International Journal of Pure and Applied Mathematics
Volume 93 No. 6 2014, 835-844
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v93i6.9
ijpam.eu

VECTOR SPACE BASES FOR THE HOMOGENEOUS PARTS IN HOMOGENEOUS IDEALS AND GRADED MODULES OVER A POLYNOMIAL RING

Natalia Dück[1 §], Karl-Heinz Zimmermann[2]
[1,2]Hamburg University of Technology
21071, Hamburg, GERMANY

Abstract: In this paper, vector space bases for the homogeneous parts of homogeneous ideals and graded modules over a commutative polynomial ring are given using Gröbner bases.

AMS Subject Classification: 13F20, 13A02
Key Words: graded ring, polynomial ring, homogeneous ideal, vector space basis

1. Introduction

Gröbner bases have originally been introduced by Buchberger for the algorithmic solution of some fundamental problems in commutative algebra [3]. Since then Gröbner bases evolved into a crucial concept in symbolic computations providing a uniform approach to solving a wide range of problems such as effective computations in residue class rings modulo polynomial ideals and in modules over polynomial rings, and calculating syzygies and graded resolutions for homogeneous ideals [2, 4, 6].

In this paper, we provide bases for the vector spaces corresponding to the homogeneous parts of homogeneous ideals or graded modules over a polynomial ring. Moreover, we show that the vector space basis for the homogenous part

Received: April 4, 2014

[§]Correspondence author

of a homogeneous ideal is also a Gröbner basis for the ideal generated by the homogeneous part if the degree of the homogeneous part is large enough. The required notions and definitions are introduced in Section 2 and the results are provided in the Sections 3 and 4.

2. Graded Rings and Gröbner Bases for Modules

Let $R = \mathbb{K}[x_0, x_1, \ldots, x_n]$ denote the commutative polynomial ring in $n+1$ indeterminates over a field $\mathbb{K}$. The *monomials* in R are denoted by $\mathbf{x}^u = x_0^{u_0} \cdots x_n^{u_n}$, where $u = (u_0, \ldots, u_n) \in \mathbb{N}_0^{n+1}$. The *total degree* of a monomial $\mathbf{x}^u$ is the sum $|u| = u_1 + \ldots u_n$. The ring R has a natural grading in the sense that it admits a direct sum decomposition

$$R = \bigoplus_{s \geq 0} R_s, \tag{1}$$

where for each integer $s \geq 0$ the set R_s is the additive subgroup of R that consists of all homogeneous polynomials of degree s plus the zero polynomial, and the complex product $R_s R_t = \{rr' \mid r \in R_s, r' \in R_t\}$ is contained in R_{s+t} for all $s, t \geq 0$. Note that $R_0 = \mathbb{K}$ and $R_0 R_s \subseteq R_s$. Thus the subgroups R_s are also $\mathbb{K}$-vector spaces.

A module M over R (or any graded ring) is a *graded module* over R if it can be decomposed as

$$M = \bigoplus_{t \in \mathbb{Z}} M_t, \tag{2}$$

where each M_t is an additive subgroup of the additive group of M with the property that the complex product $R_s M_t = \{rm \mid r \in R_s, m \in M_t\}$ lies in M_{s+t} for all $s \geq 0$ and all $t \in \mathbb{Z}$. Each additive subgroup M_t is a module over $R_0 = \mathbb{K}$ since $R_0 M_t \subset M_t$. Thus the subgroups M_t are also $\mathbb{K}$-vector spaces.

Let $m \geq 1$ be an integer. The free R-module R^m has the standard basis consisting of the canonical unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_m$. The module R^m is graded over R with the (standard) grading

$$(R^m)_t = (R_t)^m, \quad t \in \mathbb{Z}.$$

Note that $(R^m)_t = \{0\}$ if $t \leq 0$.

A *monomial* in R^m is an element of the form $\mathbf{x}^u \mathbf{e}_i$ for some $1 \leq i \leq m$ and $u \in \mathbb{N}_0^{n+1}$. Each element in R^m can be uniquely written as a $\mathbb{K}$-linear combination of monomials. For instance, let $R = \mathbb{K}[x, y]$ and take the following element in R^2,

$$\begin{pmatrix} 3x^2y + xy^2 + 1 \\ x^3y^2 + 2xy^5 - 3y \end{pmatrix} = 3x^2y\mathbf{e}_1 + xy^2\mathbf{e}_1 + \mathbf{e}_1 + x^3y^2\mathbf{e}_2 + 2xy^5\mathbf{e}_2 - 3y\mathbf{e}_2.$$

A *monomial order* on R^m is a relation $\succ$ on the set of monomials in R^m satisfying the following conditions: (1) $\succ$ is a total order, (2) $\succ$ is well-ordering, and (3) for any monomials $m, m' \in R^m$, $m \succ m'$ implies $\mathbf{x}^u m \succ \mathbf{x}^u m'$ for each monomial $\mathbf{x}^u \in R$.

