GENERAL UNIONS OF SUNDIALS (OR LINES) IMPROVE THE HILBERT FUNCTIONS OF PROJECTIVE SCHEMES

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Abstract: A sundial $T \subset \mathbb{P}^r$ is a certain flat limit of two disjoint lines with $T_{\text{red}}$ a reducible conic. Let $A \subset \mathbb{P}^r$, $r$ large, the double of a linear space. We prove that a general union of $A$ and lines or sundials has the expected postulation. Instead of $A$ we may take other low dimensional multiple structures if certain numerical conditions on their Hilbert function are satisfied.

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1. Introduction

For any reduced closed subscheme $W$ of a reduced scheme $X$ and any integer $m > 0$ let $(mW, X)$ denote the $(m - 1)$-th infinitesimal neighborhood of $W$ in $X$, i.e. the closed subscheme of $X$ with $(\mathcal{I}_{W,X})^m$ as its ideal sheaf. We often write $mW$ if $X = \mathbb{P}^r$.

For any integer $x \in \{0, \ldots, r - 1\}$ let $V_{x,r} \subset \mathbb{P}^r$ be an $x$-dimensional linear subspace. See Lemma 1 for the well-known description of the Hilbert polynomial of $mV_{x,r}$ and Lemma 4 for its Hilbert function. A sundial $A \subset \mathbb{P}^r$ is a closed subscheme such that $A_{\text{red}}$ is a reducible conic, $T$, and there is a 3-
dimensional linear linear space $V \subseteq \mathbb{P}^r$ such that $A = A \cup (2P, V)$ ([2]). $A$ is a flat limit of a family of pairs of disjoint lines ([4]). A closed subscheme $X \subset \mathbb{P}^r$ is said to have maximal rank if for every integer $t > 0$ either $h^1(I_X(t)) = 0$ or $h^0(I_X(t)) = 0$. A general union of lines has maximal rank ([4]). A general union of sundials and lines has maximal rank ([2]). Adding a general union of many disjoint lines in $\mathbb{P}^r$, $r \geq 4$, to a zero-dimensional scheme we get a scheme with maximal rank ([1]). In a few cases (e.g., for $mV_{r,0}$, $r \geq 4$ and $2V_{0,3}$ ([3]), [1], Proposition 3, for a cohomologically similar case), this is true for an arbitrary number of general disjoint lines. In the case $r \geq 4$ it is easy to extend it to general unions of sundials and lines (see Proposition 1). We prove the following results.

**Theorem 1.** Fix integers $r \geq 5$, $t \geq 0$ and $e \geq 0$. Let $X \subset \mathbb{P}^r$ be a general union of $2V_{1,r}$, $t$ sundials and $e$ lines. Then $X$ has maximal rank.

**Theorem 2.** Fix integers $x \geq 2$, $r \geq 2x + 4$, $t \geq 0$ and $e \geq 0$. Let $X \subset \mathbb{P}^r$ be a general union of $2V_{x,r}$, $t$ sundials and $e$ lines. Then $X$ has maximal rank.

We prove these results checking the numerical inequality coming in the following statement.

**Theorem 3.** Fix an integers $r \geq 4$, $m > 0$ and a closed subscheme $Z \subset \mathbb{P}^r$ such that $\dim(Z) \leq r - 2$, $h^0(I_Z(m - 1)) = 0$ and $h^i(I_Z(t)) = 0$ for all $t \geq m - i$ and all $i > 0$. Let $H \subset \mathbb{P}^r$ be a general hyperplane section. Assume $h^0(H, I_Z \cap H(m - 1)) = 0$ and $h^i(H, I_Z \cap H(t)) = 0$ for all $i > 0$ and $t \geq m - 1$. Set $f(t) := h^0(O_Z(t))$ and $f_Z = h^0(O_Z \cap H(t))$. Assume that for all $a' \geq 0$ and $b' \geq 0$ a general union $Y \subset H$ of $Z \cap H$, $a'$ sundials and $b'$ lines have maximal rank. Assume

(a) $\left(\binom{r+k-1}{r-1}\right)k(r - 1)/r - kf_Z \cap H(k) + f_Z(k - 1) > k^2(k + 1)$ for all $k > m$

(b) $\left(\binom{r+k}{r}\right) \geq (2k + 1)(k + 1) + f_Z(k)$ for all $k \geq m$.

