VECTOR SPACE BASES FOR THE HOMOGENEOUS PARTS IN HOMOGENEOUS IDEALS AND GRADED MODULES OVER A POLYNOMIAL RING

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Abstract: In this paper, vector space bases for the homogeneous parts of homogeneous ideals and graded modules over a commutative polynomial ring are given using Gröbner bases.

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1. Introduction

Gröbner bases have originally been introduced by Buchberger for the algorithmic solution of some fundamental problems in commutative algebra [3]. Since then Gröbner bases evolved into a crucial concept in symbolic computations providing a uniform approach to solving a wide range of problems such as effective computations in residue class rings modulo polynomial ideals and in modules over polynomial rings, and calculating syzygies and graded resolutions for homogeneous ideals [2, 4, 6].

In this paper, we provide bases for the vector spaces corresponding to the homogeneous parts of homogeneous ideals or graded modules over a polynomial ring. Moreover, we show that the vector space basis for the homogenous part
of a homogeneous ideal is also a Gröbner basis for the ideal generated by the homogeneous part if the degree of the homogeneous part is large enough. The required notions and definitions are introduced in Section 2 and the results are provided in the Sections 3 and 4.

2. Graded Rings and Gröbner Bases for Modules

Let \( R = \mathbb{K}[x_0, x_1, \ldots, x_n] \) denote the commutative polynomial ring in \( n + 1 \) indeterminates over a field \( \mathbb{K} \). The monomials in \( R \) are denoted by \( x^u = x_0^{u_0} \cdots x_n^{u_n} \), where \( u = (u_0, \ldots, u_n) \in \mathbb{N}_0^{n+1} \). The total degree of a monomial \( x^u \) is the sum \( |u| = u_1 + \ldots + u_n \). The ring \( R \) has a natural grading in the sense that it admits a direct sum decomposition

\[
R = \bigoplus_{s \geq 0} R_s, \tag{1}
\]

where for each integer \( s \geq 0 \) the set \( R_s \) is the additive subgroup of \( R \) that consists of all homogeneous polynomials of degree \( s \) plus the zero polynomial, and the complex product \( R_sR_t = \{rr' \mid r \in R_s, r' \in R_t \} \) is contained in \( R_{s+t} \) for all \( s, t \geq 0 \). Note that \( R_0 = \mathbb{K} \) and \( R_0R_s \subseteq R_s \). Thus the subgroups \( R_s \) are also \( \mathbb{K} \)-vector spaces.

A module \( M \) over \( R \) (or any graded ring) is a graded module over \( R \) if it can be decomposed as

\[
M = \bigoplus_{t \in \mathbb{Z}} M_t, \tag{2}
\]

where each \( M_t \) is an additive subgroup of the additive group of \( M \) with the property that the complex product \( R_sM_t = \{rm \mid r \in R_s, m \in M_t \} \) lies in \( M_{s+t} \) for all \( s \geq 0 \) and all \( t \in \mathbb{Z} \). Each additive subgroup \( M_t \) is a module over \( R_0 = \mathbb{K} \) since \( R_0M_t \subseteq M_t \). Thus the subgroups \( M_t \) are also \( \mathbb{K} \)-vector spaces.

Let \( m \geq 1 \) be an integer. The free \( R \)-module \( R^m \) has the standard basis consisting of the canonical unit vectors \( e_1, \ldots, e_m \). The module \( R^m \) is graded over \( R \) with the (standard) grading

\[
(R^m)_t = (R_t)^m, \quad t \in \mathbb{Z}.
\]

Note that \( (R^m)_t = \{0\} \) if \( t \leq 0 \).

A monomial in \( R^m \) is an element of the form \( x^u e_i \) for some \( 1 \leq i \leq m \) and \( u \in \mathbb{N}_0^{n+1} \). Each element in \( R^m \) can be uniquely written as a \( \mathbb{K} \)-linear
combination of monomials. For instance, let $R = \mathbb{K}[x, y]$ and take the following element in $R^2$:

$$
\left( \frac{3x^2y + xy^2 + 1}{x^3y^2 + 2xy^5 - 3y} \right) = 3x^2ye_1 + xy^2e_1 + e_1 + x^3y^2e_2 + 2xy^5e_2 - 3ye_2.
$$

A monomial order on $R^m$ is a relation $\succ$ on the set of monomials in $R^m$ satisfying the following conditions: (1) $\succ$ is a total order, (2) $\succ$ is well-ordering, and (3) for any monomials $m, m' \in R^m$, $m \succ m'$ implies $x^u m \succ x^u m'$ for each monomial $x^u \in R$.

