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FUNCTION CHAINS FROM UNIFORM SIGMA-1-1 WELL ORDERS

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Abstract: Appropriate restrictions may be placed on a Σ^1_1 well-order on V_κ where κ is an inaccessible cardinal, so that it gives rise to a function chain when κ is a Mahlo cardinal. Set chains may be defined for all second order formulas. Postulating that the sets in the resulting set chain are stationary yields a powerful new axiom.

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1. Introduction

From the results of [5] it is clearly of interest to devise ordinal notation systems which permit the definition of function chains. The system used in [5] is based on the infinitary Veblen function, which can be defined over any cardinal in a highly uniform manner.

An early attempt in [2] to construct set chains from a class of constructive ordinals was only partly successful. Here the notion of a "uniform" Σ^1_1 order is defined. The defining formula is required to define an order uniformly in any inaccessible cardinal. It follows that such gives rise to a function chain.

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Before giving the most general class of orders, two more specialized classes are given. These serve to illustrate the nature of these orders, and raises a variety of questions of interest.

Some notation to be used in the paper is as follows: Ord denotes the class of ordinals, Lim the limit ordinals, Card the cardinals, and Inac the (strongly) inaccessible cardinals. For $\kappa \in$ Card let Lim_κ denote $\mathrm{Lim} \cap \kappa$. For $\kappa \in$ Inac let Card_κ denote $\mathrm{Card} \cap \kappa$, and let Inac_κ denote $\mathrm{Inac} \cap \kappa$. For a well-order X, $\mathrm{Ot}(X)$ denotes its order type. Suppose $\kappa \in$ Card and f_ξ for $\xi < \kappa$ is a sequence of functions from some domain $D \subseteq \kappa$ to Ord; let $\mathrm{dsup}_{\xi<\kappa} f_\xi$ be the function f where $f(\gamma) = \sup_{\xi<\gamma} f_\xi(\gamma)$.

A binary relation $\preceq$ on some set is said to be a WPS (well-preorder on a subset) if it satisfies the axioms

T1. $A \preceq B \wedge B \preceq C \Rightarrow A \preceq C$

T2. $A \preceq B \Rightarrow A \preceq A$

T3. $A \preceq B \Rightarrow B \preceq B$

T4. $A \preceq A \wedge B \preceq B \Rightarrow (A \preceq B \vee B \preceq A)$

F. For all functions $f : \omega \mapsto \mathrm{Fld}(\preceq)$ there is an n such that $f(n) \preceq f(n+1)$

Note that $A \in \mathrm{Fld}(\preceq)$ iff $A \preceq A$. $A \prec B$ is an abbreviation for $A \preceq B \wedge \neg B \preceq A$. Write $A \equiv B$ for $A \preceq B \wedge B \preceq A$; it follows from the axioms that this is a congruence relation for $\preceq$ and $\prec$, and the quotient order is a well-order on the equivalence classes.

A well-order on a subset is a WPS, where $\equiv$ is discrete, that is, $A \equiv B \Rightarrow A = B$.

For a WPS $\preceq$ and $P \in \mathrm{Fld}(\preceq)$ let $A \preceq_P B$ iff $A \prec P \wedge B \prec P \wedge A \preceq B$. It is readily seen that $\preceq_P$ is a WPS. Note that $\preceq_P$ is defined even if $P \notin \mathrm{Fld}(\preceq)$, and equals $\emptyset$.

Lemma 1. *Suppose $\preceq$ is a WPS.*

a. *If $P \preceq Q$ then $Ot(\preceq_P) \leq Ot(\preceq_Q)$.*

b. *If $P \prec Q$ then $Ot(\preceq_P) < Ot(\preceq_Q)$.*

c. *If $P \prec Q$ and $P \prec R \Rightarrow Q \preceq R$ then $Ot(\preceq_Q) = Ot(\preceq_P) + 1$.*

d. *Suppose $\eta \leq \kappa$ is a limit ordinal and P_ξ for $\xi < \eta$ is a sequence with $P_{\xi_1} \prec P_{\xi_2}$ for $\xi_1 < \xi_2$. If $P_\xi \prec P$ for all $\xi < \eta$ and $\forall \xi < \eta (P_\xi \prec R) \Rightarrow P \preceq R$ then $Ot(\preceq_P) = \sup_{\xi<\eta} Ot(\preceq_{P_\xi})$.*

Proof. These are basic facts of order theory. Let $\preceq'$ denote the quotient well order of $\preceq$. With obvious notation, $P \preceq Q$ iff $\preceq'_P$ is an initial segment of $\preceq'_Q$, iff $\mathrm{Ot}(\preceq_P) \leq \mathrm{Ot}(\preceq_Q)$. All parts of the lemma follow readily. ☐

2. Function Chains

Let κ be any regular uncountable cardinal. The club filter is defined more generally, but is better behaved for such κ. Functions from D to κ will be considered, where D is a subset of κ, namely one of κ, Lim_κ, Card_κ, or Inac_κ. In the case of Card_κ κ will be required to be inaccessible, and in the case of Inac_κ κ will be required to be Mahlo,

For $f, g : D \mapsto \kappa$ and $S \subseteq D$ say that $f <_S g$ if $f(\alpha) < g(\alpha)$ for $\alpha \in S$. The notions $f \leq_S g$ and $f =_S g$ are defined similarly to $f <_S g$. As in definition 24.4 of [8] if $\mathcal{F}$ is a filter of subsets of D say that $f <_{\mathcal{F}} g$ if $f <_S g$ for some $S \in \mathcal{F}$. The notions $f \leq_{\mathcal{F}} g$ and $f =_{\mathcal{F}} g$ are defined similarly to $f <_{\mathcal{F}} g$.

Let $\mathcal{C}$ be the club filter on κ. For $D \subseteq \kappa$ let $\mathcal{C}_D = \{C \cap D : C \in \mathcal{C}\}$. Write $\mathcal{C}_L$, $\mathcal{C}_C$, and $\mathcal{C}_I$ for $\mathcal{C}_D$ when D is Lim_κ, Card_κ, and Inac_κ respectively. The classic Galvin-Hajnal order is the order $<_{\mathcal{C}}$ on κ^κ; it is transitive and well-founded. By a function chain is meant a chain in this order, or more generally in one of the orders $<_{\mathcal{C}_D}$. In the case of $\mathcal{C}_L$, no further restriction need be placed on κ, and $<_{\mathcal{C}_L}$ is essentially the same as $<_{\mathcal{C}}$, in that $f <_{\mathcal{C}} g$ iff $f \restriction \mathrm{Lim}_\kappa <_{\mathcal{C}_L} g \restriction \mathrm{Lim}_\kappa$. In the case of $\mathcal{C}_C$, $\kappa \in \mathrm{Inac}$ is required, so that Card_κ is club, and $<_{\mathcal{C}_C}$ is essentially the same as $<_{\mathcal{C}}$. In the case of $\mathcal{C}_I$, κ Mahlo is required, so that $\mathcal{C}_I$ is a filter. In this case it is not known to the author whether the chain lengths are the same for $<_{\mathcal{C}}$ and $<_{\mathcal{C}_I}$. As will be seen, chains in $<_{\mathcal{C}_I}$ are of interest.

Canonical functions are a well-known function chain of length κ^+. One characterization may be found in lemma 24.5 of [8], and a second in exercise 27.6. A characterization using schemes is given in [3]. Here, a fourth characterization from [1] will be given.

Suppose κ is a regular uncountable cardinal, $A \subseteq \kappa \times \kappa$ is a binary relation, and $\theta \in \mathrm{Lim}_\kappa$. Let $A \restriction \theta$ denote $A \cap \theta \times \theta$. Then V_θ is closed under ordered pairs, and $A \restriction \theta = A \cap V_\theta$. If A is a total order on a subset then $A \cap V_\theta$ is; and if A is a well-order on a subset then $A \cap V_\theta$ is.

Lemma 2. *Suppose κ is a regular uncountable cardinal. Say that a predicate which holds in V_κ reflects in a club if it holds in V_θ for a club of ordinals $\theta < \kappa$. The following reflect in a club.*

a. *A is a well order on a subset.*
b. *$Ot(A) \leq Ot(B)$.*
c. *$Ot(A) < Ot(B)$.*
d. *$Ot(A) = Ot(B) + 1$.*
e. *$Ot(A) = \sup_{\xi<\eta} Ot(A_\xi)$, where $\eta < \kappa$ and $Ot(A_\xi)$ is increasing.*
f. *$Ot(A) = \sup_\xi Ot(A_\xi)$, where $Ot(A_\xi)$ is increasing.*

Proof. This follows readily by lemma 1.3 of [1]. For convenience a detailed proof is given. Part a has already been observed.

Now, if $F : \kappa \mapsto \kappa$ then $\{\gamma : F[\gamma] \subseteq \gamma\}$ is a club subset (closure follows by elementary set theory. Unboundedness follows by letting $\gamma_0 = \gamma + 1$, $\gamma_{n+1} = \gamma \cup \sup F[\gamma]$, and $\gamma' = \sup_n \gamma_n$). Part b follows. As a corollary, if $\mathrm{Ot}(A) = \mathrm{Ot}(B)$ then $f_A =_c f_B$.