Then for all integers $\alpha \geq 0$ and $\beta \geq 0$ a general union $X \subset \mathbb{P}^r$ of $Z$, $\alpha$ sundials and $\beta$ lines have maximal rank.

We think that the inequalities in (a) and (b) hold for $Z = mV_{x,r}$ for arbitrary $m$ and $x$ if $r \gg m$ and $r \gg x$. To apply Theorem 3 need weaker numerical assumptions than a corresponding statements we could prove using using only lines and a few reducible conics as in [4] instead of using sundials.
2. The schemes $mV_{x,r}$

For any line bundle $L$ and any integer $a > 0$ we often write $aL$ as a short-hand of $L^a$.

**Lemma 1.** Fix integers $r > x \geq 0$ and $m > 0$. Then

$$O_{mV_{x,r}} \cong \oplus_{i=0}^{m-1} \binom{r-x+i-1}{i} O_{V_{x,r}}(-i)$$

as $O_{V_{x,r}}$-sheaves.

**Proof.** We use induction on $m$, the case $m = 1$ being obvious. Assume $m \geq 2$ and that the lemma is true (for all $x, r$) for all multiplicities $m' < m$. Set $I := I_{V_{x,r}}$. The $O_{V_{x,r}}$-sheaf $I/I^2$ is isomorphic to the conormal bundle of $V_{x,r}$ in $\mathbb{P}^r$. Hence $I/I^2 \cong (r-x)O_{V_{x,r}}(-1)$. Look at the following exact sequence of $O_{V_{x,r}}$ sheaves:

$$0 \to I^m \to I^{m-1} \to I^{m-1}/I^m \to 0 \quad (1)$$

Since $I^{m-1}/I^m \cong S^m(I/I^2) \cong \binom{r-x+m-1}{m} O_{V_{x,r}}(-m)$ as $O_{V_{x,r}}$-sheaves, (1) and the inductive assumption gives that $O_{mV_{x,r}}$ is isomorphic as an $O_{V_{x,r}}$-sheaf to an extension of the vector bundle $\binom{r-x+m-1}{m} O_{V_{x,r}}(-m)$ by the vector bundle

$$\oplus_{i=0}^{m-2} \binom{r-x+i-1}{i} O_{V_{x,r}}(-i).$$

If $x \neq 1$, then this extension splits. Now assume $x = 1$. We apply the lemma to $mV_{2,r+1}$ and then take the intersection of $mV_{2,r+1}$ with a general hyperplane of $\mathbb{P}^{r+1}$.

**Lemma 2.** We have $h^1(I_{mV_{x,r}}(m)) = 0$.

**Proof.** The lemma is obvious if $m = 1$. Hence we may assume $m \geq 2$ and use induction on $m$. The lemma is true if $x = r - 1$. Hence we may assume $r - x \geq 2$ and that the lemma is true for all integers $r', x', m'$ with $r' - x' < r - x$. Fix a hyperplane $M \subset \mathbb{P}^r$ such that $M \supseteq V_{x,r}$. Since $\text{Res}_M(mV_{x,r}) = (m-1)V_{x,r}$ and $M \cap mV_{x,r} \cong mV_{x,r-1}$, we have an exact sequence

$$0 \to I_{(m-1)V_{x,r}}(m-1) \to I_{mV_{x,r}}(m) \to I_{mV_{x,r-1,M}}(m) \to 0 \quad (2)$$

Use (2) and the inductive assumption on the integers $m$ and $x - r$.

**Lemma 3.** Fix integers $r > x \geq 0$, $m > 0$. Then:
(a) $h^0(\mathcal{I}_{mV_x,r}(t)) = 0$ if and only if $t \leq m - 1$.