Any monomial order on $R$ can be extended to a monomial order on $R^m$. For this, an ordering of the standard basis vectors needs to fixed, say by downward ordering $e_1 > \cdots > e_m$. Then the TOP (term over position) extension of a monomial order $\succ$ on $R$, denoted by $\succ_{TOP}$, is defined as

$$
x^u e_i \succ_{TOP} x^v e_j :\iff x^u > x^v \lor (x^u = x^v \land i < j)
$$

and the POT (position over term) extension of $\succ$, denoted by $\succ_{POT}$, is given by

$$
x^u e_i \succ_{POT} x^v e_j :\iff i < j \lor (i = j \land x^u \succ x^v).
$$

For instance, if the lex order on $R = \mathbb{K}[x, y]$ with $x > y$ is extended to a TOP order on $R^2$, then

$$x^3y^2e_2 \succ_{TOP} 3x^2ye_1 \succ_{TOP} 2xy^5e_2 \succ_{TOP} xy^2e_1 \succ_{TOP} -3ye_2 \succ_{TOP} e_1.$$

However, if the lex order on $R$ is extended to a POT order on $R^2$, then

$$3x^2ye_1 \succ_{POT} -2xy^2e_1 \succ_{POT} e_1 \succ_{POT} x^3y^2e_2 \succ_{POT} 2xy^5e_2 \succ_{POT} -3ye_2.$$

Given a monomial order $\succ$ on $R^m$, each non-zero polynomial $f \in R^m$ has a unique leading term given by the largest involved term and denoted by $\text{Lt}_{\succ}(f)$; the corresponding leading monomial is referred to as $\text{lm}_{\succ}(f)$. Each submodule $M$ of $R^m$ has a leading submodule generated as a module by the leading terms of its elements,

$$\langle \text{lt}_{\succ}(M) \rangle = \langle \{\text{lt}_{\succ}(f) \mid f \in M\} \rangle.$$

A Gröbner basis for a submodule $M$ of $R^m$ w.r.t. a monomial order $\succ$ on $R^m$ is a finite subset $\mathcal{G}$ of $M$ with the property that the leading terms of the elements in $\mathcal{G}$ generate the leading submodule of $M$, i.e.,

$$\langle \text{lt}_{\succ}(M) \rangle = \langle \{\text{lt}_{\succ}(g) \mid g \in \mathcal{G}\} \rangle.$$
Each submodule of $R^m$ has a Gröbner basis which is generally not uniquely determined. However, a unique Gröbner basis $G$ called reduced Gröbner basis can be obtained, where the leading terms of the elements in $G$ are monic and for two distinct elements $g$ and $g'$ in $G$ no term involved in $g$ is divisible by the leading term of $g'$. Gröbner bases can be computed by Buchberger’s algorithm for submodules which is available by almost every computer algebra system. More details on Gröbner bases and modules can be found in [1, 5].

3. Vector Space Bases for the Homogeneous Parts in Homogeneous Ideals

An ideal $I$ in $R$ is homogeneous if for any element $f \in I$ the homogeneous components of $f$ are also in $I$. A homogeneous ideal $I$ in $R$ is a graded submodule of $R$ with the direct sum decomposition

$$I = \bigoplus_{t \in \mathbb{Z}} I_t,$$

where the homogeneous parts are given by $I_t = I \cap R_t$ for all $t \in \mathbb{Z}$. Note that $I_t = \{0\}$ if $t \leq 0$.