Given $\delta \in \mathrm{Fld}(A)$ let $A_{<\delta}$ denote $\{\langle\beta,\gamma\rangle \in A : \langle\beta,\delta\rangle \in A$ and $\langle\gamma,\delta\rangle \in A\}$. For part c, it suffices to observe that if $\delta \in \mathrm{Fld}(A)$ then $\mathrm{Ot}(A_{<\delta} \restriction \theta) < \mathrm{Ot}(A \restriction \theta)$ provided $\delta < \theta$. For part d, if δ is the element of $\mathrm{Fld}(A)$ which is maximal in the order, then $\mathrm{Ot}(A_{<\delta} \restriction \theta) = \mathrm{Ot}(A \restriction \theta) + 1$ provided $\delta < \theta$.

For part e, let δ_ξ be such that $\mathrm{Ot}(A_{<\delta_\xi}) = \mathrm{Ot}(A_\xi)$, and suppose $\mathrm{Ot}(A_{<\delta_\xi} \restriction \theta) = \mathrm{Ot}(A_\xi \restriction \theta)$ for a club C_ξ of θ. Let $C = \cap_{\xi<\eta} C_\xi$. Then $\mathrm{Ot}(A \restriction \theta) = \sup_{\xi<\eta} \mathrm{Ot}(A_{<\delta_\xi} \restriction \theta) = \sup_{\xi<\eta} \mathrm{Ot}(A_\xi \restriction \theta)$ for $\theta \in C$. Part f is similar, except $C = \triangle_{\xi<\kappa} C_\xi$ and $\mathrm{Ot}(A \restriction \theta) = \sup_{\xi<\theta} \mathrm{Ot}(A_{<\delta_\xi} \restriction \theta) = \sup_{\xi<\theta} \mathrm{Ot}(A_\xi \restriction \theta)$ for $\theta \in C$. □

Each well-order A on κ gives rise to a function $f_A : \mathrm{Lim}_\kappa \mapsto \kappa$, where $f_A(\theta) = \mathrm{Ot}(A \cap V_\kappa)$.

Corollary 3. a. *If $Ot(A) \leq Ot(B)$ then $f_A \preceq_{\mathcal{C}_L} f_B$.*
b. *If $Ot(A) < Ot(B)$ then $f_A \prec_{\mathcal{C}_L} f_B$.*
c. *If $Ot(A) = Ot(B) + 1$ then $f_A \equiv_{\mathcal{C}_L} f_B + 1$.*
d. *if $Ot(A) = \sup_{\xi<\eta} Ot(A_\xi)$ where $\eta < \kappa$ and $Ot(A_\xi)$ is increasing then $f_A \equiv_{\mathcal{C}_L} \sup_{\xi<\eta} f_\xi$.*
e. *if $Ot(A) = \sup_{\xi<\kappa} Ot(A_\xi)$ where $Ot(A_\xi)$ is increasing then*

$$f_A \equiv_{\mathcal{C}_L} dsup_{\xi<\kappa} f_\xi.$$

Proof. This is immediate by lemma 2. □

By well-known facts, f_A is a canonical function of index $\mathrm{Ot}(A)$. The set of these, ordered by $<_{\mathcal{C}_L}$, has chains of length κ^+. In previous papers, the author defined various function chains in $<_{\mathcal{C}_I}$; here more care will be exercised in specifying the domain. It is worth observing that in the definition of f_Σ for a scheme Σ given in [5], the domain may be taken as κ.

The obvious "next" function after the f_A is $\lambda \mapsto \lambda^+$ for $\lambda \in \mathrm{Card}$. This is a function into κ only if κ is a limit cardinal. Thus, in considering function chain of length greater than κ^+, $\kappa \in \mathrm{Inac}$ will be assumed. As already observed, functions need only be defined on the domain Card_κ. For example, $\lambda \mapsto \lambda^+$ is simple to define on Card_κ.

There are 2^κ function chains, so any function chain has length less than $(2^\kappa)^+$. Since $(2^\kappa)^+$ is regular, the length of a function chain is in fact bounded below $(2^\kappa)^+$.

3. A Generalized Iterated Hull Ordinal

Countable ordinals defined by an iterated hull construction have received considerable attention since the 1970's due to their usefulness in proof theory (see [11] for example). Here, a construction in [10] will be generalized.

Let θ be a cardinal. As in [5], for $k \geq 1$ let $C_k(\eta_k, \sigma_k, \ldots, \eta_1, \sigma_1)$ denote a "Cantor normal form (CNF)" function, and $\phi_k(\zeta, \xi_1, \tau_1, \ldots, \xi_k, \tau_k)$ an "infinitary Veblen (IV)" function. In [5] a definition is given of when the arguments of a function application are proper, and of the value, which is 0 if the arguments are not proper. A function application is said to be in normal form if the value is greater than any of the arguments.

Let E denote the "ϵ-numbers" ($\mathrm{Ran}(\phi_{*1})$ in the notation of [5]). It is well-known that if $\alpha \notin E$ then there is a unique normal form function application with $\alpha = C_k(\eta_k, \sigma_k, \ldots, \eta_1, \sigma_1)$. If $\alpha \in E$ then the closure of α under C_k functions equals α, and if $\alpha \notin E$ then the closure equals the next largest element of E.

Let ϕ_d denote the "diagonal function" of the infinitary Veblen function (denoted ψ in [5]). Let L denote the fixed points of ϕ_d, and write Λ_α for the α-th element of L. It follows by lemma 6 of [5] that if $\alpha \in E$ and $\alpha \notin L$ then there is a unique normal form function application with $\alpha = \phi_k(\zeta, \xi_1, \tau_1, \ldots, \xi_k, \tau_k)$. If $\alpha \in L$ then the closure of α under ϕ_k and C_k functions equals α, and if $\alpha \notin L$ then the closure equals the next largest element of L, which will be denoted α^Λ.

For notational convenience let Ω denote θ^+. Sets B_α and a function ψ are defined as follows. B_α is the closure of $\theta \cup \{\Omega\}$ under the functions C_k, ϕ_k, and $\psi \restriction \alpha$. $\psi(\alpha) = \min\{\xi : \xi \notin B_\alpha\}$. Let T be the set of terms, whose leaves are ordinals $\alpha < \theta$ or Ω, and whose interior nodes are functions C_k, ϕ_k, or ψ. Then B_α is those ordinals given by such terms, when ψ is interpreted as $\psi \restriction \alpha$.

The definition of B_α given in [10] requires only closure under the binary Veblen function. Here, closure under the ϕ_k is assumed, so that it is easy to see that the ordinal being defined is greater than Λ_0.

Lemma 4. *a. $|B_\alpha| = \theta$, and $\psi(\alpha) < \Omega$.*
b. If $\alpha \leq \beta$ then $B_\alpha \subseteq B_\beta$ and $\psi(\alpha) \leq \psi(\beta)$.
c. $B_\alpha \cap \Omega = \psi(\alpha)$.

d. $\psi(\alpha) \in L$.
e. If $\alpha \in Lim$ *then* $B_\alpha = \cup_{\beta<\alpha} B_\beta$ *and* $\psi(\alpha) = \sup_{\beta<\alpha} \psi(\beta)$.

Proof. For part a, $|T| = \theta$ by standard arguments, and $\psi(\alpha) < \Omega$ follows because Ω is a regular cardinal. Part b is straightforward. For part c, $B_\alpha \cap \Omega \subseteq \psi(\alpha)$ follows by the definition and part a. If $\beta < \psi(\alpha)$, it follows by induction on terms that $\beta \in B_\alpha$. This is clear if the term is a single node. If the root is a C_k or ϕ_k function then the value is either 0 or no less than the arguments, which are therefore in B_α inductively. If the root is ψ, say $\beta = \psi(\gamma)$ where $\gamma \in B_\alpha \cap \alpha$, then by part b $\psi(\gamma) \leq \psi(\alpha)$, that is $\beta \leq \psi(\alpha)$; and since $\beta \in B_\alpha$, $\beta \neq \psi(\alpha)$. Part d follows because $\psi(\alpha)$ is closed under the C_k and ϕ_k by part c. For part e, $\cup_{\beta<\alpha} B_\beta \subseteq B_\alpha$ by part b. If $\gamma \in B_\alpha$, it follows by induction on terms that $\gamma \in B_\beta$ for some $\beta < \alpha$. This is clear if the term is a single node. If the root is a C_k function then inductively the arguments are in some B_β, whence the value is. If the root is ψ, say $\gamma = \psi(\delta)$ where $\delta \in B_\alpha \cap \alpha$, then using the induction hypothesis β can be chosen so that $\delta \in B_\beta \cap \beta$, and it follows that $\gamma \in B_\beta$. $\sup_{\beta<\alpha} \psi(\beta) \leq \psi(\alpha)$ by part b. If $\gamma < \psi(\alpha)$ then by part c $\gamma \in B_\alpha \cap \Omega$, whence by what has already been proved $\gamma \in B_\beta \cap \Omega$ for some $\beta < \alpha$, whence again by part c $\gamma < \psi(\beta)$. □

Lemma 5. *Let* $\Lambda'(0)$ *be the first fixed point of* $\alpha \mapsto \Lambda_\alpha$.
a. If $\alpha \notin B_\alpha$ *then* $B_{\alpha+1} = B_\alpha$ *and* $\psi(\alpha+1) = \psi(\alpha)$.
b. If $\alpha \in B_\alpha$ *then* $\psi(\alpha+1) = \psi(\alpha)^\Lambda$.
c. If $\xi < \Lambda'(0)$ *then* $\psi(\xi) = \Lambda_\xi$.
d. If $\Lambda'(0) \leq \xi \leq \Omega$ *then* $\psi(\xi) = \Lambda'(0)$.
e. For any α, $B_\alpha \subseteq B_{\Omega^\Lambda}$ *and* $\psi(\alpha) \leq \psi(\Omega^\Lambda)$.