(b) $h^1(\mathcal{I}_{mV_x,r}(t)) = 0$ for all $t \geq m - 1$.

(c) $h^i(\mathcal{I}_{mV_x,r}(t)) = 0$ for all $i \geq 2$ and all $t \geq m$.

**Proof.** Part (a) is obvious. The case $x = 0$ of the lemma is trivial. Assume $x > 0$ and that part (b) is true for the integer $x - 1$ in an $(r - 1)$-dimensional projective space. Let $H \subset \mathbb{P}^r$ be a general hyperplane. We have $H \cap mV_{x,r} = (m(H \cap V_{x,r}), H)$ (as schemes). Look at the exact sequence

$$0 \to \mathcal{I}_{mV_x,r}(t-1) \to \mathcal{I}_{mV_x,r}(t) \to \mathcal{I}_{(mV_{x,r} \cap H),H}(t) \to 0 \quad (3)$$

Since $h^0(H, \mathcal{I}_{(mV_{x,r} \cap H),H})(m-1) = 0$ and $h^1(H, \mathcal{I}_{(mV_{x,r} \cap H),H})(t) = 0$ for all $t \geq m - 1$ by the inductive assumption, for all $t \geq m$ the maps $H^1(\mathcal{I}_{mV_x,r}(t-1)) \to H^1(\mathcal{I}_{mV_x,r}(t))$ are surjective. Lemma 2 gives $h^1(\mathcal{I}_{mV_x,r}(t)) = 0$. Hence $h^1(\mathcal{I}_{mV_x,r}(t)) = 0$ for all $t \geq m$. Since every element of $|\mathcal{I}_{mV_x,r}(m)|$ is a cone with vertex containing $V_{x,r}$, the restriction map

$$H^0(\mathcal{I}_{mV_x,r}(m)) \to H^0(\mathcal{O}_{mV_x,r} \cap H,H)(m)$$

is surjective. Hence $h^1(\mathcal{I}_{mV_x,r}(m-1)) \leq h^1(\mathcal{I}_{mV_x,r}(m)) = 0$. For all $i \geq 2$ we have $h^i(\mathcal{I}_{mV_x,r}(t)) = h^{i-1}(\mathcal{O}_{mV_x,r}(t))$ and the latter integer is zero if either $i \geq x + 2$ or $i \leq x$ or $i = x + 1$ and $t \geq m - x + 1$ by Lemma 1. \hfill $\Box$

**Remark 1.** For all integers $r > x > 0$, $t > m > 0$ we have $h^0(\mathcal{I}_{mV_x,r}(t)) = h^0(\mathcal{I}_{mV_x,r}(t-1)) + h^0(\mathcal{I}_{mV_{x-1,r-1}}(t))$ and $h^0(\mathcal{O}_{mV_x,r}(t)) = h^0(\mathcal{O}_{mV_x,r}(t-1)) + h^0(\mathcal{O}_{mV_{x-1,r-1}}(t))$ and $h^0(\mathcal{I}_{mV_x,r}(t)) = h^0(\mathcal{I}_{mV_x,r}(t-1)) + h^0(\mathcal{I}_{mV_{x-1,r-1}}(t))$ (use (3)). From Lemma 1 we get

$$h^0(\mathcal{O}_{mV_x,r}(t)) = \sum_{i=0}^{m-1} \binom{r-x+i-1}{i} \binom{x+t-i}{x}$$

or all $t \geq m$.

**Lemma 4.** Fix integers $r > x \geq 0$ and $m > 0$. We have $h^0(\mathcal{I}_{mV_x,r}(t)) = 0$ if $t \leq m - 1$ and $h^0(\mathcal{I}_{mV_x,r}(t)) = \binom{r+t}{r} - \sum_{i=0}^{m-1} \binom{r-x+i-1}{i} \binom{x+t-i}{x}$ if $t \geq m$.

