ON THE TYPICAL RANK OF REAL CLOSED FIELDS

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Abstract: In this note we introduce the notion of typical rank for any real closed field R , mimicking the case $R = \mathbb{R}$. We show that the typical ranks are the same if we take a larger real closed field and in particular that for every format $(n_1, \dots, n_s)$ the typical ranks of tensors of format $(n_1, \dots, n_s)$ are the same over R and over $\mathbb{R}$.

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Let R be a real closed field. In this note we introduce the notion of typical rank for R , mimicking the case $R = \mathbb{R}$. We show that the typical ranks are the same if we take a larger real closed field and in particular that for every format $(n_1, \dots, n_s)$ the typical ranks of tensors of format $(n_1, \dots, n_s)$ are the same over R and over $\mathbb{R}$.

For any field K let $\overline{K}$ denote its algebraic closure. For any field K , any non-degenerate and geometrically integral variety $X \subset \mathbb{P}^r$ defined over K and any $P \in \mathbb{P}^r(K)$ let $r_{X,K}(P)$ (the K - X -rank of P) be the minimal cardinality of a set $S \subseteq X(K)$ such that $P \in \langle S \rangle_K$ (where $\langle \cdot \rangle_K$ denote the linear span over K), with the convention $r_{X,K}(P) = +\infty$ if there is no such set S . Notice that if $L \supset K$ is any field, then $r_{X,K}(P) \geq r_{X,L}(P)$, and that $n\langle S \rangle_L \cap \mathbb{P}^r(K) \cap n\langle S \rangle_K$

for any set $S \subseteq X(K)$. For each integer $b \geq 0$ let $\sigma_b(X)$ denote the secant variety of order b of X , i.e. the geometrically integral variety whose $\overline{K}$ -points are the closure in $\mathbb{P}^r(\overline{K})$ of the union of all sets $\langle S \rangle_{\overline{K}}$ with $S \subset X(\overline{K})$ and $\sharp(S) = b$ (it is defined over K). The *generic X -rank* is the first integer b such that $\sigma_b(X) = \mathbb{P}^r$. This integer is the only integer such that

Let $U_b(X)(R) \subseteq X(R)^b$ be the set of all b -ples of distinct points of $X(R)$. Set $V_b(X)(R) = \{a \in U_b(X)(R) : \dim(\langle a \rangle_R) = b - 1\}$. For any integer b set $S(X, R, b) := \{P \in \mathbb{P}^r(R) : r_{X,R}(P) = r, S(X, R, \leq b) := \cup_{i=1}^b S(X, R, i)$ and $S(X, R, \geq b) = \cup_{i \geq b} S(X, R, i)$. There is a regular map $\rho_{b,R} : V_b(X)(R) \times \mathbb{P}^{b-1}(R) \rightarrow \mathbb{P}^r(R)$. The image $\text{Im}(\rho_{b,R})$ is a semialgebraic subset of $\mathbb{P}^r(R)$ ([2], Proposition 2.2.7). Since $S(X, r, \leq b) := \cup_{i=1}^b \text{Im}(\rho_i)$, we get that each set $S(X, r, \leq b)$ is semialgebraic. Since $S(X, R, 1) = X(R)$, by induction on b we get that each set $S(X, R, b)$ is semialgebraic. Hence we get that in the decomposition of $\mathbb{P}^r(R)$ into pairwise disjoint sets $S(X, R, b)$, $b \geq 1$, only finitely many sets $S(X, R, b)$ have dimension r ; we say that these integers b are the *typical X -ranks* over R . We prove the following result .

Theorem 1. *Let $X \subset \mathbb{P}^r$ be a non-degenerate geometrically irreducible variety defined over R . Let $L \supset R$ be another real closed field. Assume $X_{\text{reg}}(R) \neq \emptyset$. Then the generic X -rank is the minimum typical X -ranks and the typical X -ranks over R and over L are the same.*

With our definition (not the unique possible definition) it is easy to check that any typical X -rank is at least the generic X -rank and that the generic X -rank is a typical rank if and only if $X_{\text{reg}}(R) \neq \emptyset$ (see also [1], Theorem 2). The integer b is a typical X -rank if and only if $S(X, b, R)$ is Zariski dense in $\mathbb{P}^r(\overline{R})$.

Since any finite products $\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n_s}$ of projective space s is defined over $\mathbb{Q}$ and its Segre embedding $u_{n_1, \dots, n_s} : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s} \rightarrow \mathbb{P}^N$ is defined over $\mathbb{Q}$, we get the following corollary.

Corollary 1. *For all integers $s \geq 2$, $n_i > 0$, $1 \leq i \leq s$, the typical ranks of tensors of format $(n_1, \dots, n_s)$ are the same for all real closed fields.*

See [4] and [5] for recent results on the typical ranks of three-ways tensors. In the following observation we recall a case in which the generic rank is also the maximal rank and so the generic rank is the only typical rank.

Remark 1. Fix integers $s \geq 2$ and $n_i > 0$, $1 \leq i \leq s$. Assume $n_s \geq \prod_{i=1}^{n_1} (n_i + 1) - 1$. Then every tensor of format $(n_1, \dots, n_s)$ has rank at most $\prod_{i=1}^{n_1} (n_i + 1)$ and $\prod_{i=1}^{n_1} (n_i + 1)$ is the generic rank ([3], formula (††) at p. 267 and part (1) of Theorem 3.1). Hence $\prod_{i=1}^{n_1} (n_i + 1)$ is the only typical rank for

formats of type $(n_1, \dots, n_s)$.

There are many other examples for varieties $X \subset \mathbb{P}^r$ (e.g. most space curves and almost all curves in $\mathbb{P}^4$).

Proof of Theorem 1. The condition “ $X_{\text{reg}}(R) \neq \emptyset$ ” is equivalent to require that $X(R)$ is Zariski dense in $X(\overline{R})$ and it implies that $V_b(X)(R)$ is Zariski dense in $V_b(X)(\overline{R})$. Since $X(R)$ spans $\mathbb{P}^r(\overline{R})$, we have $S(X, R, b) = \emptyset$ for all $b > r$. We get that for each $b > 0$ the set $\sigma_b(X)(R)$ is Zariski dense in $\sigma_b(\overline{R})$. Since $\sigma_b(\overline{R}) = \mathbb{P}^r(\overline{R})$ if and only if b is at least the generic X -rank, while $\dim(\sigma_b(\overline{R})) < r$ if b is smaller, we get that the generic X -rank is the minimal typical X -rank. Using the sets $S(X, R, b)$ and $S(X, L, b)$ and that the dimensions of semialgebraic sets don't change extending the real closed base field (use [2], Proposition 5.2.1), we get that typical X -ranks are the same over R and over L . $\square$

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