Proof. These correspond to lemma 23.9.ii, lemma 23.9.i, lemma 23.10.i, lemma 23.10.ii, and theorem 23.12 of [10] respectively. The proofs given there may be readily adapted. The adaptations of the proofs include replacing SC by L, etc. □

$\psi(\Omega^\Lambda)$ is thus the ordinal which has been constructed. As already mentioned, it is greater that Λ_0 (in fact the ordinal $\Lambda'(0)$ defined above). When necessary to specify the base θ, the notation $\psi(\Omega^\Lambda_\theta)$ will be used.

A function application $\alpha = \psi(\beta)$ is said to be in normal form if $\beta \in B_\beta$. The notation $\alpha =_N \psi(\beta)$ will be used to denote that this is the case.

Lemma 6. *a. Suppose* $\alpha_i =_N \psi(\beta_i)$ *for* $i = 1, 2$. *Then* $\alpha_1 = \alpha_2$ *iff* $\beta_1 = \beta_2$, *and* $\alpha_1 < \alpha_2$ *iff* $\beta_1 < \beta_2$.

b. *Given $\alpha < \Omega^\Lambda$ there is a $\beta \in B_\alpha$ such that $\beta \geq \alpha$. Let α_N be the least such; then α_N is the unique ordinal such that $\psi_(\alpha_N) =_N \psi(\alpha)$.*

c. *Given $\alpha \in L \cap \psi(\Omega^\Lambda)$ there is a unique ordinal $\beta \in B_{\Omega^\Lambda}$ such that $\alpha =_N \psi(\beta)$.*

Proof. The parts correspond respectively to lemma 23.14, 23.15, and 23.16 of [10]. The proofs are readily adapted. □

Say that a term is in normal form if every function application is.

Theorem 7. *For every $\alpha \in B_{\Omega^\Lambda}$ there is a unique normal form term whose value is α.*

Proof. Say that a term is in pre-normal form if all applications of C_k or ϕ_k are in normal form. Given any term, improper function applications may be replaced by 0, and then non-normal applications of C_k or ϕ_k replaced by a son in a bottom-up manner. Thus, only pre-normal form terms need be considered. It will be shown by induction on γ that any pre-normal form term for a value in B_δ may be transformed to a normal form term. The basis $\delta = 0$ is clear, and the claim for $\delta \in \mathrm{Lim}$ follows by lemma 4.e. Thus, suppose $\gamma = \gamma^- + 1$.

Suppose $\alpha \in B_\gamma - B_{\gamma^-}$, and suppose t is a pre-normal form term for α. Let n be a node of t which is not normal, which is closest to the root among such. If α_n is the value at n then $\alpha_n = \psi(\beta)$ for some β. There is an ordinal β_0 such that $\{\xi : \psi(\xi) = \psi(\gamma^-)\}$ equals the closed interval $[\beta_0, \gamma^-]$. If $\beta < \beta_0$ then $\alpha_n \in B_{\gamma^-}$, so inductively there is a normal form term u for α_n, and the subtree of t rooted at n may be replaced by u. If $\beta \in [\beta_0, \gamma^-]$ then there is a normal form term u for $\gamma^=$; letting m be the son of n, the subtree of t rooted at m may be replaced by u. The resulting modified version of t has fewer non-normal nodes, and the step may be repeated until a normal form term is obtained. □

Number the cases of the nodes of a term t as follows:

1. an element of θ
2. Ω
3. $C_k(\eta_1, \sigma_1, \ldots, \eta_k, \sigma_k)$
4. $\phi_k(\zeta, \xi_1, \tau_1, \ldots, \xi_k, \tau_k)$
5. $\psi(\beta)$

Let $A(t)$ denote the list of values of the arguments of type 4 nodes. Write $A(t) < \gamma$ if $\delta < \gamma$ for all δ in the list.

Lemma 8. *Suppose t is a term for α.*

a. *$\alpha \in B_\beta$ iff $A(t) < \beta$.*

b. *$\beta =_N \psi(\alpha)$ iff $A(t) < \alpha$.*

Proof. Both parts follow readily from the definitions. □

The results of a comparison between normal form terms may be broken in to the following cases, depending on the cases of the root nodes:

1,1: use an oracle
1,2: less than
2,2: equal
1,3: less than
2,3: compare Ω to η_1
3,3: use lexicographic order
1,4: less than
2,4: compare Ω to all arguments (greater than if all greater than, else less than)
3,4: compare η_1 to the second term
4,4: use reverse lexicographic order
1,5: less than
2,5: greater than
3,5: compare η_1 to the second term
4,5: compare all arguments to the second term (less than if all less than, else greater than)
5,5: compare the two β's

Using the above recursion, facts stated in the proof of lemma 10 of [5], and lemma 8 an algorithm for determining if two terms are in normal form and comparing them is obtained. It is readily seen that it runs in polynomial time, where oracle queries require 1 time unit.

Say that a tree with interior nodes labeled with C_k, ϕ_k, and ψ (with correct valencies) is a pre-term. If each leaf is labeled with an element of θ or Ω, call the result an o-term (over θ) (called simply a term above). If each leaf is labeled with a well-order on a subset of κ or Ω, call the result a w-term (over κ). A w-term over κ may be converted to an o-term over κ^+ by replacing each well-order by its length. Each o-term (and thus each w-term) represents an ordinal, namely that given by the recursive definition of the value of a term.

Given an o-term or w-term $\boldsymbol{\alpha}$ let α denote its value. Given a cardinal θ, the o-terms over θ which are in normal form and whose value is less than Ω, form a univalent system of notation for the ordinals less than $\psi(\Omega_\theta^\Lambda)$. The algorithm given above defines a binary relation $\preceq$ on the terms, and $\boldsymbol{\alpha} \prec \beta$ iff $\alpha < \beta$. In particular, $\preceq$ is a well-order on a subset.

Given an inaccessible cardinal κ, a w-term $\boldsymbol{\alpha}$ over κ yields an o-term over κ^+. The order $\preceq$ may be defined on the w-terms in an obvious manner, and is a

WPS. If the value is less than Ω then $\boldsymbol{\alpha}$ determines a function $f_{\boldsymbol{\alpha}} : \mathrm{Card}_\kappa \mapsto \kappa$, namely where $f_{\boldsymbol{\alpha}}(\theta)$ is the value of the term obtained by replacing each leaf well-order A by $A \cap V_\theta$. By lemma 2, if $\alpha < \beta$ then $f_{\boldsymbol{\alpha}} <_{\mathcal{C}_C} f_{\boldsymbol{\beta}}$. In particular, there are chains of length $\psi(\Omega^{\Lambda}_{\kappa^+})$ in the order $<_{\mathcal{C}}$ over κ.

4. Ordinal Comparison Oracle Machines

A comparison oracle machine is a Turing machine M with four additional states. M enters state Q to make a comparison. In the next step, the state is L(ess than), E(qual), or G(reater than). For $\theta \in \mathrm{Card}$ M operates over θ on pairs $\boldsymbol{\alpha}_1 = \langle c_1, k_1, \gamma_1, \ldots, \gamma_{k_1}\rangle$ and $\boldsymbol{\alpha}_2 = \langle c_2, k_2, \gamma_{k_1+1}, \ldots, \gamma_{k_1+k_2}\rangle$, where c_i is an integer code for each i and $\gamma_j \in \theta$ for each j. M enters state Q with the head to the left of a string for $\langle j_1, j_2\rangle$ where $1 \le j_i \le k_1 + k_2$; the result of the query is the result of the comparison of γ_{j_1} to γ_{j_2}. M halts with the value of the predicate $\boldsymbol{\alpha}_1 \preceq \boldsymbol{\alpha}_2$.

Say that M runs properly at θ if for any input pair $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, all queries are well-formed and the computation halts. Further the predicate $\preceq$ is a well-order on a subset.

