International Journal of Pure and Applied Mathematics
Volume 93 No. 6 2014, 807-812
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v93i6.7

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ON THE UNIQUENESS FOR THE SYMMETRIC TENSOR RANK OF TRIVARIATE POLYNOMIALS; A LOCAL UNIQUENESS FOR MULTIVARIATE POLYNOMIALS

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Abstract: For all integers $m \ge 2$ and $d \ge 3$ and $x > 0$. Let $\nu_d : \mathbb{P}^m \to \mathbb{P}^N$, $N := \binom{m+d}{m} - 1$, be the Veronese embedding. We discuss the uniqueness (only for trivariate polynomials) and the local uniqueness of a decomposition of a polynomial into powers of linear forms in the following sense. Take $P \in \mathbb{P}^N$. Let $S(m, x, d, P)$ be the set of all $S \subset \mathbb{P}^m$ such that $\sharp(S) = x$, $P \in \langle \nu_d(S) \rangle$ (where $\langle\ \rangle$ is the linear span), and $P \notin \langle \nu_d(S') \rangle$ for any $S' \subsetneq S$. We prove that $S(m, x, d, P) = \{S\}$ (resp. S is a isolated point of $S(m, x, d, P)$) if $m = 2$, $x < (d^2 + 3d)/8$ and S has the general uniform position (resp. $\sharp(S) \le \binom{m+\lfloor (d-1)/2 \rfloor}{m}$ and S has general postulation). We do the same for zero-dimensional schemes (scheme rank or cactus rank).

AMS Subject Classification: 14N05
Key Words: symmetric tensor rank, trivariate polynomial, zero-dimensional scheme, multivariate polynomial

1. Introduction

This note is in part a continuation of [1] and we do not repeat the proofs con-

Received: March 1, 2014

tained therein. The problem considered in [1] and in this note may be summarized as the search for explicit criteria to show the uniqueness of a solution (in this case a finite set of $\mathbb{P}^2$ evincing the symmetric tensor rank of a trivariate homogeneous polynomial). We recall the following terminology. Fix integers $d \geq 3$ and $m \geq 1$. Let K be an algebraically closed field with either characteristic zero or characteristic $> d$. Let $K[x_0, \dots, x_m]_d$ be the vector space of all homogeneous polynomials of degree d. For each $f \in K[x_0, \dots, x_m]_d \setminus \{0\}$ the rank $r_{m,d}(f)$ is the minimal integer t such that $f = \sum_{i=1}^{t} \ell_i^t$ for some $\ell_i \in K[x_0, \dots, x_m]_1$. This notion may be translated in the following way. Let $\nu_d : \mathbb{P}^m \to \mathbb{P}^N$, $N := \binom{m+d}{m} - 1$, be the the order d Veronese embedding of $\mathbb{P}^m$, i.e. the embedding of $\mathbb{P}^m$ induced by the complete linear system associated to $K[x_0, \dots, x_m]_d$. Each $P \in \mathbb{P}^N$ corresponds to a polynomial $f \in K[x_0, \dots, x_m]_d \setminus \{0\}$, which is unique, up to a multiplicative constant, and we set $r_{m,d}(P) := r_{m,d}(f)$. The integer $r_{m,d}(P)$ is called the rank (or the symmetric tensor rank) of P. The integer $r_{m,d}(P)$ is the minimal cardinality of a set $S \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(S) \rangle$, where $\langle\ \rangle$ denote the linear span. Using the Veronese embedding it is possible to introduce another invariant of P, the minimal degree $z_{m,d}(P)$ of a zero-dimensional scheme $Z \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(Z) \rangle$. Nowadays the integer $z_{m,d}(P)$ is often called the *cactus rank* of P (or of any homogeneous polynomial) ([5]); it was introduced in [8],Definition 5.1, p. 135, Definition 5.66, p. 198, with the name *scheme length*. We say that Z evinces the cactus rank (resp. the rank) of P if $P \in \nu_d(Z)$ and $\deg(Z) = z_{m,d}(P)$ (resp. Z is reduced and $\sharp(Z) = r_{m,d}(P)$). As in [1] we are only able to handle the case $m = 2$ (see Theorem 2 and Remark 1). The local uniqueness problem is easier and we give a local uniqueness criterion in the multivariate case. To state it we need to introduced some jargon. Fix integers $x > 0$ and $d > 0$. Set $U(m, x) := \{S \subset \mathbb{P}^m : \sharp(S) : x\}$ and $U_d(m, x) := \{S \in U(m, x) : \dim(\langle \nu_d(S) \rangle) = x - 1\}$. Let $Z(m, x)$ be the set of all zero-dimensional schemes $Z \subset \mathbb{P}^m$ such that $\deg(Z) = x$. Set $Z_d(m, x) := \{S \in Z(m, x) : \dim(\langle \nu_d(S)) = x - 1\}$. Fix $P \in \mathbb{P}^N$,. $N := \binom{m+d}{m} - 1$. Set $Z'(m, x, d, P) := \{S \in Z(m, x) : P \in \langle \nu_d(S) \rangle$, $Z''(m, x, d, P) := Z'(m, x, d, P) \cap Z_d(m, x)$, $S'(m, x, d, P) := \{S \in U(m, x) : P \in \langle \nu_d(S) \rangle$, $S''(m, x, d, P) := S'(m, x, d, P) \cap Z_d(m, x)$. Let $Z(m, x, d, P)$ (resp. $S(m, x, d, P)$) be the set of all $S \in Z'(m, x, d, P)$ (resp. $S \in Z'(m, x, d, P)$) such that $P \notin \langle \nu_d(S') \rangle$ for any $S' \subsetneq S$. Notice that $Z(m, x, d, P) \subseteq Z''(m, x, d, P)$ and $S(m, x, d, P) \subseteq S''(m, x, d, P)$. The schemes $U(m, x)$ and $Z(m, x)$ have a Zariski topology and hence each of their subsets inherits a topology (the induced topology of the Zariski topology of $Z(m, x)$). Fix $S \in Z'(m, x, d, P)$ (resp. $S \in S'(m, x, d, P)$). We say that $Z'(m, x, d, P)$ (resp. $S'(m, x, d, P)$) has *local uniqueness* at S if S is an isolated point of $Z'(m, x, d, P)$ (resp. $S'(m, x, d, P)$)