**Proof.** Apply Remark 1 and parts (a) and (b) of Lemma 3. \hfill $\Box$
3. The proofs

For all integers \( r \geq x + 3 \geq 3 \) and \( k \geq m > 0 \) define the integers \( a_{r,x,m,k} \) and \( b_{r,x,m,k} \) by the relations

\[
(k + 1)a_{r,x,m,k} + b_{r,x,m,k} = \binom{r + k}{r} - h^0(\mathcal{O}_{mV_{x,r}}(k)), \quad 0 \leq b_{r,x,m,k} \leq k
\]  

(4)

For the values of the integers \( h^0(\mathcal{O}_{mV_{x,r}}(k)) \), see Remark 1. If \( x > 0 \) and \( k > m \), then Remark 1 and (1) for the integers \( k \) and \( k - 1 \) give the following equality

\[
a_{r,x,m,k-1} + (k + 1)(a_{r,x,m,k} - a_{r,x,m,k-1}) + b_{r,x,m,k} - b_{r,x,m,k-1} \\
= \binom{r + k - 1}{r - 1} - h^0(\mathcal{O}_{mV_{x-1,r-1}}(k))
\]  

(5)

The equality (5) is true also if \( x = 0 \), just setting \( h^0(\mathcal{O}_{mV_{x-1,r-1}}(k)) = 0 \) (i.e. taking \( V_{-1,n} = \emptyset \) for any \( n > 0 \)).

**Lemma 5.** Fix integers \( x \geq 0, r \geq 4 + x, m > 0, t \geq 0 \) and \( e \geq 0 \). Let \( W \subset \mathbb{P}^r \) be a general union of \( t \) sundials and \( e \) lines. Set \( X := V_{x,r} \cup W \). Then either \( h^0(\mathcal{I}_X(m)) = 0 \) or \( h^1(\mathcal{I}_X(m)) = 0 \).

**Proof.** Let \( \ell : \mathbb{P}^r \setminus V_{x,r} \to \mathbb{P}^{r-x-1} \) be the linear projection from \( V_{x,r} \). Since \( W \) is general and \( r \geq x + 4 \), we have \( V_{x,r} \cap W = \emptyset \) and \( \ell(W) \) is a general disjoint union of \( t \) sundials and \( e \) lines. Since \( r - x - 1 \geq 3 \), \( \ell(W) \) has maximal rank ([2]). Hence either \( h^0(\mathbb{P}^{r-x-1}, \mathcal{I}_\ell(W)(k)) = 0 \) or \( h^1(\mathbb{P}^{r-x-1}, \mathcal{I}_\ell(W)(k)) = 0 \). Since \( |\mathcal{I}_{mV_{x,r}}(m)| \) is the set of all degree \( m \) cones with vertex containing \( V_{x,r} \), we have \( h^0(\mathcal{I}_X(m)) = h^0(\mathbb{P}^{r-x-1}, \mathcal{I}_\ell(W)(k)) \). We also have \( h^1(\mathcal{I}_X(m)) = h^1(\mathbb{P}^{r-x-1}, \mathcal{I}_\ell(W)(k)) \) by part (b) of Lemma 3. \( \square \)

**Proposition 1.** Fix integers \( r \geq 4, m > 0, t \geq 0 \) and \( e \geq 0 \). Let \( X \subset \mathbb{P}^r \) be a general union of \( mV_{0,r} \), \( t \) sundials and \( e \) lines. Then \( X \) has maximal rank.

**Proof.** By Lemma 4 we may assume \( (t,e) \neq (0,0) \). Since a general union of a prescribed number of lines and sundials and one point has maximal rank ([2]), it is sufficient to do the case \( m \geq 2 \). Since a sundial is a flat limit of a family of disjoint lines ([4]), it is sufficient to do the case \( e = 0 \) and the case \( e = 1 \). Let \( k \) be the minimal positive integer such that \( \binom{r+m-1}{r} + (2t+e)(k+1) \leq \binom{r+k}{r} \) (this integer is called the critical value of the triple \( (r,m,2t+e) \)). Since \( 2t + e > 0 \) we have \( k \geq m \). For the case \( k = m \) see the case \( x = 0 \) of Lemma 5. Hence we may assume \( k > m \) and that Proposition 1 is true (for fixed integers \( r \) and...
m) for the pairs \((t',e')\) such that \((r,m,2t'+e')\) has critical value \(<k\). By the Castelnuovo-Mumford’s lemma it is sufficient to prove that \(h^1(I_X(k)) = 0\) and \(h^0(I_X(k-1)) = 0\). Fix \(P \in \mathbb{P}^r\). Since \(\text{Aut}(\mathbb{P}^r)\) is transitivity, without losing the generality of \(X\) we may assume that \(\{P\} = V_{0,r}\). Let \(H \subset \mathbb{P}^r\) be a hyperplane such that \(P \notin H\).