Any monomial order on R can be extended to a monomial order on R^m. For this, an ordering of the standard basis vectors needs to fixed, say by downward ordering $\mathbf{e}_1 > \cdots > \mathbf{e}_m$. Then the TOP (term over position) extension of a monomial order $\succ$ on R, denoted by $\succ_{TOP}$, is defined as

$$\mathbf{x}^u\mathbf{e}_i \succ_{TOP} \mathbf{x}^v\mathbf{e}_j \quad :\Longleftrightarrow \quad \mathbf{x}^u \succ \mathbf{x}^v \ \vee \ (\mathbf{x}^u = \mathbf{x}^v \ \wedge \ i < j) \tag{3}$$

and the POT (position over term) extension of $\succ$, denoted by $\succ_{POT}$, is given by

$$\mathbf{x}^u\mathbf{e}_i \succ_{POT} \mathbf{x}^v\mathbf{e}_j \quad :\Longleftrightarrow \quad i < j \ \vee \ (i = j \ \wedge \ \mathbf{x}^u \succ \mathbf{x}^v)\,. \tag{4}$$

For instance, if the lex order on $R = \mathbb{K}[x, y]$ with $x > y$ is extended to a TOP order on R^2, then

$$x^3y^2\mathbf{e}_2 \succ_{TOP} 3x^2y\mathbf{e}_1 \succ_{TOP} 2xy^5\mathbf{e}_2 \succ_{TOP} xy^2\mathbf{e}_1 \succ_{TOP} -3y\mathbf{e}_2 \succ_{TOP} \mathbf{e}_1.$$

However, if the lex order on R is extended to a POT order on R^2, then

$$3x^2y\mathbf{e}_1 \succ_{POT} -2xy^2\mathbf{e}_1 \succ_{POT} \mathbf{e}_1 \succ_{POT} x^3y^2\mathbf{e}_2 \succ_{POT} 2xy^5\mathbf{e}_2 \succ_{POT} -3y\mathbf{e}_2.$$

Given a monomial order $\succ$ on R^m, each non-zero polynomial $f \in R^m$ has a unique *leading term* given by the largest involved term and denoted by $\mathrm{lt}_\succ(f)$; the corresponding leading monomial is referred to as $\mathrm{lm}_\succ(f)$. Each submodule M of R^m has a *leading submodule* generated as a module by the leading terms of its elements,

$$\langle \mathrm{lt}_\succ(M)\rangle = \langle\{\mathrm{lt}_\succ(f) \mid f \in M\}\rangle. \tag{5}$$

A Gröbner basis for a submodule M of R^m w.r.t. a monomial order $\succ$ on R^m is a finite subset $\mathcal{G}$ of M with the property that the leading terms of the elements in $\mathcal{G}$ generate the leading submodule of M, i.e.,

$$\langle \mathrm{lt}_\succ(M)\rangle = \langle\{\mathrm{lt}_\succ(g) \mid g \in \mathcal{G}\}\rangle. \tag{6}$$

Each submodule of R^m has a Gröbner basis which is generally not uniquely determined. However, a unique Gröbner basis $\mathcal{G}$ called *reduced Gröbner basis* can be obtained, where the leading terms of the elements in $\mathcal{G}$ are monic and for two distinct elements g and g' in $\mathcal{G}$ no term involved in g is divisible by the leading term of g'. Gröbner bases can be computed by *Buchberger's algorithm* for submodules which is available by almost every computer algebra system. More details on Gröbner bases and modules can be found in [1, 5].

3. Vector Space Bases for the Homogeneous Parts in Homogeneous Ideals

An ideal I in R is *homogeneous* if for any element $f \in I$ the homogeneous components of f are also in I. A homogeneous ideal I in R is a graded submodule of R with the direct sum decomposition

$$I = \bigoplus_{t \in \mathbb{Z}} I_t,$$

where the homogeneous parts are given by $I_t = I \cap R_t$ for all $t \in \mathbb{Z}$. Note that $I_t = \{0\}$ if $t \leq 0$.