with respect to the topology just described. The same definition applies to $Z''(m,x,d,P)$, $S''(m,x,d,P)$, $Z(m,x,d,P)$, and $S(m,x,d,P)$ (see Remarks 2 and 3 for a collection of elementary properties). Fix $Z \in Z(m,x)$. We say that Z has *general Hilbert function* or *general postulation* if for each $t \in \mathbb{N}$ either $h^1(\mathcal{I}_Z(t)) = 0$ or $h^1(\mathcal{I}_Z(t)) = 0$. Let v_x be the maximal $t \in \mathbb{N}$ such that $\binom{m+t}{m} \leq x$. The scheme $Z \in Z(m,x)$ has general Hilbert function if and only if $h^0(\mathcal{I}_Z(t)) = 0$ for all $t \leq v_x$ and $h^1(\mathcal{I}_Z(t)) = 0$ for all $t > v_x$. We say that Z has *general uniform position* if every $Z' \subseteq Z$ has general postulation. We prove the following results.

Theorem 1. *Fix integers $m \geq 2$ and $d \geq 3$. Fix an integer $x \leq \binom{m+\lfloor (d-1)/2 \rfloor}{m}$ and assume the existence of $Z \in Z(m,x,d,P)$ with general postulation. Then Z is isolated in $Z(m,x,d,P)$, $Z'(m,x,d,P)$, and $Z''(m,x,d,P)$. If $Z \in S(m,x,d,P)$, then it is isolated in $S(m,x,d,P)$, $S'(m,x,d,P)$, and $S''(m,x,d,P)$.*

Theorem 2. *Fix an integer $d \geq 3$ and let $S \subset \mathbb{P}^2$ be a finite set with general uniform position and with $k := \sharp(S) < (d^2+2d)/8$. Fix any $P \in \langle \nu_d(S) \rangle$ such that $P \notin \langle S' \rangle$ for any $S' \subset S$ with $\sharp(S') = k-1$. Then S is the only zero-dimensional scheme $Z \subset \mathbb{P}^2$ such that $P \in \langle \nu_d(Z) \rangle$ and $\deg(Z) \leq k$.*

