ON THE STRATIFICATION BY X-RANKS OF A PROJECTIVE SPACE (MAXIMAL AND SUBMAXIMAL X-RANKS)

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Abstract: Let $X \subset \mathbb{P}^N$ be an integral and non-degenerate variety. For each $P \in \mathbb{P}^N$ the X-rank $r_X(P)$ of P is the minimal cardinality of a subset of X whose linear span contains P . For each $x > 0$ set $A_X(x) := \{P \in \mathbb{P}^N : r_X(P) = x\}$. We prove that if $r_{\max}$ is the maximum of all $r_X(P)$ and $a := \dim(X) \leq N - 2$, then $\dim(A_X(r_{\max} - 1)) \geq \max\{1 + a, \dim(A_X(r_{\max})) + 2\}$.

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1. Introduction

Let $X \subset \mathbb{P}^N$ be an integral and non-degenerate variety. For each $P \in \mathbb{P}^N$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \rangle$ denote the linear span. The open X-rank $w_X(P)$ of P is the minimal integer t with the following property. Fix a closed proper subset $B \subsetneq X$. Then there is a set $S \subset X \setminus B$ such that $P \in \langle S \rangle$ and $\sharp(S) = t$ ([4]). The open X-rank $w_X(P)$ measures the following game. You fix P . Then your opponent fix $B \subsetneq X$. Then you need to describe P only using points of $X \setminus B$. Call $r_{\max}$ (resp. $w_{\max}$) the maximum of the integers $r_X(P)$ (resp. $w_X(P)$), $P \in \mathbb{P}^N$. In many cases the X-rank is important for real life applications (e.g. if X is a Veronese variety for the decomposition of a homogeneous polynomial

into a sum of power of linear forms and if X is a Segre embedding for the tensor decomposition of a tensor into rank one tensors) ([3], [5]). Hence it is useful to have upper bounds for the integer $r_{\max}$. For every P we have $w_X(P) \geq r_X(P)$. There is a non-empty open subset U of $\mathbb{P}^N$ such that $r_X(P) = r_X(Q)$ for all $P, Q \in U$. The integer $r_X(P)$, $P \in U$, is called the *generic X-rank* r_{gen} . Using that the image of a constructible subset by an algebraic map is constructible (Chevalley's theorem), one can also show the existence of a non-empty open subset W of $\mathbb{P}^N$ such that $w_X(P) = w_X(Q)$ for all $P, Q \in W$. We call this integer $w_{\max} := w_X(P)$, $P \in W$, the *generic open X-rank*. It is well-know that the integer $r_{\max}$ is computed in the following way. For each integer $b \geq 1$ let $\sigma_b(X)$ denote the closure of the union of all linear spaces $\langle S \rangle$ for all $S \subset X$ with $\sharp(S) = b$. The variety $\sigma_b(X)$ is called the b -secant variety of X . The integer r_{gen} is the minimal integer b such that $\sigma_b(X) = \mathbb{P}^N$. In many cases (e.g. if X is the Veronese variety) this observation gives an explicit solution for the integer r_{gen} . Recently G. Blekherherman and Z. Teitler proved that $r_{\max} \leq 2r_{\text{gen}}$ ([2, Theorem 1]). In some cases they also give a better bound under stronger assumptions and studied the corresponding concept over the field $\mathbb{R}$ ([2, Theorem 3]). We observe that the proof of [2, Theorem 1] works also for the open X-rank (see Proposition 1). For each integer $x > 0$ set $A_X(x) := \{P \in \mathbb{P}^N : r_X(P) = x\}$, $A_X(\geq x) := \{P \in \mathbb{P}^N : r_X(P) \geq x\}$ and $A_X(\leq x) := \{P \in \mathbb{P}^N : r_X(P) \leq x\}$. By Chevalley's theorem the sets $A_X(x)$, $A_X(\leq x)$ and $A_X(\geq x)$ are locally constructible. For any two irreducible subsets A, B of $\mathbb{P}^N$ the join $J(A, B) \subseteq \mathbb{P}^N$ is the closure in $\mathbb{P}^N$ of the union of all lines $\langle \{O, P\} \rangle$ with $O \in A$, $P \in B$ and $O \neq P$, with the convention $J(A, B) := A$ if A and B are just points and $A = B$. For any subset $E \subseteq \mathbb{P}^N$ let $\overline{E}$ denote the closure of E in $\mathbb{P}^N$.

In this note we prove the following result.

Theorem 1. *Fix an integer $x \geq 2$ and fix any irreducible algebraic set $T \subseteq A_X(x)$ such that $\overline{T}$ is an irreducible component of $\overline{A_X(x)}$. Then there is an irreducible subset $E \subset A_X(x-1) \cup A_X(\geq x+1)$ such $J(X, \overline{T}) \subseteq \overline{E}$. We have $\dim(E) \geq \max\{\dim(T) + 2, \dim(X) + 1\}$.*

Theorem 1 is more interesting in the range $x \geq 2 + w_{\text{gen}}$, because $J(X, \sigma_b(X)) = \sigma_{b+1}(X)$ for all $b > 0$. Notice that $A_X(\geq r_{\max} + 1) = \emptyset$. Hence Theorem 1 gives the following corollary.

Corollary 1. *Assume $X \neq \mathbb{P}^N$. Then $\dim(A_X(r_{\max} - 1)) \geq 1 + \dim(X)$ and $\dim(A_X(r_{\max} - 1)) \geq \min\{N, \dim(A_X(r_{\max})) + 2\}$.*

In particular both the integers $r_{\max}$ and $r_{\max} - 1$ are the X-ranks of some points of $\mathbb{P}^N$.

We work over an algebraically closed field $\mathbb{K}$.