Theorem 9. *If a comparison oracle machine M runs properly at ω then it runs properly at any cardinal θ.*

Proof. Given θ, for given c_1, c_2 the possible computations of M form a tree, with ternary nodes where queries are made. The branch through the tree depends only on the results of the queries. Integer values for the γ_j can be found, so that the same branch is taken. It follows that if queries are well-formed and the computation always halts over ω then this is true over any θ. For axiom T1, there are three inputs $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$ involved; integer values may be chosen for all the γ values, so that the behavior is mimicked on all three input pairs. Axioms T2-T5 follow similarly. Finally, given a descending chain $\boldsymbol{\alpha}_i$, again integer values for all the γ_j involved may be chosen, to mimic the behavior. $\square$

A machine which runs properly at all $\theta \in \mathrm{Card}$ will be said to be in the class $\mathcal{U}_M$. Given a $\mathcal{U}_M$ machine M, for each $\theta \in \mathrm{Card}$ a binary relation $\preceq_\theta$ is induced, on inputs of the form $\boldsymbol{\alpha} = \langle c, k, G_1, \ldots, G_k\rangle$ where G_j is a well-order on a subset of θ; this relation is a WPS. Suppose $\kappa \in \mathrm{Inac}$. For $\boldsymbol{\alpha}$ an input over κ, a value α may be associated with $\boldsymbol{\alpha}$, namely, $\mathrm{Ot}(\preceq_{\boldsymbol{\alpha}})$. A function $f_{\boldsymbol{\alpha}} : \mathrm{Card}_\kappa \mapsto \kappa$ may also be associated, where $f_{\boldsymbol{\alpha}}(\theta)$ equals $\mathrm{Ot}((\preceq_\theta)_{\boldsymbol{\alpha}\cap V_\theta})$.

Theorem 10. *Suppose M is a $\mathcal{U}_M$ machine and $\kappa \in$ Inac. If $\boldsymbol{\alpha} \preceq_\kappa \boldsymbol{\beta}$ then $f_{\boldsymbol{\alpha}} <_{\mathcal{C}_C} f_{\boldsymbol{\beta}}$.*

Proof. Using lemma 2 and the fact that $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ is forced by finitely many queries, it follows that $\{\theta : \boldsymbol{\alpha} \cap V_\theta \prec_\theta \boldsymbol{\beta} \cap V_\theta$ is in the club filter. The theorem follows using lemma 1. $\square$

In showing that an order is in $\mathcal{U}_M$, an informal algorithm may be given. This has been done for the terms of section 3, showing that $\psi(\Omega^\Lambda)$ is a $\mathcal{U}_M$ ordinal.

5. $\boldsymbol{\Sigma^1_1}$ Formulas

Let L_2 be the language of second order set theory. Usually, L_2 has a single sort, and a definable predicate which specifies which classes are sets. An alternative, which results in various simplifications, and will be adopted here, is to have two sorts, sets and classes. This raises its own complications, but they are readily handled as follows. There are two predicates, $x \in y$, and $x \in X$. There is an "equi-extensional" predicate $x \cong X$ which satisfies appropriate axioms.

An L_2, formula involving only first order quantifiers will be said to be Δ^1_0. A Δ^1_0 formula preceded by a block of existential (universal) class quantifiers will be said to be Σ^1_1 (Π^1_1).

For $\theta \in$ Lim, V_θ is a structure for L_2, with second order variables ranging over subsets of V_κ. Suppose $\kappa \in$ Inac. Now, a class is the same thing as an additional predicate symbol; in particular if $X \subseteq V_\kappa$ and $\theta \in \mathrm{Lim}_\kappa$ then in the structure V_θ, X is interpreted as $X \cap V_\theta$.

Theorem 11. *Suppose $\kappa \in$ Inac and $X_1, \ldots, X_k \subseteq V_\kappa$ are class parameters. Then there is a club subset $C \subseteq \mathrm{Lim}_\kappa$ such that for $\theta \in C$, V_θ is an elementary substructure of V_κ, in the language with symbols for the unary predicates $X_1, \ldots, X_k$.*

Proof. Unboundedness follows by a variant of the downward Lowenheim-Skolem theorem (see also Theorem 9.1.3 of [6]). Closedness is a well-known fact about elementary substructures. $\square$

Corollary 12. *If $\exists \vec{W} \phi(\vec{W}, \vec{X})$ is a Σ^1_1 sentence with class parameters, which holds in V_κ, then it holds in V_θ for a club of limit ordinals below κ.*

Proof. Instantiate the W_i and apply the theorem to $\vec{W}, \vec{X}$. $\square$

6. $\mathcal{U}_{\Sigma^1_1}$ Orders

Suppose $\kappa \in \mathrm{Inac}$, and $\phi(X,Y,\vec{P})$ and $\psi(X,Y,\vec{P})$ are Σ^1_1 formulas with class parameters $P_i \subseteq V_\kappa$. $\langle\phi,\psi\rangle$ is said to be a $\mathcal{U}_{\Sigma^1_1}$ formula pair at κ if ϕ defines a WPS $\preceq$ in V_κ and a WPS $\preceq_\lambda$ in V_λ for $\lambda \in \mathrm{Inac}_\kappa$; and ψ defines $\prec$ in V_κ and $\prec_\lambda$ in V_λ for $\lambda \in \mathrm{Inac}_\kappa$. $\preceq$ will be said to be a $\mathcal{U}_{\Sigma^1_1}$ order if it is the relation defined by the formula ϕ if a $\mathcal{U}_{\Sigma^1_1}$ formula pair.

Alternatively, one could require ϕ to define a WPS in V_θ for any $\theta \in \mathrm{Card}$. This stronger requirement would yield a function chain in $<_{\mathcal{C}_C}$. It is a question of interest whether the chains in $\preceq_{\mathcal{C}_C}$ are as long as those in $\preceq_{\mathcal{C}_I}$.

Various other specializations can be considered, such as requiring ϕ to be uniformly Δ^1_1, requiring ϕ to define a WPS in V_κ for all $\kappa \in \mathrm{Inac}$ and all values of $\vec{P}$, or requiring this last fact to be provable in NBG (see [9] for NBG, which is readily adapted to L_2). Whether such restrictions result in smaller ordinals is a question of interest. Further specializations will be described below.

Given $\kappa \in \mathrm{Inac}$ and a $\mathcal{U}_{\Sigma^1_1}$ order $\preceq$, let $\boldsymbol{\alpha}$ be an element of $\mathrm{Fld}(\preceq)$. As in section 4, given $\boldsymbol{\alpha}$, α may be used to denote $\mathrm{Ot}(\preceq_{\boldsymbol{\alpha}})$. The function $f_{\boldsymbol{\alpha}}$: $\mathrm{Inac}_\kappa \mapsto \kappa$ is that where $f_{\boldsymbol{\alpha}}(\lambda) = \mathrm{Ot}((\preceq_\lambda)_{\boldsymbol{\alpha}\cap V_\lambda})$. $f_{\boldsymbol{\alpha}}(\lambda) < (2^\lambda)^+$, which is less than κ since κ is inaccessible. It is a question of interest whether this bound can be improved. Note that $\mathcal{U}_{\Sigma^1_1}$ orders and $f_{\boldsymbol{\alpha}}$ are defined for any inaccessible cardinal; however $<_{\mathcal{C}_I}$ is defined only if κ is Mahlo.

Theorem 13. *Suppose $\preceq$ is a $\mathcal{U}_{\Sigma^1_1}$ order over κ where κ is a Mahlo cardinal.*

a. *If $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ then $f_{\boldsymbol{\alpha}} \leq_{\mathcal{C}_I} f_{\boldsymbol{\beta}}$.*
b. *If $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ then $f_{\boldsymbol{\alpha}} <_{\mathcal{C}_I} f_{\boldsymbol{\beta}}$.*

Proof. Let $\vec{P}$ be the parameters in the formula ϕ definition of $\preceq$. Let C be a club as in corollary 12 for the parameters $\boldsymbol{\alpha},\boldsymbol{\beta},\vec{P}$, so that $C \subseteq \{\theta : \boldsymbol{\alpha}\cap V_\theta \preceq_\theta \boldsymbol{\beta}\cap V_\theta$. Part a follows using lemma 1. Part b follows similarly, using ψ rather than ϕ. □

By part b, for Mahlo cardinals, if there is a $\mathcal{U}_{\Sigma^1_1}$ order of length α then there is a chain of length α in the order $<_{\mathcal{C}_I}$.

7. Set Chains for $\mathcal{U}_{\Sigma^1_1}$ Orders

As in [3], for $\kappa \in \mathrm{Inac}$ and $X, Y \subseteq \kappa$ say that $X \subseteq_{\mathrm{t}} Y$ if $X - Y$ is thin. For $X \subseteq \mathrm{Inac}_\kappa$ let $\mathrm{H}(X) = \{\lambda \in X\colon X \cap \lambda$ is a stationary subset of $\lambda\}$. This operation is uninteresting unless κ is Mahlo, but for technical reasons it is defined for $\kappa \in \mathrm{Inac}$. For X, Y stationary subsets of Inac_κ say that $X <_R Y$ if $Y \subseteq_{\mathrm{t}} \mathrm{H}(X)$. It is well-known that this relation is transitive and well-founded; let ρ_R denote the rank function. Note that $<_R$ is empty unless κ is Mahlo.

Suppose $\preceq$ is a WPS on κ. For $\boldsymbol{\alpha} \in \mathrm{Fld}(\preceq)$ and $X \subseteq \mathrm{Inac}_\kappa$, say that $\lambda \in \mathrm{H}^{\boldsymbol{\alpha}}(X)$ iff $\lambda \in X$ and $\mathrm{H}^{\gamma}(X \cap \lambda)$ is a stationary subset of λ for all $\boldsymbol{\gamma} \in \mathrm{Fld}(\preceq_\lambda)$ where $\gamma < f_{\boldsymbol{\alpha}}(\lambda)$, or equivalently $\boldsymbol{\gamma} \prec_\lambda \boldsymbol{\alpha} \cap V_\lambda$.