(a) In this step we prove that \(h^1(I_X(k)) = 0\). Increasing if necessary either \(t\) or \(e\) we may assume \((r+m-1) + (2t+e)(k+1) \geq (r^+k) - k\), i.e. we may assume that \(t = \lfloor a_{r,0,m,k}/2 \rfloor\) and \(e = a_{r,0,m,k} - 2t\). Set \(e' := a_{r,0,m,k-1} - 2\lfloor a_{r,0,m,k-1}/2 \rfloor \in \{0,1\}\). Let \(Y \subset \mathbb{P}^r\) be a general union of \([a_{r,0,m,k-1}/2]\) sundials and \(e'\) lines. Since \(a_{r,0,m,k-1} > a_{r,0,m,k-2}\) (Lemma 7), the inductive assumption gives that \(h^1(I_Y(k-1)) = 0\) and \(h^0(I_Y(k-2)) = 0\). Since \(h^1(I_Y(k-1)) = 0\), we have \(h^0(I_Y(k-1)) = b_{r,0,m,k-1}\). Let \(S \subset H\) be a general subset with \(\sharp(S) = b_{r,0,m,k-1}\). Since \(h^0(I_Y(k-2)) = 0\), \(\text{Res}_H(Y) = Y\) and \(h^0(I_Y(k-1)) = b_{r,0,m,k-1}\), we have \(h^0(I_Y \cup S(k-1)) = 0\). The case \(x = 0\) and \(k' = k-1\) of (5) gives \(h^1(I_Y \cup S(k-1)) = 0\).

(a1) First assume \(e' = e\). Let \(E \subset H\) be a general union of \(t - \lfloor a_{r,0,m,k-1}/2 \rfloor\) reducible conics and write \(E = E_1 \cup E_2\) with \(E_2\) a general union of \(b_{r,0,m,k-1}\) conics. Let \(G_1 \subset H\) be a general union of sundials with \((G_1)_{\text{red}} = E_1\). Let \(G_2 \subset \mathbb{P}^r\) be a general union of sundials with \((G_2)_{\text{red}} = E_2\). Set \(X' := Y \cup G_1 \cup G_2\). Since \(E_2\) is a general union of \(b_{r,0,m,k-1}\) reducible conics contained in \(H\), we may assume that \(S\) is the support of the nilpotent sheaf of \(G_2\). By the semicontinuity theorem for cohomology it is sufficient to prove that \(h^1(I_{X'}(k)) = 0\). We have \(X' \cap H = (Y \cap H) \cup G_1 \cup E_2\) and \(\text{Res}_H(Y') = Y \cup S\). Since \(h^1(I_{Y \cup S}(k-1)) = 0\), it is sufficient to prove that \(h^0(H, I_{X' \cap H}(k)) = 0\). We have \(h^0(O_{X' \cap H}(k)) = a_{r,0,m,k-1} + h^0(I_{G_1} \cup E_2(k)) = a_{r,0,m,k-1} + (k+1)(a_{r,0,m,k-1} - a_{r,0,m,k-1} - b_{r,0,m,k-1}) \leq (r^+k-1)\) by (5). Since \(Y \cap H\) is a general subset of \(H\) with cardinality \(\sharp(Y \cap H)\), it is sufficient to prove that \(h^1(H, I_{G_1} \cup E_2(k)) = 0\). Let \(F_2 \subset H\) be a general union of sundials with \((F_2)_{\text{red}} = E_2\). Since \(h^1(I_{\text{Res}_H(Y')} (k-1)) = 0\), a Castelnuovo’s sequence shows that it is sufficient to prove that \(h^1(H, I_{G_1} \cup E_2(k)) = 0\). \(G_1 \cup F_2\) is a general union of sundials. Hence by [2] it is sufficient to check that \(h^0(O_{G_1} \cup F_2(k)) \leq (r^+k-1)\). By (5) it is sufficient to use that \(\sharp(Y \cap H) = a_{r,0,m,k-1} \geq b_{r,0,m,k-1}\). This inequality is true by Lemma 6, because \(b_{r,0,m,k-1} \leq k-1\).