A $\mathbb{K}$-basis for the homogeneous part R_t is given by all monomials of total degree t and so we have

$$\dim_{\mathbb{K}} R_t = \binom{t+n-1}{n-1}, \quad t \in \mathbb{N}_0.$$

Thus the homogeneous part I_t is a finite-dimensional vector space for each $t \in \mathbb{Z}$.

The quotient module R/I has also a graded module structure defined by

$$(R/I)_t = R_t/I_t = R_t/(I \cap R_t), \quad t \in \mathbb{Z}.$$

By the dimension formula,

$$\dim_{\mathbb{K}} R_t = \dim_{\mathbb{K}} I_t + \dim_{\mathbb{K}} (R/I)_t\,, \quad t \in \mathbb{N}_0,$$

and thus the quotient spaces $(R/I)_t$ are also finite dimensional. Moreover, the ideal of leading terms of I fulfills

$$\dim_{\mathbb{K}} R_t/I_t = \dim_{\mathbb{K}} R_t/\langle \mathrm{lt}_{\succ}(I)\rangle_t, \quad t \in \mathbb{N}_0,$$

where $\langle \mathrm{lt}_{\succ}(I)\rangle_t = \langle \mathrm{lt}_{\succ}(I)\rangle \cap R_t$.

Note that the additive subgroups I_t are not ideals. Nonetheless, we can consider the ideal $\langle I_t \rangle$ generated by the elements in I_t.

Proposition 1. *Let I be a homogeneous ideal in R. Let $\mathcal{G}$ be the reduced Gröbner basis for I w.r.t. any monomial order $\succ$ on R and let*

$$lm_\succ(I_t) = \{lm_\succ(f) \mid f \in I_t\} = \{\mathbf{x}^{a_1}, \ldots, \mathbf{x}^{a_s}\}, \quad t \in \mathbb{N}_0.$$

Then a $\mathbb{K}$-basis for the vector space I_t is given by the set of binomials

$$B_t = \{\mathbf{x}^{a_1} - r_1, \ldots, \mathbf{x}^{a_s} - r_s\},$$

where r_i is the remainder of $\mathbf{x}^{a_i}$ on division by $\mathcal{G}$ for $1 \le i \le s$.

The vector space I_t is non-trivial if and only if $t \ge \min\{\deg(lt_\succ(g)) \mid g \in \mathcal{G}\}$. If $t \ge \max\{\deg(lt_\succ(g)) \mid g \in \mathcal{G}\}$, then the set $lm_\succ(I_t)$ consists of all monomial multiples of the elements in $\{lt_\succ(g) \mid g \in \mathcal{G}\}$ which are of total degree t, and B_t is the reduced Gröbner basis for the homogeneous ideal $\langle I_t\rangle$.

Proof. Note that the reduced Gröbner basis $\mathcal{G}$ for a homogeneous ideal always consists of homogeneous polynomials and the remainder of a homogeneous polynomial f divided by a set of homogeneous polynomials is either zero or homogeneous of the same total degree as f. It follows that if a monomial $\mathbf{x}^{a_i}$ of total degree t is divided by the Gröbner basis $\mathcal{G}$ giving the remainder r_i, then the difference $\mathbf{x}^{a_i} - r_i$ will be a polynomial of total degree t with leading term $\mathbf{x}^{a_i}$ which lies in I_t, $1 \le i \le s$. Hence, B_t is contained in I_t.

By rearranging and deleting duplicates, we may assume that $\mathbf{x}^{a_1} \succ \ldots \succ \mathbf{x}^{a_s}$. Let $f_i = \mathbf{x}^{a_i} - r_i \in I_t$, $1 \le i \le s$, and claim that the elements $f_1, \ldots, f_s$ form a $\mathbb{K}$-basis of I_t. Indeed, consider a nontrivial linear combination $k_1 f_1 + \ldots + k_t f_t$ with $k_i \in \mathbb{K}$ and take the smallest index i such that $k_i \neq 0$. By the ordering of the leading monomials, there is nothing to cancel $k_i f_i$ and so the linear combination is nonzero. Hence, $f_1, \ldots, f_s$ are linearly independent.