Remark 1. Look at the statement of Theorem 2. If we assume $\deg(Z) = k$, then this is the union of [1], Theorem 1.4 and Remark 2.5, except that a key assumption of [1] is almost remove. In [1] it was assumed that $r_{2,d}(P) = k$. In the statement of Theorem 2 we only assume that $P \notin \langle S' \rangle$ for any $S' \subset S$ with $\sharp(S') = x-1$. Hence we get for free that all points of $\langle \nu_d(S) \rangle$ have rank x, except the points of x hyperplanes for which obviously the rank is at most $x-1$. Fix any $P\langle S \rangle$ and let $S_1 \subseteq S$ be a minimal subset of S such that $P \in \langle S_1 \rangle$. Since every subset of S is in uniform position, we may apply Theorem 2 to S_1 and get that P has rank $\sharp(S_1)$ and that S_1 is the unique zero-dimensional scheme $A \subset \mathbb{P}^2$ such that $P \in \langle \nu_d(A) \rangle$ and $\deg(A) \leq \sharp(S_1)$, i.e. P has both rank and scheme-rank equal to $\sharp(S_1)$ and S_1 is the only subscheme of $\mathbb{P}^2$ evincing the scheme-rank of P.

2. Proofs and related results

Outline of the proof of Theorem 2. Decreasing if necessary Z we may assume that $P \notin \langle \nu_d(Z') \rangle$ for any $Z' \subsetneq Z$. For any zero-dimensional scheme $U \subset \mathbb{P}^2$ let h_U denotes the Hilbert function of U and Dh_U its first difference. Set $W := S \cup Z$ and $w := \deg(W)$. By assumption we have $\deg(W) \leq 2k$. Let

u be the only integer such that $(u^2+3u+2)/2 \le k < ((u+1)^2+3(u+1)+2)/2$. Since S has general uniform position we have $h^0(\mathcal{I}_S(u)) = 0$ and for each $E \subseteq A$ and each integer $t \le u$ we have $h^0(\mathcal{I}_E(t)) = \max\{0, \binom{t+2}{2} - \sharp(E)\}$. As in [1], Claims 2.1, 2.2, 2.3, there is a positive integer $j \le d$ such that $Dh_W(j) = Dh_W(j+1) > 0$ and the minimal, m, of such integers j satisfies $m \le u$. By [1], Claim 2.4, (which uses in an essential way either [6] or [7], Proposition at page 112), there is a degree m curve $M \subset \mathbb{P}^2$ such that $a := \deg(M \cap W) \ge (m+1)d - m^2 + m + 2$. Set $A := M \cap W$. Let B be the residual scheme of W with respect to M, i.e. the closed subscheme of $\mathbb{P}^2$ with $\mathcal{I}_W : \mathcal{I}_M$ as its ideal sheaf. Set $A := M \cap D$. We have $\deg(B) + \deg(A) = w$. By [7], page 112, we also have $Dh_A(i) + Dh_B(i-m) = Dh_W(i)$ for all i. Since $Dh_W(d) \le m$, we have $Dh_A(d) = \min\{m, Dh_W(d)\} = Dh_W(d)$ and hence $Dh_B(d-m) = 0$. Hence $\mathrm{Res}_M(S) = \mathrm{Res}_M(Z)$ and there is a unique $Q \in \langle \nu_d(S \cap M) \cap \langle \nu_d(Z) \rangle$ such that $P \in \langle \nu_d(S \cap M) \cup \{Q\} \rangle$ ([4], Lemma 8, or [3], Lemma 8). Since $\deg(\mathrm{Res}_M(Z)) = \deg(\mathrm{Res}_M(S)$, we have $\deg(Z \cap M) \le \sharp(S \cap M)$. Since $S \cap M$ has general uniform position and $\sharp(S \cap M) < k$, we may apply the inductive assumption to Q and $S \cap M$ and get a contradiction. □

Remark 2. Fix $S \in U(m,x)$. Since being reduced is an open condition in flat families of zero-dimension schemes, there is a neighborhood Ω of S is $Z(m,x)$ such that $\Omega \subset U(m,x)$. Hence the local uniqueness property at a reduced set S is the same if we look at all schemes near S or just to finite sets with cardinality x.