A $\mathbb{K}$-basis for the homogeneous part $R_t$ is given by all monomials of total degree $t$ and so we have

$$\dim_{\mathbb{K}} R_t = \binom{t + n - 1}{n - 1}, \quad t \in \mathbb{N}_0.$$ 

Thus the homogeneous part $I_t$ is a finite-dimensional vector space for each $t \in \mathbb{Z}$. The quotient module $R/I$ has also a graded module structure defined by

$$(R/I)_t = R_t/I_t = R_t/(I \cap R_t), \quad t \in \mathbb{Z}.$$ 

By the dimension formula,

$$\dim_{\mathbb{K}} R_t = \dim_{\mathbb{K}} I_t + \dim_{\mathbb{K}} (R/I)_t, \quad t \in \mathbb{N}_0,$$

and thus the quotient spaces $(R/I)_t$ are also finite dimensional. Moreover, the ideal of leading terms of $I$ fulfills

$$\dim_{\mathbb{K}} R_t/I_t = \dim_{\mathbb{K}} R_t/\langle \text{lt}_\succ (I) \rangle_t, \quad t \in \mathbb{N}_0,$$

where $\langle \text{lt}_\succ (I) \rangle_t = \langle \text{lt}_\succ (I) \rangle \cap R_t$.

Note that the additive subgroups $I_t$ are not ideals. Nonetheless, we can consider the ideal $\langle I_t \rangle$ generated by the elements in $I_t$. 
Proposition 1. Let $I$ be a homogeneous ideal in $R$. Let $\mathcal{G}$ be the reduced Gröbner basis for $I$ w.r.t. any monomial order $\succ$ on $R$ and let

$$\text{lm}_<(I_t) = \{\text{lm}_<(f) \mid f \in I_t\} = \{x^{a_1}, \ldots, x^{a_s}\}, \quad t \in \mathbb{N}_0.$$ 

Then a $\mathbb{K}$-basis for the vector space $I_t$ is given by the set of binomials

$$B_t = \{x^{a_1} - r_1, \ldots, x^{a_s} - r_s\},$$

where $r_i$ is the remainder of $x^{a_i}$ on division by $\mathcal{G}$ for $1 \leq i \leq s$.

The vector space $I_t$ is non-trivial if and only if $t \geq \min\{\deg(\text{lt}_<(g)) \mid g \in \mathcal{G}\}$. If $t \geq \max\{\deg(\text{lt}_<(g)) \mid g \in \mathcal{G}\}$, then the set $\text{lm}_<(I_t)$ consists of all monomial multiples of the elements in $\{\text{lt}_<(g) \mid g \in \mathcal{G}\}$ which are of total degree $t$, and $B_t$ is the reduced Gröbner basis for the homogeneous ideal $\langle I_t \rangle$.

Proof. Note that the reduced Gröbner basis $\mathcal{G}$ for a homogeneous ideal always consists of homogeneous polynomials and the remainder of a homogeneous polynomial $f$ divided by a set of homogeneous polynomials is either zero or homogeneous of the same total degree as $f$. It follows that if a monomial $x^{a_i}$ of total degree $t$ is divided by the Gröbner basis $\mathcal{G}$ giving the remainder $r_i$, then the difference $x^{a_i} - r_i$ will be a polynomial of total degree $t$ with leading term $x^{a_i}$ which lies in $I_t$, $1 \leq i \leq s$. Hence, $B_t$ is contained in $I_t$.

By rearranging and deleting duplicates, we may assume that $x^{a_1} \succ \ldots \succ x^{a_s}$. Let $f_i = x^{a_i} - r_i \in I_t$, $1 \leq i \leq s$, and claim that the elements $f_1, \ldots, f_s$ form a $\mathbb{K}$-basis of $I_t$. Indeed, consider a nontrivial linear combination $k_1f_1 + \ldots + k_tf_t$ with $k_i \in \mathbb{K}$ and take the smallest index $i$ such that $k_i \neq 0$. By the ordering of the leading monomials, there is nothing to cancel $k_if_i$ and so the linear combination is nonzero. Hence, $f_1, \ldots, f_s$ are linearly independent.

Moreover, let $U$ be the subspace of $I_t$ spanned by $f_1, \ldots, f_s$. Suppose $U$ is a proper subspace of $I_t$. Pick an element $f \in I_t \setminus U$ whose leading monomial is minimal. By definition, the leading monomial of $f$ equals the leading monomial of $f_i$ for some $i$ and $\text{lt}(f) = k\text{lt}(f_i)$ for some $k \in \mathbb{K}$. It follows that $f - k_f_i$ lies in $I_t$ and has a smaller leading monomial. Thus $f - k_f_i \in U$ by the minimality of the leading monomial of $f$ and so $f \in U$, a contradiction. Hence, $U = I_t$ and the claim follows.