Theorem 14. *If $\beta \leq \alpha$ then for any $X \subseteq Inac_\kappa$, $H^{\boldsymbol{\beta}}(X) \supseteq_t H^{\boldsymbol{\alpha}}(X)$.*

Proof. The proof is by induction on κ. For the basis, κ is the smallest inaccessible cardinal, X is always empty, and the claim is trivial. For arbitrary κ, there is a thin set T such that if $\lambda \in \mathrm{Inac}_\kappa$ and $\lambda \notin T$ then $f_{\boldsymbol{\beta}}(\lambda) \leq f_{\boldsymbol{\alpha}}(\lambda)$. For such λ, if $\mathrm{H}^{\boldsymbol{\beta}}(X \cap \lambda)$ is stationary for $\beta < f_{\boldsymbol{\alpha}}(\lambda)$, then by the induction hypothesis $\mathrm{H}^{\boldsymbol{\beta}}(X \cap \lambda)$ is stationary for $\beta < f_{\boldsymbol{\beta}}(\lambda)$. □

Lemma 15. *Suppose $\boldsymbol{\alpha} \in Fld(\preceq)$, $X \subseteq Inac_\kappa$, and $\lambda \in Inac_\kappa$. Then $H^{\boldsymbol{\alpha} \cap V_\lambda}(X \cap \lambda) = H^{\boldsymbol{\alpha}}(X) \cap \lambda$.*

Proof. Suppose $\mu \in X \cap \lambda$. Then $\mu \in \mathrm{H}^{\boldsymbol{\alpha} \cap V_\lambda}(X \cap \lambda)$ iff $\mathrm{H}^{\gamma}(X \cap \mu)$ is stationary for $\boldsymbol{\gamma} \prec_\mu \boldsymbol{\alpha} \cap V_\lambda \cap V_\mu$ iff $\mathrm{H}^{\gamma}(X \cap \mu)$ is stationary for $\boldsymbol{\gamma} \prec_\mu \boldsymbol{\alpha} \cap \cap V_\mu$ iff $\mu \in \mathrm{H}^{\boldsymbol{\alpha}}(X)$. □

Theorem 16. *If $\beta < \alpha$ then for any $X \subseteq Inac_\kappa$, $H(H^{\boldsymbol{\beta}}(X)) \supseteq_t H^{\boldsymbol{\alpha}}(X)$.*

Proof. By theorem 14, except for a thin set of λ, if $\lambda \in \mathrm{H}^{\boldsymbol{\alpha}}(X)$ then $\lambda \in \mathrm{H}^{\boldsymbol{\beta}}(X)$. Also, except for a thin set of λ, $\boldsymbol{\beta} \cap V_\lambda \prec \boldsymbol{\alpha} \cap V_\lambda$, and for such λ, $\mathrm{H}^{\beta \cap V_\lambda}(X \cap \lambda)$ is stationary. By lemma 15, $\mathrm{H}^{\boldsymbol{\beta}}(X) \cap \lambda$ is stationary. □

8. Enforceability

If ϕ is an L_2 formula and s is a set let $\phi^{(s)}$ be ϕ, with first (second) order bound variables constrained to range over elements (subsets) of s.

Lemma 17. *Fix a Godel numbering $\ulcorner\phi\urcorner$ of the L_2 formulas ϕ in the free second order variables $\vec{P}$.*

a. *There is a Δ^1_0 formula $Sat_s(s, f)$ in the free second order variables $\vec{P}$ such that $\vdash_{NBG} \phi^{(s)} \Leftrightarrow Sat_s(s, \ulcorner\phi\urcorner)$.*
b. *Write $\vec{x}$ for some of the free variables. For $\kappa \in Inac$, $\lambda \in Inac_\kappa$, and $x_i \subseteq V_\lambda$, $\models_{V_\lambda} \phi(\vec{x})$ iff $\models_{V_\kappa} Sat_s(V_\lambda, \ulcorner\phi\urcorner, \vec{x})$.*

Proof. Part a follows by standard methods; an outline will be given. Let u be the set of sentences of L_2 expanded with members (subsets) of s, to instantiate first (second) order variables. Let $t : u :\mapsto \{0,1\}$ be the assignment of truth values. There is a Δ^1_0 formula $\mathrm{TA}(s,t)$ stating that t is the truth assignment. Then $\mathrm{Sat}_s(s,f)$ iff $\exists t(\mathrm{TA}(s,t) \wedge t(f) = 1)$. Part b follows, since $\models_s \phi$ iff $\phi^{(s)}$ iff $\models_{V_\kappa} \phi^{(s)}$. $\square$

Suppose for the rest of the section that $\kappa \in \mathrm{Inac}$ and $\preceq$ is a $\mathcal{U}_{\Sigma^1_1}$ order over κ, defined by the formula pair $\langle \phi, \psi \rangle$, with parameters $\vec{P}$.

Lemma 18. *There is a Δ^1_0 formula defining the function $\lambda \mapsto S_\lambda$ with domain $Inac_\kappa$, where $S_\lambda = \{\langle b, x, y\rangle : b \in Fld(\preceq_\lambda) \wedge x \subseteq Inac_\lambda \wedge y \subseteq Inac_\lambda \wedge y = H^b(x)\}$.*

Proof. The function may be defined by recursion on λ. The clause $b \in \mathrm{Fld}(\preceq_\lambda)$ may be expressed as $\mathrm{Sat}_s(V_\lambda, \ulcorner\phi\urcorner, b, b, \vec{P})$. The clause $y = \mathrm{H}^b(x)$ may be expressed as $\forall \mu < \lambda(\mu \in y \Leftrightarrow \mu \in x \wedge \forall c \forall y'(\mathrm{Sat}_s(V_\mu, \ulcorner\psi\urcorner, c, b \cap V_\mu) \wedge \langle c, x \cap V_\mu, y'\rangle \in S_\mu \Rightarrow$ "y' is stationary")). $\square$

Theorem 19. *There is a Π^1_1 formula $\Phi_\preceq(A)$ which holds in V_κ iff $H^A(Inac)$ is stationary.*

Proof. The formula may be stated as $\forall C, X, Y, Z($ $A \in \mathrm{Fld}(\preceq) \wedge$ "C is club" $\wedge$ $X = \mathrm{Inac} \wedge Y = \mathrm{H}^A(X) \wedge Z = Y \cap C \Rightarrow Z \neq \emptyset)$. The clause $A \in \mathrm{Fld}(\preceq)$ is $A \preceq A$, which is Σ^1_1. The clause $Y = \mathrm{H}^A(X)$ may be expressed as $\forall \lambda(\lambda \in Y \Leftrightarrow \lambda \in X \wedge \forall c \forall y'($ $\mathrm{Sat}_s(V_\lambda, \ulcorner\psi\urcorner, c, A \cap V_\lambda) \wedge \langle c, X \cap V_\lambda, y'\rangle \in S_\lambda \Rightarrow$ "y' is stationary" $))$. This is Δ^1_0, as are the remaining clauses. $\square$

Let $\Phi^-_\preceq$ denote the formula $\forall B(B \prec A \Rightarrow \Phi_\preceq(B)$.

Theorem 20. *If κ is weakly compact then $\models_{V_\kappa} \Phi_\preceq(A)$ for any $A \in Fld(\preceq)$.*

Proof. This may be proved by induction along $\preceq$ on A. Inductively, $\models_{V_\kappa} \Phi^-_\preceq$ may be assumed. Since κ is weakly compact, $\{\lambda \in \mathrm{Inac}_\kappa :\models_{V_\lambda} \Phi^-_\preceq$ is stationary. It follows that for a stationary set of λ, $\mathrm{H}^b(\mathrm{Inac} \cap \lambda)$ is stationary for all $b \prec_\lambda A \cap V_\lambda$; and so $\mathrm{H}^A(X)$ is stationary. $\square$

9. New Axioms

Suppose $\phi(X, Y, \vec{P})$ is a formula of L_2. The statement that ϕ defines a WPS, and does so in V_κ for any $\kappa \in \text{Inac}$, is a formula of L_2, with free variables for the parameters. Letting $\preceq$ denote the order, $\preceq$ will be said to be a $\mathcal{U}_{\Delta^1_\infty}$ order if the formula holds in V, and a $\mathcal{U}_{\Delta^1_\infty}$ order in V_κ if it holds in V_κ, where $p_j \subseteq V_\kappa$.

Let $A_\preceq$ be the L_2 statement, "if $\preceq$ is a WPS then $\forall A \Phi_\preceq(A)$". A justification of adopting $A_\preceq$ as an axiom is as follows. Suppose the universe is sufficiently large. Inductively, assume $\Phi^-_\preceq$ holds in V_λ for a stationary class of λ. Since the universe is sufficiently large, there is a $\kappa \in \text{Inac}$ such that $\Phi^-_\preceq$ holds in V_λ for a stationary set $\lambda < \kappa$. In fact, there is a stationary class of such κ. Thus, there is a stationary class of κ such that $\Phi_\preceq$ holds in V_κ.

This argument is fairly strong as it is. It should be investigated whether it can be strengthened by giving more details. Further discussion is omitted here.