Moreover, let U be the subspace of I_t spanned by $f_1, \ldots, f_s$. Suppose U is a proper subspace of I_t. Pick an element $f \in I_t \setminus U$ whose leading monomial is minimal. By definition, the leading monomial of f equals the leading monomial of f_i for some i and $\mathrm{lt}(f) = k\mathrm{lt}(f_i)$ for some $k \in \mathbb{K}$. It follows that $f - kf_i$ lies in I_t and has a smaller leading monomial. Thus $f - kf_i \in U$ by the minimality of the leading monomial of f and so $f \in U$, a contradiction. Hence, $U = I_t$ and the claim follows.

Let $t \ge \min\{\deg(\mathrm{lt}_\succ(g)) \mid g \in \mathcal{G}\}$. Then $t \ge \deg(g)$ for some element $g \in \mathcal{G}$ and thus the leading monomial of $g\mathbf{x}^u$ with $|u| + \deg(g) = t$ lies in $\mathrm{lm}_\succ(I_t)$. Hence, the vector space I_t is non-trivial. Conversely, let $f \in I_t$. Then $f \in I$ and there is an element $g \in \mathcal{G}$ such that the leading term of f is divisible by the leading term of g. Hence, $t \ge \deg(g)$.

Finally, let $t \geq \max\{\deg(\mathrm{lt}_\succ(g)) \mid g \in \mathcal{G}\}$ and claim that B_t is the reduced Gröbner basis for $\langle I_t \rangle$. Indeed, the set B_t generates I_t as a vector space and so generates also the ideal $\langle I_t \rangle$. It remains to show that the leading terms of the elements in B_t generate the leading ideal $\langle \mathrm{lt}_\succ(\langle I_t \rangle)\rangle$. To this end, let $f \in \langle I_t \rangle$. Since $f \in I$, there is a Gröbner basis element $g \in \mathcal{G}$ such that $\mathrm{lt}_\succ(g)$ divides $\mathrm{lt}_\succ(f)$. By the choice of t, $\deg(g) \leq t$ and so all monomial multiples of $\mathrm{lt}_\succ(g)$ of total degree t appear as leading terms in B_t. But the leading term of f is also a monomial multiple of $\mathrm{lt}_\succ(g)$ (possibly of total degree larger than t) and so must be divisible by at least one monomial $\mathbf{x}^{a_i}$ where $1 \leq i \leq s$. This proves the claim. □

The set $\mathrm{lm}_\succ(I_t)$ can be constructed from the reduced Gröbner basis $\mathcal{G}$ for I w.r.t. any monomial order $\succ$ as follows. Starting with the empty set add for each element $g \in \mathcal{G}$ with leading term of total degree $s \leq t$ all monomial multiples of $\mathrm{lt}_\succ(g)$ that are of degree t, i.e., the set $\{\mathrm{lt}_\succ(g)\mathbf{x}^u \mid |u| = t - s\}$. This will give the set $\mathrm{lm}_\succ(I_t)$ in a finite number of steps, since the Gröbner basis is finite.

Example 1. Consider the homogeneous ideal

$$I = \left\langle z^3 - yw^2, yz - xw, y^3 - x^2z, xz^2 - y^2w \right\rangle \subset \mathbb{K}[x, y, z, w] = R.$$

The above set is the reduced Gröbner basis w.r.t. the grevlex order $\succ$ on R with $x \succ y \succ z \succ w$. Thus

$$\langle \mathrm{lt}_\succ(I)\rangle = \left\langle z^3,\ yz,\ y^3,\ xz^2 \right\rangle.$$

Note that all monomials in this generating set have degree greater than or equal to 2 and so $I_i = \{0\}$ for $i \leq 1$. By the above remark, the leading ideals for I_2, I_3 and I_4 are generated as follows,

$$\begin{aligned}
\mathrm{lm}_\succ(I_2) &= \{yz\}, \\
\mathrm{lm}_\succ(I_3) &= \left\{z^3, xyz, y^2z, yz^2, yzw, y^3, xz^2\right\}, \\
\mathrm{lm}_\succ(I_4) &= \{xz^3, yz^3, z^4, wz^3, x^2yz, y^3z, yzw^2, xy^2z, y^2z^2, yz^2w, \\
&\qquad xyz^2, y^2zw, xyzw, xy^3, y^4, y^3w, x^2z^2, xz^2w\}.
\end{aligned}$$