(a2) Now assume \(e' = 0\) and \(e = 1\). Instead of \(E = E_1 \cup E_2\) we take \(E = E_1 \cup E_2 \cup L\) with \(L\) a general line of \(H\). We work in \(H\) as in step (a1).

(a3) Now assume \(e' = 1\) and \(e = 0\). Write \(Y = Y' \cup R\) with \(Y' = mP \cup W\), \(W\) general union of \((a_{r,0,m,k-1} - 1)/2\) sundials and \(L\) a general line. Take a general \(E_1 \cup E_2 \cup L \subset H\) with \(E_1\) general union of \((a_{r,0,m,k-1} - 1)/2 - b_{r,0,m,k-1}\) reducible conics, \(E_2\) a general union of \(b_{r,0,m,k-1}\) reducible conics and \(L\) a general
line of \( H \) through the point \( R \cap H \). For a general line \( R \) the point \( R \cap H \) is a general point of \( H \). Hence \( E_1 \sqcup E_2 \sqcup L \) has the Hilbert function of a general union of \( t - (a_r,0,m,k-1 - 1)/2 \) reducible conics and one line. Let \( L' \subset H \) be a general +line of \( H \) with \( L \) as its support and with \( P \) as the support of its ideal sheaf, i.e. a general scheme \( L \cup \nu \subset H \) with \( \nu \subset H \) a general degree 2 scheme with \( \nu_{\text{red}} = \{ P \} \). Let \( G_1 \subset H \) be a general union of sundials with \((G_1)_{\text{red}} = E_1 \). Let \( G_2 \subset \mathbb{P}^r \) be a general union of sundials with \((G_2)_{\text{red}} = E_2 \). Set \( X' := Y \cup G_1 \cup G_2 \cup L' \). Since \( R \cup L' \) is a sundial of \( \mathbb{P}^r \) and \( \text{Res}_H(X') = Y \cup S \), it is sufficient to prove that \( h^1(Y, \mathcal{I}_{X'(\cap H)}(k)) = 0 \). Let \( F_2 \subset H \) be a general union of sundials with \((F_2)_{\text{red}} = E_2 \). Take a general line \( D \subset H \) through \( R \cap H \) and set \( T := L' \cup D \), i.e. take a general sundial of \( H \) with \( R \cap H \) as the support of the nilpotent sheaf and \( L \) as an irreducible component of \( T_{\text{red}} \). In this step we need \( a_{r,0,m,k-1} \geq b_{r,0,m,k-1} + k \) (see Lemma 6).