There is an axiom scheme whose formulas are all the formulas $A_\preceq$. This axiom scheme states that Card is $\mathcal{U}_{\Delta^1_\infty}$-Mahlo, and $\kappa \in \text{Inac}$ is said to be $\mathcal{U}_{\Delta^1_\infty}$-Mahlo if the axiom scheme holds in V_κ.

Restricting $\preceq$ to be $\mathcal{U}_{\Sigma^1_1}$ yields the notion of a $\mathcal{U}_{\Sigma^1_1}$-Mahlo cardinal. As seen in section 8 a weakly compact cardinal is $\mathcal{U}_{\Sigma^1_1}$-Mahlo. It seems likely that the Mahlo-ness of a weakly compact cardinal cannot be raised much higher than this. It is a question of considerable interest whether the existence of $\mathcal{U}_{\Delta^1_\infty}$-Mahlo cardinals implies the existence of weakly compact cardinals. The former axiom has been justified, so if this is so then the existence of weakly compact cardinals would be justified.

10. L_{OS} and OS_θ

In this and the following few sections a discussion will be given of a class of orders intermediate between $\mathcal{U}_M$ and $\mathcal{U}_{\Sigma^1_1}$ orders. This topic was considered in [2], but the discussion there was incomplete. It is of interest since it provides a further example, and the uniformity requirement holds automatically,

Let L_{OS} be a language with two sorts, one for ordinals (which will be denoted α, etc.), and one for sequences of ordinals whose length is an ordinal (which will be denoted s, etc.). The language includes the functions and relations 0, 1, $\alpha + \beta$, and $\alpha < \beta$. In addition there are the functions $\alpha = \text{Dom}(s)$, $\beta = \text{Eval}(s, \alpha)$, and $t = \text{Rstr}(s, \alpha, \beta)$.

We define a structure OS_θ for the language L_{OS} for any cardinal θ. The ordinals are interpreted as θ, and the sequences as $\theta^{<\theta}$. $0, 1, +, <$ are interpreted as usual. The additional functions are interpreted as follows.

- Dom : $\theta^{<\theta} \mapsto \theta$. Dom$(s)$ is the domain of s.
- Eval : $\theta^{<\theta} \times \theta \mapsto \theta$. Eval$(s, \alpha) = s(\alpha)$.
- Rstr : $\theta^{<\theta} \times \theta \times \theta \mapsto \theta^{<\theta}$. Rstr$(s, \alpha, \beta)$ equals s, restricted to $\{\gamma : \alpha \leq \gamma < \alpha + \beta\}$.

We may write $|s|$ for Dom(s) and $s(\alpha)$ for Eval(s, α).

Bounded quantifiers in L_{OS} are those of the form $\forall\gamma < \beta$ or $\exists\gamma < \beta$ where β is a term. Δ_0, Σ_1, and Π_1 formulas are defined as usual, where free variables and unbounded quantifiers may be of either sort. Note that $<$ is definable; but it is convenient to have it in the language for the specification of the bounded quantifiers.

In the notation of [12], let $R_0(\beta_1, \beta_2, \beta_1', \beta_2')$ denote the Godel well order on ordered pairs, and let J_0 denote the Godel pairing function.

Lemma 21. *a. J_0 is Δ_1.*
b. The Σ_1 predicates are closed under bounded quantification.
c. If $G : \theta^{<\theta} \mapsto \theta$ is a Σ_1 function then there is a Σ_1 function $F : \theta \mapsto \theta$ such that $F(\alpha) = G(F \upharpoonright \alpha)$.

Proof. For part a, let $P_1(s, t)$ be the predicate which is true iff $|s| = |t|$ and the sequence of ordered pairs $\langle s(\gamma), t(\gamma)\rangle$ for $\gamma < |s|$ is the enumeration of an initial segment of R_0. It is readily verified that P_1 is Δ_0; and $J(\beta, \gamma) = \delta$ iff $\exists s, t(|s| = |t| = \delta + 1 \wedge s(\delta) = \beta \wedge t(\delta) = \gamma)$. For part b, first, $\forall\gamma < \beta\exists\delta R(\gamma, \delta)$ can be rewritten as $\exists t(|t| = \beta \wedge \forall\gamma < \beta R(\gamma, t(\gamma)))$. Second, let $P_1(t, u)$ hold iff $|t| = |u|$ and for all $\gamma < |t|$, $t(\gamma)$ is the sum of the $u(\delta)$ for $\delta < \gamma$; P_1 is Δ_0, and $\forall\gamma < \beta\exists s R(\gamma, s)$ can be rewritten as $\exists t, u, v(|t| = \beta \wedge P_1(t, u) \wedge \forall\gamma < \beta R(\gamma, \mathrm{Rstr}(v, t(\gamma), u(\gamma))))$, Part b now follows. For part c, the predicate $F(\beta) = \gamma$ can be written as $\exists s(|s| = \beta + 1 \wedge \forall\delta < \beta + 1(s(\delta) = G(\mathrm{Rstr}(s, 0, \delta))) \wedge s(\beta) = \gamma)$. $\square$

It is clear from the foregoing that OS_θ is a suitable setting for recursion theory. For further evidence, it is shown in [2] that a relation is Σ_1 over L_α iff it is Σ_1 over $\mathrm{OS}_\theta \cap L$.

The case $\theta = \kappa^+$ will be of special interest. In this case, by Hausdorff's formula (see [8]) the number of sequences equals 2^κ.

11. An Axiom System

An axiom system for the theory of OS_θ can be given. An axiom system for ordinal addition is of independent interest, and will be given first.

Axioms for $0, 1, +$:

- The axioms of equality (for the ordinal sort)
- $\alpha + 0 = \alpha$
- $0 + \alpha = \alpha$
- $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
- $\alpha + \beta = \alpha + \gamma \Rightarrow \beta = \gamma$
- $\alpha + \beta = 0 \Rightarrow \beta = 0$
- $\neg \exists \delta(\beta = \alpha + \delta) \Rightarrow \exists \delta(\delta \neq 0 \wedge \alpha = \beta + \delta)$
- $\alpha + \beta = 1 \Rightarrow (\alpha = 0 \vee \beta = 0)$

Axiom defining $<$:

- $\alpha < \beta$ iff $\exists \delta(\delta \neq 0 \wedge \beta = \alpha + \delta)$

Theorems concerning $<$ provable from the preceding:

- $\neg(\alpha < \alpha)$
- $\alpha < \beta \wedge \beta < \gamma \Rightarrow \alpha < \gamma$
- $\alpha + \beta < \alpha + \gamma \Leftrightarrow \beta < \gamma$
- $\neg(\alpha < 0)$
- Exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ holds
- $\neg \exists \beta(\alpha < \beta < \alpha + 1)$
- $\alpha < \beta + 1 \Leftrightarrow \alpha < \beta \vee \alpha = \beta$
- $\alpha < \beta \Leftrightarrow \alpha + 1 < \beta \vee \alpha + 1 = \beta$
- $\alpha + 1 < \beta + 1 \Leftrightarrow \alpha < \beta$

Axiom defining $\leq$:

- $\alpha \leq \beta$ iff $\alpha < \beta \wedge \alpha = \beta$

Theorems concerning $\leq$ provable from the preceding:

- $\alpha \leq \alpha$
- $\alpha \leq \beta \wedge \beta \leq \gamma \Rightarrow \alpha \leq \gamma$
- $\alpha \leq \beta \wedge \beta \leq \alpha \Rightarrow \alpha = \beta$
- $\alpha \leq \beta \vee \beta \leq \alpha$
- $\alpha + \beta \leq \alpha + \gamma \Leftrightarrow \beta \leq \gamma$
- $0 \leq \alpha$
- $\alpha < \beta + 1 \Leftrightarrow \alpha \leq \beta$
- $\alpha < \beta \Leftrightarrow \alpha + 1 \leq \beta$

Axiom of continuity of $+$:

- $\forall \beta' < \beta(\alpha + \beta' < \gamma) \Rightarrow \alpha + \beta < \gamma$

Induction axiom (for formulas involving only the ordinal sort)

- $\forall \alpha(\forall \gamma < \alpha \phi_{\gamma/\alpha} \Rightarrow \phi) \Rightarrow \phi$.

The preceding system of axioms will be referred to as A_{O}. It is a question of interest whether there are any redundancies, but this topic is omitted here. It is a straightforward observation that Pressburger arithmetic is interpretable in A_{O}. To obtain an axiom system A_{OS} for L_{OS} the equality and induction axioms allow variables of either sort; and the following axioms are added. to A_{O}.

- $\gamma \geq \mathrm{Dom}(s) \Rightarrow \mathrm{Eval}(s,\gamma) = 0$
- $\mathrm{Dom}(s) = \mathrm{Dom}(t) \wedge \forall\gamma(\mathrm{Eval}(s,\gamma) = \mathrm{Eval}(t,\gamma)) \Rightarrow s = t$.
- $t = \mathrm{Rstr}(s,\alpha,\beta) \wedge 0 \leq \gamma \wedge \alpha + \gamma < \beta \Rightarrow \mathrm{Eval}(t,\gamma) = \mathrm{Eval}(s,\alpha+\gamma)$.
- (Comprehension) If F is an ordinal values function whose domain is an ordinal α then there is a sequence s such that $\mathrm{Dom}(s) = \alpha$ and $\forall\gamma < \alpha(F(\gamma) = \mathrm{Eval}(s,\gamma))$.

