, y) = \log_a \mu(xy^{-1}) - \log_a \mu(e)$ for every $x, y \in X$ where $0 < a < 1$. Then by Lemma 2.11, (X, d) is a complete metric space. In general, if $f : G \longrightarrow G$ is continuous, then we can prove that it is continuous on metric space (X, d). Since $\mu(f(x_n)(f(x))^{-1}) \longrightarrow \mu(e)$, we have that $\log_a \mu(f(x_n)(f(x))^{-1}) \longrightarrow \log_a \mu(e)$. Therefore, $d(f(x_n), f(x)) \longrightarrow 0$, i.e. $f(x_n) \xrightarrow{d} f(x)$. From (iii), we get

$$\begin{aligned}\log_a \mu(Sx(Ty)^{-1}) - \log_a \mu(e) \leq k \max\{&\log_a \mu(Ax(By)^{-1}) - \log_a \mu(e),\\ &\log_a \mu(Ax(Sx)^{-1}) - \log_a \mu(e), \log_a \mu(By(Ty)^{-1}) - \log_a \mu(e)\},\end{aligned}$$

that is,

$$d(Sx, Ty) \leq k \max\left\{\ d(Ax, By), d(Ax, Sx), d(By, Ty)\ \right\}.$$

Therefore, all the conditions of Theorem 2.16 hold. We get the desired result : the conclusion of Theorem 2.17 follows from an application of Theorem 2.16 . □

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