By the division algorithm, a vector space basis for I_2 is $B_2 = \{yz - xw\}$, a vector space basis for I_3 is

$$\begin{aligned}
B_3 = \{&z^3 - yw^2, xyz - x^2w, y^2z - xyw, yz^2 - xzw, \\
&yzw - xw^2, y^3 - x^2z, xz^2 - y^2w\},
\end{aligned}$$

and a vector space basis for I_4 is

$$B_4 = \left\{ \begin{array}{lll} xz^3 - xyw^2, & yz^3 - y^2w^2, & z^4 - xw^3, \\ z^3w - yw^3, & x^2yz - x^3w, & y^3z - xy^2w, \\ yzw^2 - xw^3, & xy^2z - x^2yw, & y^2z^2 - x^2w^2, \\ yz^2w - xzw^2, & xyz^2 - x^2zw, & y^2zw - xyw^2, \\ xyzw - x^2w^2, & xy^3 - x^3z, & y^4 - x^3w, \\ y^3w - x^2zw, & x^2z^2 - xy^2w, & xz^2w - y^2w^2 \end{array} \right\}.$$

The homogeneous ideal $\langle I_3 \rangle$ has the generating set B_3 and one can show that it is also the reduced Gröbner basis for this ideal w.r.t. the above grevlex order. However, a Gröbner basis for the ideal I w.r.t. the grevlex order with $w \succ y \succ z \succ x$ is

$$\{z^4 - xw^3, yw^2 - z^3, yz - xw, y^2w - xz^2, y^3 - x^2z\}.$$

It follows that a basis for the $\mathbb{K}$-vector space I_3 is

$$B_3' = \{yw^2 - z^3, xyz - x^2w, y^2z - xyw, yz^2 - xzw, \\ yzw - xw^2, y^2w - xz^2, y^3 - x^2z\}.$$

Since B_3' differs from B_3' only by scalar multiples, it is also a generating set for the ideal $\langle I_3 \rangle$. However, it is not the reduced Gröbner basis w.r.t. the grevlex order with $w \succ y \succ z \succ x$ since the S-polynomial

$$S\left(yw^2 - z^3, yzw - xw^2\right) = z(yw^2 - z^3) - w(yzw - xw^2) = -z^4 + xw^3$$

has the leading term z^4 which is not divisible by any of the leading terms in B_3'. This confirms the necessity of the condition $t \geq \max\{\deg(\mathrm{lt}_\succ(g)) \mid g \in \mathcal{G}\}$ for B_t to form a Gröbner basis for the ideal $\langle I_t \rangle$.

4. Vector Space Bases for the Homogeneous Parts in Graded Modules

Let $m \geq 1$ be an integer. The graded submodules M of R^m can be characterized as follows [5]:

- The standard grading on R^m induces a graded module structure on M, which is given by $M_t = (R^m)_t \cap M$ for all $t \in \mathbb{Z}$.

- There are elements $f_1, \dots, f_r$ in R^m, whose components are homogeneous polynomials of the same degree, such that $M = \langle f_1, f_2, \dots, f_r \rangle \subset R^m$ for all $t \in \mathbb{Z}$.
- A reduced Gröbner basis for M (w.r.t. any monomial order on R^m) consists of vectors of homogeneous polynomials whose components have the same degree.

Using these facts, we obtain the following result.

Proposition 2. *Let $M \subset R^m$ be a graded module over R. Let $\mathcal{G}$ be a Gröbner basis for M w.r.t. any monomial order $\succ$ and let*

$$lm_\succ(M_t) = \{lm_\succ(f) \mid f \in M_t\} = \{\mathbf{x}^{a_1}\mathbf{e}_{i_1}, \dots, \mathbf{x}^{a_s}\mathbf{e}_{i_s}\}, \quad t \in \mathbb{N}_0,$$

where $\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_s}$ are unit vectors in R^m. Then a $\mathbb{K}$-basis for the vector space M_t is given by

$$B_t = \{\mathbf{x}^{a_1}\mathbf{e}_{i_1} - r_1, \dots, \mathbf{x}^{a_s}\mathbf{e}_{i_s} - r_s\},$$

where $r_j \in R^m$ is the remainder of $\mathbf{x}^{a_j}\mathbf{e}_{i_j}$ on division by $\mathcal{G}$ for each $1 \leq j \leq s$. The vector space M_t is non-trivial if and only if $t \geq \min\{\deg(lt_\succ(g)) \mid g \in \mathcal{G}\}$.

The proof is the same as that of Prop. 1 since all statements used there are also applicable to submodules of R^m (see for instance [5, Chapter 5, 2]). Moreover, the construction of $\mathrm{lm}_\succ(M_t)$ in the module case is analogous to that in the ideal case (see the remark after the proof of Prop. 1).