(b) In this step we prove that \( h^0(\mathcal{I}_X(k-1)) = 0 \). For the case \( k - 1 = m \) see Lemma 5. Hence we may assume \( k \geq m + 2 \). By the definition of the integer \( k \) we have \( 2t + e > a_{r,0,m,k-1} \). Since a line is contained in a sundial, it is sufficient to do the case \( 2t + e = a_{3,0,m,k-1} + 1 \). Let \( Y' \subset \mathbb{P}^r \) be a general union of \( \lfloor a_{r,0,m,k-2}/2 \rfloor \) sundials and \( e'' := a_{r,0,m,k-2}/2 \) lines. Since \( a_{r,0,m,k-1} > a_{r,0,m,k-2} \) (Lemma 7), the inductive assumption gives that \( h^1(Y',(k-2)) = 0 \) and \( h^0(\mathcal{I}_{Y'}(k-3)) = 0 \) (even if \( k = m + 2 \), because \( h^0(\mathcal{I}_{Y',0}(m-1)) = 0 \)). Since \( h^1(Y'(k-1)) = 0 \), we have \( h^0(\mathcal{I}_{Y'}(k-1)) = b_{r,0,m,k-1} \). Let \( S' \subset H \) be a general subset with \( \sharp(S) = b_{r,0,m,k-2} \). Since \( h^0(\mathcal{I}_{Y'}(k-3)) = 0 \), \( \text{Res}_H(Y') = Y' \) and \( h^0(\mathcal{I}_{Y'}(k-2)) = b_{r,0,m,k-2} \), we have \( h^0(\mathcal{I}_{Y' \cup S'}(k-2)) = 0 \). We make the construction in step (a) with minimal modifications. We need the same numerical checks.

Proofs of Theorems 1, 2, and 3. Look at the proof of Proposition 1. Take \( Z, H \) and \( Z \cap H \) as in Theorem 3. For all integers \( k \geq m \) define the integers \( u_k \) and \( v_k \) by the relations

\[
f_Z(k) + (k + 1)u_k + v_k = \binom{r + k}{r}, 0 \leq v_k \leq k \quad (6)
\]

From (6) for the integers \( k \) and \( k - 1 \) we get

\[
f_{Z \cap H}(k) + u_{k-1} + (k + 1)(u_k - u_{k-1}) + v_k - v_{k-1} = \binom{r + k - 1}{r - 1} \quad (7)
\]

Fix \((t,e) \in \mathbb{N}^2\) and suppose you want to prove that a general union \( X \) of \( Z \), \( t \) sundials and \( e \) lines are maximal rank. It is sufficient to do all cases with \( e \in \{0,1\} \). Let \( k \) be the minimal integer such that \( u_k \geq 2t + e \). To prove
that \( h^1(I_X(k)) = 0 \) it is sufficient to do it for the case \( 2t + e = u_k \). This is assumed to be true if \( k = m \). Hence we may assume \( k > m \) and use induction on the integer \( k \). Condition (a) of Theorem 3 implies \( u_k - u_{k-1} \geq k + 1 \) for all \( k > m \) (see the proof of Lemma 7). A weaker form of Condition (b) gives the inequality \( u_{k-1} \geq k - 1 \) used at the end of step (a1) (as well silently in (a2) and (b)). Condition (b) gives the inequality \( u_{k-1} \geq 2k - 1 \) used in step (a3). In the set-up of Theorems 1 and 2 we have \( m = 2 \). 

Multiplying it by \( k \) (mod 3) and \( a_r + 1 \) and \( m \) increasing function of \( b \).

Lemma 7. Fix integers \( r \geq 4 \) and \( k \geq m \geq 2 \). Then \( a_{r,0,m,k} - a_{r,0,m,k-1} \geq k + 1 \).

Proof. We first check the case \( k = m \). Notice that \( a_{r,0,m,m-1} = 0 \) and \( a_{r,0,m,m} = \lfloor (r + m - 1)/(m + 1) \rfloor \). Hence for fixed \( m \) the integer \( a_{r,0,m,m} \) is an increasing function of \( m \). We have \( a_{4,0,m,m} = (m + 3)(m + 2)/6 \) if \( m \equiv 0, 1 \) (mod 3) and \( a_{4,0,m,m} = (m^2 + 5m - 2)/6 \) if \( m \equiv 0 \) (mod 3). Hence \( a_{4,0,2,2} \geq m + 1 \) and \( a_{4,0,m,m} \geq m \) if and only if \( m \geq 6 \).