A_{OS} is interpretable in NBG, without power set or replacement.

12. Interpreting $\mathbf{OS}_{\kappa^+}$ in Classes

Interpreting ordinals and sequences as classes, L_{OS} may be interpreted in L_2 by means of the predicates below. Unless otherwise indicated these predicates are Δ^1_0. π_i denotes a projection function.

$I_{Ord}(A)$. A is a class of pairs of ordinals and as a binary relation A is transitive, satisfies trichotomy on its field, and $\neg\exists f(f : \omega \mapsto \mathrm{Fld}(A)$ and $\forall n(A(f(n+1), f(n))))$.

$I_{Seq}(S)$. $S = \{0\} \times S_0 \cup \{1\} \times S_1$ where S_0 satisfies I_{Ord} and S_1 is a class of triples $\langle\zeta,\alpha,\beta\rangle$ and $\pi_1[S_1] = \mathrm{Fld}(S_0)$ and $\forall\zeta \in \mathrm{Fld}(S_0)$ $S_{1\zeta}$ satisfies I_{Ord}, where $S_{1\zeta} = \{\langle\alpha,\beta\rangle : \langle\xi,\alpha,\beta\rangle \in S_1\}$.

$I_{Zero}(A)$. A is the empty relation.

$I_{One}(A)$. A consists of a single pair $\langle 0,1\rangle$.

$I_{Plus}(C,A,B)$. If $\alpha = \beta + n$ where β is a limit ordinal and n is an integer, let $\alpha^{(i)} = \beta + 2n + i$ for $i = 0,1$. Using obvious notation, $\mathrm{Fld}(C) = \mathrm{Fld}(A)^{(0)} \cup \mathrm{Fld}(B)^{(1)}$ and $\langle\alpha^{(i)},\beta^{(j)}\rangle \in C$ iff $i < j$ or $i = j = 0$ and $\langle\alpha,\beta\rangle \in A$ or $i = j = 1$ and $\langle\alpha,\beta\rangle \in B$.

$I_<(A,B)$. A Σ^1_1 formula for $A \leq B$ is $\exists F(F : \mathrm{Fld}(A) \mapsto \mathrm{Fld}(B)$ and F is monotone). A Σ^1_1 formula for $A < B$ adds the conjunct "F is not surjective" to the matrix. $\neg(B \leq A)$ is a Π^1_1 formula for $A < B$. Thus, $I_<(A,B)$ is a Δ^1_1 predicate.

$I_{=_{Ord}}(A,B)$. A formula for this is $A \leq B \wedge B \leq A$. By the preceding paragraph, $I_{=_{Ord}}(A,B)$ is a Δ^1_1 predicate.

$I_{=_{Seq}}(S,T)$. This holds iff there is an order isomorphisms $F_0 : S_0 \mapsto T_0$ such that $S_{1\zeta} \equiv T_{1,F_0(\zeta)}$. The Π^1_1 form is $S_0 \equiv T_0 \wedge \forall F_0 \ldots$. Thus, $I_{=_{Seq}}(S,T)$ is a Δ^1_1 predicate.

$I_{Dom}(A,S)$. $\langle\alpha,\beta\rangle \in A$ iff $\langle 0,\langle\alpha,\beta\rangle\rangle \in S$.

$I_{Eval}(B, S, A)$. The predicate $\mathrm{IsoIS}(F, A, B, \alpha)$ stating that F is an order isomorphism from A to $\{\langle\beta_1, \beta_2\rangle \in B : \langle\beta_1, \alpha\rangle \in B$ and $\langle\beta_2, \alpha\rangle \in B\}$ is readily seen to be Δ^1_0. Let ϕ_1 be $\mathrm{IsoIS}(F, A, S_0, \alpha)$. Let ϕ_2 be $\langle\beta_1, \beta_2\rangle \in B \Leftrightarrow \langle\alpha, \beta_1, \beta_2\rangle \in S_1$. A Σ^1_1 formula for I_{Eval} is $\exists F\exists\alpha(\phi_1 \wedge \phi_2) \vee A \geq S_0 \wedge B = \emptyset$. A Π^1_1 formula for I_{Eval} is $A < S_0 \wedge \forall F\forall\beta(\phi_1 \Rightarrow \phi_2) \vee A \geq S_0 \wedge B = \emptyset$. Thus, $I_{Eval}(B, S, A)$ is a Δ^1_1 predicate.

$I_{Rstr}(T, S, A, B)$. Let ϕ_1 be $\mathrm{IsoIS}(F_1, A, S_0, \alpha)$. Let ϕ_2 be $\mathrm{IsoIS}(F_2, A, S_0, \beta)$. Let ϕ_3 be $\langle\xi_1, \xi_2\rangle \in T_0 \Leftrightarrow \langle\xi_1, \xi_2\rangle \in S_0 \wedge (\xi_1 = \alpha \vee \langle\alpha, \xi_1\rangle \in S_0) \wedge \langle\xi_2, \beta\rangle \in S_0$. Let ϕ_4 be $\langle\xi, \gamma_1, \gamma_2\rangle \in T_1 \Leftrightarrow \langle\xi, \gamma_1, \gamma_2\rangle \in S_1 \wedge \xi \in \mathrm{Fld}(T_0)$. Let ϕ'_3 be $\langle\xi_1, \xi_2\rangle \in T_0 \Leftrightarrow \langle\xi_1, \xi_2\rangle \in S_0 \wedge (\xi_1 = \alpha \vee \langle\alpha, \xi_1\rangle \in S_0)$. A Σ^1_1 formula for I_{Rstr} is $A \geq B \wedge S = \emptyset \vee A < B \wedge (\exists F_1\exists F_2\exists\alpha\exists\beta(\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4) \vee B \geq S_0 \wedge \exists F_1\exists\alpha(\phi_1 \wedge \phi'_3 \wedge \phi_4) \vee A \geq S_0 \wedge S = \emptyset)$. Transforming the Σ^1_1 formula to a Π^1_1 formula is routine. Thus, $I_{Rstr}(T, S, A, B)$ is a Δ^1_1 predicate.

Using these predicates, using well-known methods (see [7] for example), each formula ϕ over L_{OS} can be translated to an "equivalent" formula ϕ^I over L_2. Formulas over L_2 which are translations of Δ_0 formulas over L_{OS} will be called Δ^I_0 formula, and similarly for Σ^I_1 and Π^I_1. Predicates are said to be Σ^I_1, Π^I_1, or Δ^I_1, analogously to the Σ^1_1 case.

The (standard) translation of atomic formulas is as follows. If t is a term let $\phi_{V=t}$ denote the formula $V_1 = t_1 \wedge \cdots \wedge V_k = t_k$, where $V = V_k$ and each conjunct is an interpreting formula of a function symbol with appropriate free variables. Let $\phi_{V<W}$ be the interpreting formula for $<$ with appropriate free variables For an atomic formula $t < u$, a Σ^1_1 translation is $\exists\vec{V}\vec{W}(\phi_{V=t} \wedge \phi_{W=u} \wedge \phi_{V<W})$. A Π^1_1 translation is $\forall\vec{V}\vec{W}(\phi_{V=t} \wedge \phi_{W=u} \Rightarrow \phi_{V<W})$. Atomic formulas involving the equality predicates are translated similarly.

For $\kappa \in \mathrm{Inac}$, V_κ is a structure for L_2. I^κ_{Ord} will be used to denote the classes satisfying $I_{Ord}(A)$ in V_κ. Likewise, I^κ_{Seq} denotes the classes satisfying $I_{Seq}(S)$. These families are closed under the functions $+$, Dom, Eval, and Rstr. In this manner, V_κ may be considered as a structure for L_{sOS}. I^κ_{Ord} consists of the well-orders on κ; this will be proved below.

13. Congruence

The interpreting formula for $=$ is not equality; however, as will be seen, it is a congruence relation, in the structures V_κ, $\kappa \in \mathrm{Inac}$. Write $A \equiv B$ for $I_{=_{Ord}}(A, B)$, and and $S \equiv T$ for $I_{=_{Seq}}(S, T)$.