Example 2. Let $R = \mathbb{K}[x, y, z, w]$ and consider the submodule M of R^4 generated by the vectors

$$\begin{pmatrix} y^2 \\ xz \\ yw \\ z^2 \end{pmatrix}, \begin{pmatrix} z \\ w \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ y \\ -z \\ -w \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ x \\ y \end{pmatrix}.$$

The generators are vectors of homogeneous monomials of the same total degree and so by the above remark, the module M is graded.

The reduced Gröbner basis for the module M w.r.t. the POT-extension of the grevlex order $\succ$ with $x \succ y \succ z \succ w$ is given by the columns of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & z & x & y^2 \\ 0 & yz - xw & xz^2 - y^2w & y^3 - x^2z & w & y & xz \\ x & -z^2 & yzw & -y^2z & 0 & -z & yw \\ y & -zw & z^3 & -xz^2 & 0 & -w & z^2 \end{pmatrix}.$$

Thus the leading ideal of M is

$$\langle \mathrm{lt}_{\succ_{POT}}(M)\rangle = \left\langle y^2\mathbf{e}_1,\ x\mathbf{e}_1,\ z\mathbf{e}_1,\ y^3\mathbf{e}_2,\ xz^2\mathbf{e}_2,\ yz\mathbf{e}_2,\ x\mathbf{e}_3\right\rangle$$

and therefore

$$\begin{aligned}\mathrm{lm}_{\succ_{POT}}(M_1) &= \{x\mathbf{e}_1,\ z\mathbf{e}_1,\ x\mathbf{e}_3\},\\ \mathrm{lm}_{\succ_{POT}}(M_2) &= \{x^2\mathbf{e}_1,\ xy\mathbf{e}_1,\ xz\mathbf{e}_1,\ xw\mathbf{e}_1,\ y^2\mathbf{e}_1,\ yz\mathbf{e}_1,\ z^2\mathbf{e}_1,\\ &\qquad zw\mathbf{e}_1,\ yz\mathbf{e}_2, x^2\mathbf{e}_3,\ xy\mathbf{e}_3,\ xz\mathbf{e}_3,\ xw\mathbf{e}_3\}.\end{aligned}$$

The bases for the $\mathbb{K}$-vector spaces B_1 and B_2 for M_1 and M_2, respectively, are

$$B_1 = \left\{\begin{pmatrix}0\\0\\x\\y\end{pmatrix},\begin{pmatrix}z\\w\\0\\0\end{pmatrix},\begin{pmatrix}x\\y\\-z\\-w\end{pmatrix}\right\},$$

$$B_2 = \left\{\begin{pmatrix}x^2\\xy\\-xz\\-xw\end{pmatrix},\begin{pmatrix}xy\\y^2\\-yz\\-yw\end{pmatrix},\begin{pmatrix}xz\\xw\\0\\0\end{pmatrix},\begin{pmatrix}xw\\yw\\-zw\\-w^2\end{pmatrix},\begin{pmatrix}y^2\\xz\\yw\\z^2\end{pmatrix},\begin{pmatrix}yz\\yw\\0\\0\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}z^2\\zw\\0\\0\end{pmatrix},\begin{pmatrix}zw\\w^2\\0\\0\end{pmatrix},\begin{pmatrix}0\\yz-xw\\-z^2\\-zw\end{pmatrix},\begin{pmatrix}0\\0\\x^2\\xy\end{pmatrix},\begin{pmatrix}0\\0\\xy\\y^2\end{pmatrix},\begin{pmatrix}0\\0\\xz\\yz\end{pmatrix},\begin{pmatrix}0\\0\\xw\\yw\end{pmatrix}\right\}.$$

References

[1] W. Adams and P. Loustaunau, *An Introduction to Gröbner Bases*, American Mathematical Society, USA (1994).

[2] T. Becker and V. Weispfenning, *Gröbner Bases – A Computational Approach to Commutative Algebra*, Springer, New York (1998).

[3] B. Buchberger, *An Algorithm for Finding the Bases Elements of the Residue Class Ring Modulo a Zero Dimensional Polynomial Ideal*, PhD thesis, University of Innsbruck (1965).

[4] D. Cox and J. Little and D. O'Shea, *Ideals, Varieties, and Algorithms*, Springer, New York (1996).

[5] D. Cox and J. Little and D. O'Shea, *Using Algebraic Geometry*, Springer, New York (1998).

[6] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, American Mathematical Society, USA (1996).