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POISSON APPROXIMATION FOR INDEPENDENT BINOMIAL RANDOM VARIABLES

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Abstract: This paper gives a bound for the total variation distance between the distribution of a sum of independent binomial random variables and an appropriate Poisson distribution with mean $\sum_{i=1}^{n} m_i p_i$, where m_i and $p_i = 1 - q_i$ are parameters of each binomial distribution. With this bound, it is indicated that the distribution of the sum can be approximated by the Poisson distribution when each $m_i p_i$ is small.

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1. Introduction

Let $X_1, ..., X_n$ be n independently distributed binomial random variables, each with probability $P(X_i = k) = \binom{m_i}{k} p_i^k q_i^{m_i - k}$, $k \in \{0, 1, ..., m_i\}$, mean $\mu_i = m_i p_i$ and variance $\sigma_i^2 = m_i p_i q_i$ where $q_i = 1 - p_i$. Let $\mathbf{S}_n = \sum_{i=1}^{n} X_i$ and $\mathbf{P}_\lambda$ denote

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the the Poisson random variable with mean λ. Let $\lambda = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mu_i$, if all λ_i are small, then the distribution of $\mathbf{S}_n$ converges to the Poisson distribution with mean λ. Therefore, the distribution of $\mathbf{S}_n$ can be approximated by the Poisson distribution with mean λ. In this paper, we are interest to give a bound on the total variation distance between the distributions of $\mathbf{S}_n$ and $\mathbf{P}_\lambda$

$$d_{TV}(\mathbf{S}_n, \mathbf{P}_\lambda) = \sup_{A \subseteq \mathbb{N} \cup \{0\}} |P(\mathbf{S}_n \in A) - P(\mathbf{P}_\lambda \in A)| \tag{1.1}$$

for the Poisson mean $\lambda = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} m_i p_i$, which is derived in Section 2. In Section 3, the conclusion of this study is also presented.

2. Result

Before giving the main result, it should be defined the w-function associated with each negative binomial random variable X_i as follows.

Lemma 2.1. *For $1 \le i \le n$, let w_i be the w-function associated with the negative binomial random variable X_i, then we have the following:*

$$w_i(k) = \frac{(m_i - k)p_i}{\sigma_i^2}, \ \ k \in \mathbb{N} \cup \{0\} \quad [1]. \tag{2.1}$$

Theorem 2.1. *Let $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$. Then the following inequality holds:*

$$d_{TV}(\mathbf{S}_n, \mathbf{P}_\lambda) \le \frac{1 - e^{-\lambda}}{\lambda} \sum_{i=1}^{n} m_i p_i^2. \tag{2.2}$$

Proof. Since $\lambda_i - \sigma_i^2 w_i(k) = m_i p_i - (m_i - k)p_i = kp_i \ge 0$ for every $k \ge 0$, it follows from [2] that

$$\begin{aligned} d_{TV}(\mathbf{S}_n, \mathbf{P}_\lambda) &\le \frac{1 - e^{-\lambda}}{\lambda} \left| \lambda - \sigma^2 \right| \\ &= \frac{1 - e^{-\lambda}}{\lambda} (\lambda - \sigma^2) \\ &= \frac{1 - e^{-\lambda}}{\lambda} \sum_{i=1}^{n} (m_i p_i - m_i p_i q_i) \\ &= \frac{1 - e^{-\lambda}}{\lambda} \sum_{i=1}^{n} m_i p_i^2. \end{aligned}$$

Hence (2.2) holds. □

Corollary 2.1. *For* $m_1 = m_2 = \cdots = m_n = 1$*, then* $\lambda = \sum_{i=1}^{n} p_i$ *and*

$$d_{TV}(\mathbf{S}_n, \mathbf{P}_\lambda) \leq \frac{1 - e^{-\lambda}}{\lambda} \sum_{i=1}^{n} p_i^2. \tag{2.3}$$

The result (2.3) is a Poisson approximation for a sum of independent Bernoulli random variables, which is the same result as in [2].

When all X_i are identically distributed random variables, thus immediately from the Theorem 2.1, we have the following Corollary.

Corollary 2.2. *If* $m_1 = m_2 = \cdots = m_n = m$ *and* $p_1 = p_2 = \cdots = p_n = p$*, then* $\lambda = nmp$ *and the following inequality holds:*

$$d_{TV}(\mathbf{S}_n, \mathbf{P}_\lambda) \leq (1 - e^{-\lambda})p. \tag{2.4}$$

3. Conclusion

In this study, a bound on the total variation distance between the distribution of a sum of independent binomial random variables and an appropriate Poisson distribution was obtained. With this bound, it can be seen that the distribution of the summands can be approximated by the Poisson distribution with mean $\lambda = \sum_{i=1}^{n} m_i p_i$ when $m_i p_i$ is small for every $i \in \{1, ..., n\}$.

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