Lemma 22. *In any V_κ, $\equiv$ (of either sort) is a congruence relation.*

Proof. First, $\leq$ is readily seen to be reflexive and transitive, by properties of the functions involved. That $\equiv$ on I^{κ}_{Ord} is an equivalence relation follows at once. That $<$ respects this relation follows by properties of the functions. Verification that $\equiv$ on I_{Seq} is an equivalence relation is straightforward, using the claim for $\equiv$ on ordinals, and the identity function, the inverse function, and composition of functions to construct the function F_0 of the definition, in the cases of the reflexive, symmetric, and transitive laws respectively. For the function symbols, given the (unique) isomorphisms witnessing that the arguments are equivalent, the isomorphism witnessing that the values are equivalent may be readily constructed. $\square$

Lemma 23. *Suppose ϕ is a Δ^I_0 formula with free variables for the values $X_1, \ldots, X_k$. Suppose $X_1 \equiv X'_1, \ldots, X_k \equiv X'_k$. Then $\phi(X_1, \ldots, X_k)$ iff $\phi(X'_1, \ldots, X'_k)$. The same holds if ϕ is Σ^I_1 or Π^I_1.*

Proof. The proof is a routine induction using lemma 22. Using the lemma and the translation described in section 4, it follows for a term that given equivalent arguments, the values are equivalent. The theorem for atomic formulas now follows using the lemma. The induction step for the propositional connectives is routine. The translation of $\exists Y < t\psi(Y, X_1, \ldots, X_k)$ is $\exists V \exists Y(\phi_{V=t} \wedge Y < V \wedge \psi(Y, X_1, \ldots, X_k))$. By the claim for terms, $V \equiv V'$. It is easily seen that Y' may be taken as Y, and the claim follows by induction. The claim for the unbounded existential quantifier follows also. $\square$

14. Quantifier Complexity

Lemma 24. *A Δ^I_0 predicate is Δ^1_1.*

Proof. It suffices to show that it is Σ^1_1. This is observed for atomic formulas in section 12. It follows at once for negated atomic formulas; it may be assumed for the induction that all negations apply to atomic formulas. The claim follows for $\wedge$ and $\vee$ be standard methods. It follows for bounded existential quantification by standard methods, noting that $X < t$ has already been shown to be Δ^1_1. Suppose ϕ is $\forall A < t \exists W_1 \cdots W_k \psi(A, \vec{W})$ where ψ is Δ^1_0. Using theorem 23 it follows that ϕ is equivalent to $\exists \vec{V} \exists \vec{S}(\phi_{V=t} \wedge \forall x(x \in \mathrm{Fld}(A) \Rightarrow \psi(V_{<x}, S_{1x}, \ldots, S_{kx})))$, where $w \in A_{<x}$ iff $\langle w, x\rangle \in V$ and $w \in S_{ix}$ iff $\langle x, w\rangle \in S_i$. $\square$

Corollary 25. *A Σ_1^I predicate is Σ_1^1. A Π_1^I predicate is Π_1^1. A Δ_1^I predicate is Δ_1^1.*

Proof. Immediate. □

15. Absoluteness

Lemma 26. *a. For $A \subseteq V_\kappa$, $\models_V I_{Ord}(A)$ iff $\models_{V_\kappa} I_{Ord}(A)$.*
b. For $S \subseteq V_\kappa$, $\models_V I_{Seq}(S)$ iff $\models_{V_\kappa} I_{Seq}(S)$.

Proof. Most claims follow by standard absoluteness arguments and the fact that $\alpha, \beta \in V_\kappa$ iff $\langle \alpha, \beta \rangle \in V_\kappa$. For part a, if f is a descending chain in V_κ then it is a descending chain in V. Conversely, if f is a descending chain in V then f is a bounded subset of V_κ, whence $f \in V_\kappa$. Part b follows using part a. □

Lemma 27. *a. For $A, B \in I_{Ord}^\kappa$, $\models_{V_\kappa} A < B$ iff $\models_V A < B$.*
b. For $A, B \in I_{Ord}^\kappa$, $\models_{V_\kappa} A \equiv B$ iff $\models_V A \equiv B$.
c. For $S, T \in I_{Seq}^\kappa$, $\models_{V_\kappa} S \equiv T$ iff $\models_V S \equiv T$.

Proof. Note that if $F : A \mapsto B$ then $F \subseteq V_\kappa$. Parts a and b follow readily. Part c follows similarly. □

Lemma 28. *Suppose $\phi(Y, X_1, \ldots, X_n)$ is the interpreting formula of a function symbol of L_{OS}. Suppose $X_1, \ldots, X_k \subseteq V_\kappa$. If $\models_V G(Y, X_1, \ldots, X_k)$ then $Y \subseteq V_\kappa$. For $Y \subseteq V_\kappa$, $\models_V G(Y, X_1, \ldots, X_k)$ iff $\models_{V_\kappa} G(Y, X_1, \ldots, X_k)$.*

Proof. If G is Δ_0^1 the claim follows by routine verification. If $A, B \subseteq V_\kappa$ and $\mathrm{IsoIS}(F, A, B, \alpha)$ then $F \subseteq V_\kappa$ and $\alpha \in V_\kappa$. Using the Σ_1^1 form for G and lemma 27, the claim follows for Eval and Rstr as well. □

Theorem 29. *Suppose ϕ is a formula with free variables for the values $X_1, \ldots, X_k$. Suppose $\kappa \in$ Inac, and $X_i \subseteq V_\kappa$ for $1 \leq i \leq k$.*
a. If ϕ is Δ_0^I then $\models_{V_\kappa} \phi(X_1, \ldots, X_k)$ iff $\models_V \phi(X_1, \ldots, X_k)$.
b. If ϕ is Σ_1^I then if $\models_{V_\kappa} \phi(X_1, \ldots, X_k)$ then $\models_V \phi(X_1, \ldots, X_k)$.
c. If ϕ is Π_1^I then if $\models_V \phi(X_1, \ldots, X_k)$. then $\models_{V_\kappa} \phi(X_1, \ldots, X_k)$
The same claims hold if V and V_κ are replaced by V_κ and V_λ where $\lambda \in \mathrm{Inac}_\kappa$.

Proof. For a term t, using the translation described in section 4 and lemma 28, $\models_{V_\kappa} Y = t$ iff $\models_V Y = t$. The lemma for atomic formulas follows by lemma 27. The induction step for the propositional connectives is routine. If $\models_{V_\kappa} \exists Y < t\psi(Y, X_1, \ldots, X_k)$ then clearly (using the induction hypothesis) $\models_V \exists Y < t\psi(Y, X_1, \ldots, X_k)$. If $\models_V \exists Y < t\psi(Y, X_1, \ldots, X_k)$ then for some $Z \subseteq V_\kappa$ and some $Y < Z$, $\psi(Y, X_1, \ldots, X_k)$. Clearly there is a $Y' \subseteq V_\kappa$ such that $Y' \equiv Y$. By lemma 23, $\psi(Y', X_1, \ldots, X_k)$; it follows that $\models_{V_\kappa} \exists Y < t\psi(Y, X_1, \ldots, X_k)$. This proves part a. Part b follows by a variant of the argument for the bounded existential quantifier. Part c follows from part b by contraposition. The last claim follows by essentially the same argument. $\Box$

Part a can be strengthened. Say that a predicate is uniformly Δ^I_1 if it is defined by a Σ^I_1 formula ϕ, and there is a Π^I_1 formula ψ with the same free variables, such that $\models_{V_\kappa} \phi \Leftrightarrow \psi$ for any $\kappa \in$ Inac.

Theorem 30. *Suppose ϕ is a formula with free variables for the values $X_1, \ldots, X_k$, which is uniformly Δ^I_1. Suppose $\kappa \in$ Inac, $\lambda \in$ Inac$_\kappa$, and $X_i \subseteq V_\lambda$ for $1 \le i \le k$. Then $\models_{V_\lambda} \phi(X_1, \ldots, X_k)$ iff $\models_{V\kappa} \phi(X_1, \ldots, X_k)$.*

Proof. If $\models_{V_\lambda} \phi$ then $\models_{V_\kappa} \phi$ by theorem 29. If $\models_{V_\kappa} \phi$ then $\models_{V_\kappa} \psi$ so $\models_{V_\lambda} \psi$ by theorem 29, so $\models_{V_\lambda} \phi$. $\Box$

16. $\mathcal{U}_{\mathrm{OS}}$ Orders

Say that $\langle \phi, \psi \rangle$ is a $\mathcal{U}_{sOS}$ formula pair at $\kappa \in$ Inac if the following hold:

1. $\phi(A, B, \vec{P})$ is Σ^1_1, $\psi(A, B, \vec{P})$ is Π^1_1, and A, B are restricted to satisfy I_{Ord}.
2. In V_κ and V_λ for $\lambda \in$ Inac$_\kappa$, ϕ and ψ define the same predicate.
3. For each $\vec{P}$ the predicate on A, B defined in V_κ and each V_λ is a WPS.

Letting $\preceq$ denote the order, by lemma 23, $\mathrm{Ot}(\preceq) < \kappa^{++}$.

Theorem 31. *Suppose requirements 1 and 2 above hold, and $\preceq$ is a WPS in V_κ. Then $\preceq$ is a $\mathcal{U}_{sOS}$ order.*

Proof. Axioms T1-T4 are Π^I_1, and the claim for these follows using theorem 30. For axiom F, suppose $X_i \subseteq V_\lambda$ for $i \in \omega$, and in V_λ, $X_{i+1} \prec X_i$ for all i. Then this is true in V_κ, again by theorem 30. $\Box$

Theorem 32. *A $\mathcal{U}_M$ machine may be transformed to an equivalent $\mathcal{U}_{OS}$ formula pair.*

Proof. Recall the definition of I_{Plus} from section 12. Given an input $\langle c, k, A_1, \dots, A_k\rangle$, let $A_0 = \{\langle d, d\rangle\}$ where d is the integer pairing function applied to c, k; the input can be coded as $A_0 + A_1 + \cdots + A_k$. It is readily seen that the predicate "w is the computation when the inputs are I_1 and I_2" is Δ_0^I. The theorem follows. $\square$

It is worth mentioning that the preceding theorem does not hold in V_ω; $\aleph_1$ is a $\mathcal{U}_M$ order, but is not Σ_1^1.

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