From now on we assume \( k > m \). Assume \( a_{r,0,m,k} - a_{r,0,m,k-1} \leq 2k \). From (5) we get

\[
a_{r,0,m,k-1} + (k + 1)k + b_{r,0,m,k} - b_{r,0,m,k-1} \geq \binom{r + k - 1}{r - 1}
\]

Multiplying it by \( k \), using (5), that \( k\binom{r + k - 1}{r - 1} - \binom{r + k - 1}{r - 1} = \binom{r + k - 1}{r - 1}k(r - 1)/r \), that \( b_{r,0,m,k} \leq k \) and that \( b_{r,0,m,k-1} \geq 0 \) we get

\[
k^2(k + 1) + k^2 \geq \binom{r + k - 1}{r - 1}k(r - 1)/r + \binom{r + m - 1}{r}
\]

Call \( e(r, k, m) \) the difference between the right hand side and the left hand side of (8) and write \( f(r, k) := e(r, k, m) - \binom{r + m - 1}{r} \). To prove the lemma for the
triple \((r, m, k)\) it is sufficient to prove that \(e(r, k, m) > 0\). Hence it is sufficient to prove that \(f(r, k) > 0\). For a fixed integer \(k\) the function \(f(r, k)\) is an increasing function of \(r\). We have \(f(5, k)/k = (k + 4)(k + 3)(k + 2)(k + 1)/32 - k^2 - 2k > 0\) for all \(k > 2\). We have \(f(4, k)/k = (k + 3)(k + 2)(k + 1)/8 - k^2 - 2k > 0\) for all \(k \geq 5\). Now assume \(k \leq 4\). We need to check the cases \((r, m, k) \in \{(4, 2, 3), (4, 2, 4), (4, 3, 4)\}\). We have \(a_{4,0,2,2} = 3, a_{4,0,2,3} = 7, a_{4,0,2,4} = 13, a_{4,0,3,3} = 5, a_{4,0,3,4} = 11\). In all cases we have \(a_{r,0,m,k} \geq a_{r,0,m,k-1} + k + 1\). \(\Box\)

**Lemma 8.** For all integers \(r \geq x + 4 \geq 5\) and \(t \geq m \geq 2\) we have \(a_{r,x,m,t} \geq 2t + 1\).

**Proof.** It is easy to check that \(a_{r,x,m,k} \geq a_{r-x,0,m,k}\). Apply Lemma 6. \(\Box\)

**Lemma 9.** Fix integers \(r \geq 2x + 4 \geq 8\) and \(k \geq 3\). Then \(k^2(k + 2) < (r + k - 1)(r - 1)/r - k(x + k - 1)/x - k(r - x)(x + k - 2)/x - (x + k - 2)(x - 1)(x - 2)/(x - 1)\).

**Proof.** Since the right hand side of the inequality in the lemma is \(k\) times \((r + k - 1)(r - 1)/r + (x + k - 1)(x - 1)/x - (r - x)(x + k - 2)(x - 1)(x - 2)/(x - 1)\), it is sufficient to prove the inequality

\[
k(k + 2) < \left(\frac{r + k - 1}{r - 1}\right)(r - 1)/r + \left(\frac{x + k - 1}{x - 1}\right)(x - 1)/x - (r - x)\left(\frac{x + k - 2}{x - 1}\right)(x - 2)/(x - 1)
\]

This inequality is obvious if \(r \geq 2x + 4\). \(\Box\)

**Lemma 10.** For all integers \(r \geq 5\) and \(k \geq 3\) we have \(k(k + 2) < (r + k - 1)(r - 1)/r + r(k - 1) + 1 - rk\).

**Proof.** Set \(u(r, k) := (r + k - 1)(r - 1)/r - r + 1 - k(k + 2)\). It is sufficient to prove that \(u(r, k) > 0\). We have \(u(5, k) = (k + 4)(k + 3)(k + 2)(k + 1)/32 - 4 - k(k + 2) > 0\) for all \(k \geq 3\). Then we use that \(u(r + 1, k) \geq u(r, k)\) for all \(k \geq 4\) and that \(u(r, 3) = (r + 2)(r + 1)(r - 1)/6 - r - 15 > 0\) for all \(r \geq 5\). \(\Box\)

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References

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