2. Proofs and remarks

Proposition 1. *We have $w_{\max} \leq 2w_{\max}$.*

Proof. Let $W \subset \mathbb{P}^N$ be any nonempty open subset such that $w_X(Q) = w_{\text{gen}}$ for all $Q \in W$. Fix $P \in \mathbb{P}^N$ and a closed subset $B \subsetneq X$. Let $L \subset \mathbb{P}^N$ be a line through P and intersecting W . Since $W \cap L$ is non-empty and open in L , there are $P_1, P_2 \in W \cap L$ with $P_1 \neq P_2$. By the definition of the integer $w_X(P_i)$, $i = 1, 2$, there is $S_i \subset X \setminus B$ such that $P_i \in \langle S_i \rangle$ and $\sharp(S_i) \leq w_{\text{gen}}$. We have $P \in \langle S_1 \cup S_2 \rangle$, $S_1 \cup S_2 \subset X \setminus B$ and $\sharp(S_1 \cup S_2) \leq 2w_{\text{gen}}$. Hence $w_X(Q) \leq 2w_{\text{gen}}$. $\square$

We may improve Proposition 1 in the following way ([2, Theorem 3]).

Proposition 2. *Fix an integer $\rho < w_{\text{gen}}$. Assume the existence of a closed irreducible hypersurface $T \subset \mathbb{P}^N$ and a non-empty open subset $U \subset T$ such that $w_X(P) \leq \rho$ for all $P \in U$. Then $w_X(Q) \leq w_{\text{gen}} + \rho$ for all $Q \in \mathbb{P}^N \setminus T$.*

Proof. Fix $Q \in \mathbb{P}^N \setminus T$ and a closed subset $B \subsetneq X$. Let $W \subset \mathbb{P}^N$ a non-empty open subset such that $w_X(P) = w_{\text{gen}}$ for all $P \in W$. Let L be a general line through Q . Since T is a hypersurface, we have $L \cap T \neq \emptyset$. Since U is open and dense in T , for a general L we have $L \cap U \neq \emptyset$. Since $T \subseteq \mathbb{P}^N \setminus W$, we have $P_1 \neq P_2$. Take $S_i \subset X \setminus B$, $i = 1, 2$, such that $P_i \in \langle S_i \rangle$, $\sharp(S_1) \leq w_{\text{gen}}$ and $\sharp(S_2) \leq \rho$. Since $\sharp(S_1 \cup S_2) \leq \rho + w_{\text{gen}}$, we have $w_X(Q) \leq w_{\text{gen}} + \rho$. $\square$

Proposition 3. *Fix an integer $\rho < w_{\text{gen}}$. Assume the existence of a closed irreducible hypersurface $T \subset \mathbb{P}^N$ and a non-empty open subset $U \subset T$ such that $w_X(P) \leq \rho$ for all $P \in U$ and $d := \deg(T) \geq 2$. Fix $Q \in \mathbb{P}^N \subset T$. If $\text{char}(\mathbb{K}) > 0$ assume that T is not a strange variety with Q as one of its strange points. Then $w_X(Q) \leq 2\rho$.*

Proof. Let $\ell : \mathbb{P}^N \setminus \{Q\} \rightarrow \mathbb{P}^{N-1}$ denote the linear projection. Since $Q \notin T$, $\ell|_T$ is a morphism. The morphism $\ell|_T$ is a finite morphism of degree d . Since Q is not a strange point of T , $\ell|_T$ is separable. Hence for a general line L through P the scheme $T \cap L$ is formed by d distinct points, all of them contained in U . Taking $P_1, P_2 \in L \cap U$ with $P_1 \neq P_2$ we get $w_X(Q) \leq 2\rho$. $\square$

Remark 1. Assume $p := \text{char}(\mathbb{K}) > 0$. Let $T \subset \mathbb{P}^N$ be an integral hypersurface of degree $d \geq 2$. Fix $Q \in \mathbb{P}^N \setminus T$. If T is strange, then $p|d$. The proof of Proposition 3 gives $w_X(Q) \leq 2\rho$ if $\ell|_T$ is not purely inseparable. If $\ell|_T$ is purely inseparable, then d is a p -power.

Proof of Theorem 1: Fix a general $Q \in T$. Since $x \geq 2$, we have $Q \notin X$. Fix a general $O \in X$ and fix any $P \in \langle \{O, Q\} \rangle \setminus \{O\}$. Since $Q \in \langle \{P, O\} \rangle$ (even in the case $P = Q$), we have $r_X(Q) \leq 1 + r_X(P)$. Hence a general element of the irreducible variety $J(X, T)$ has X -rank at least $x - 1$. Since X is non-degenerate and $\overline{T} \neq \mathbb{P}^N$, a general $O \in X$ is not a vertex of $\overline{T}$. Hence $\dim(J(\{O\}, \overline{T})) = \dim(T) + 1$ ([1, part (i) of Proposition 1.3]). If $b := \dim(T) = N - 1$, then we are done. Now assume $b \leq N - 2$. Since X is non-degenerate, the variety $J(\{O\}, \overline{T})$ is not a cone with vertex containing X . Hence $\dim(J(X, \overline{T})) \geq b + 2$ ([1, part (ii) of Proposition 1.3]). Now we reverse the role of T and X . Fix a general $Q \in T$. Since $r_X(Q) = x > 1$, we have $Q \notin X$. Hence $\dim(J(T, X)) \geq \dim(J(\{Q\}, X)) = \dim(X) + 1$. $\square$

Assume $a := \dim(X) \leq N - 2$ and $b := \dim(T) \geq 2$. To get $\dim(J(X, \overline{T})) \geq \dim(X) + 2$ it would be sufficient to assume that for a general $O \in T$ the cone $J(\{O\}, X)$ has not a vertex containing T . We know no natural condition which assures it.

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