Remark 3. Fix $Z \in Z(m,x)$ and assume that Z has general postulation (resp. general uniform position). By the semicontinuity theorem for cohomology there is an open neighborhood Ω of Z such that W has general postulation (resp. general uniform position) for all $W \in \Omega$. Hence Theorem 2 is proved if we prove the following result.

Proposition 1. *Fix positive integers m, x, y, a, b, d such that $a + b \le d - 1$, $x \le \binom{m+a}{m}$, $y \le \binom{m+b}{m}$. Fix $P \in \mathbb{P}^N$ and assume the existence of $A \in Z(m,x,d,P)$, $B \in Z(m,y,d,P)$ such that A and B have general postulation; assume that for all $Q \in A_{\mathrm{red}} \cap B_{\mathrm{red}}$ (if any) B is reduced at Q. Then $x = y$ and $A = B$.*

Proof. Assume $A \ne B$. We have $h^1(\mathcal{I}_{A\cup B}(d)) > 0$ ([2], Lemma 1). Since A has general postulation and $x \le \binom{m+a}{m}$, we have $h^1(\mathcal{I}_A(a)) = 0$. Castelnuovo-Mumford's lemma implies that $\mathcal{I}_A(a+1)$ is spanned by its global sections. Hence a general $T \in |\mathcal{I}_A(a+1)|$ meets no connected component of B which does not intersect A. Since B is reduced at each point of $A_{\mathrm{red}} \cap B_{\mathrm{red}}$, we get that $T \cap (A \cup B) = A$ as schemes. Hence we have an exact sequence

$$0 \to \mathcal{I}_{B \setminus B \cap A_{\mathrm{red}}}(d-a-1) \to \mathcal{I}_{A \cup B}(d) \to \mathcal{I}_{A,T}(d) \to 0 \tag{1}$$

We have $h^1(T, \mathcal{I}_{A,T}(d)) = h^1(\mathcal{I}_A(d)) = 0$. Since $b \le d-a-1$, $y \le \binom{m+b}{m}$ and B has general postulation, we have $h^1(\mathcal{I}_B(d-a-1)) = 0$. Since $B \setminus B \cap A_{\mathrm{red}} \subseteq B$, we have $h^1(\mathcal{I}_{B \setminus B \cap A_{\mathrm{red}}}(d-a-1)) = 0$. Hence (1) gives $h^1(\mathcal{I}_{A \cup B}(d)) = 0$, a contradiction. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico and L. Chiantini, A criterion for detecting the identifiability of symmetric tensors of size three, Differential Geom. Appl. 30 (2012), no. 3, 233–237.

[2] E. Ballico and A. Bernardi, Decomposition of homogeneous polynomials with low rank, Math. Z. 271 (2012), 1141–1149; DOI 10.1007/s00209-011-0907-6.

[3] E. Ballico and A. Bernardi, A partial stratification of secant varieties of Veronese varieties via curvilinear subschemes, Sarajevo J. Math. 8 (2012), no. 1, 33–52.

[4] E. Ballico and A. Bernardi, Stratification of the fourth secant variety of Veronese variety via the symmetric rank, Adv. Pure Appl. Math. 4 (2013), no. 2, 215–250; DOI: 10.1515/apam-2013-0015

[5] A. Bernardi and K. Ranestad, The cactus rank of cubic forms, J. Symbolic. Comput. 50 (2013) 291–297. DOI: 10.1016/j.jsc.2012.08.001

[6] E. D. Davis, 0-dimensional subschemes of $\mathbf{P}^2$: new application of Castelnuovo's function. Ann. Univ. Ferrara Sez. VII (N.S.) 32 (1986), 93–107 (1987).

[7] Ph. Ellia and Ch. Peskine, Groupes de points de $\mathbf{P}^2$: caractère et position uniforme, in: Algebraic geometry (L' Aquila, 1988), 111–116, Lecture Notes in Math., 1417, Springer, Berlin, 1990.

[8] A. Iarrobino and V. Kanev. Power sums, Gorenstein algebras, and determinantal loci. Lecture Notes in Mathematics, vol. 1721, Springer-Verlag, Berlin, 1999, Appendix C by Iarrobino and Steven L. Kleiman.