Let $t \geq \min\{\deg(\text{lt}_<(g)) \mid g \in \mathcal{G}\}$. Then $t \geq \deg(g)$ for some element $g \in \mathcal{G}$ and thus the leading monomial of $g x^u$ with $|u| + \deg(g) = t$ lies in $\text{lm}_<(I_t)$. Hence, the vector space $I_t$ is non-trivial. Conversely, let $f \in I_t$. Then $f \in I$ and there is an element $g \in \mathcal{G}$ such that the leading term of $f$ is divisible by the leading term of $g$. Hence, $t \geq \deg(g)$. 

Finally, let \( t \geq \max \{ \deg(\text{lt}_{\succ}(g)) \mid g \in G \} \) and claim that \( B_t \) is the reduced Gröbner basis for \( \langle I_t \rangle \). Indeed, the set \( B_t \) generates \( I_t \) as a vector space and so generates also the ideal \( \langle I_t \rangle \). It remains to show that the leading terms of the elements in \( B_t \) generate the leading ideal \( \langle \text{lt}_{\succ}(\langle I_t \rangle) \rangle \). To this end, let \( f \in \langle I_t \rangle \).

Since \( f \in I_t \), there is a Gröbner basis element \( g \in G \) such that \( \text{lt}_{\succ}(g) \) divides \( \text{lt}_{\succ}(f) \). By the choice of \( t \), \( \deg(g) \leq t \) and so all monomial multiples of \( \text{lt}_{\succ}(g) \) of total degree \( t \) appear as leading terms in \( B_t \). But the leading term of \( f \) is also a monomial multiple of \( \text{lt}_{\succ}(g) \) (possibly of total degree larger than \( t \)) and so must be divisible by at least one monomial \( x^{a_i} \) where \( 1 \leq i \leq s \). This proves the claim.

The set \( \text{lm}_{\succ}(I_t) \) can be constructed from the reduced Gröbner basis \( G \) for \( I \) w.r.t. any monomial order \( \succ \) as follows. Starting with the empty set add for each element \( g \in G \) with leading term of total degree \( s \leq t \) all monomial multiples of \( \text{lt}_{\succ}(g) \) that are of degree \( t \), i.e., the set \( \{ \text{lt}_{\succ}(g)x^u \mid |u| = t - s \} \). This will give the set \( \text{lm}_{\succ}(I_t) \) in a finite number of steps, since the Gröbner basis is finite.

**Example 1.** Consider the homogeneous ideal

\[
I = \langle z^3 - yw^2, yz - xw, y^3 - x^2 z, xz^2 - y^2 w \rangle \subset K[x, y, z, w] = R.
\]

The above set is the reduced Gröbner basis w.r.t. the grevlex order \( \succ \) on \( R \) with \( x \succ y \succ z \succ w \). Thus

\[
\langle \text{lt}_{\succ}(I) \rangle = \langle z^3, yz, y^3, xz^2 \rangle.
\]

Note that all monomials in this generating set have degree greater than or equal to 2 and so \( I_i = \{0\} \) for \( i \leq 1 \). By the above remark, the leading ideals for \( I_2 \), \( I_3 \) and \( I_4 \) are generated as follows,

\[
\text{lm}_{\succ}(I_2) = \{yz\},
\]

\[
\text{lm}_{\succ}(I_3) = \{z^3, xyz, y^2 z, yzw, y^3, xz^2\},
\]

\[
\text{lm}_{\succ}(I_4) = \{xz^3, yz^3, z^4, wz^3, x^2 yz, y^3 z, yzw^2, xy^2 z, y^2 z^2, yz^2 w, xz^2, y^2 zw, xyzw, xy^3, y^4, y^3 w, x^2 z^2, xz^2 w\}.
\]

By the division algorithm, a vector space basis for \( I_2 \) is \( B_2 = \{yz - xw\} \), a vector space basis for \( I_3 \) is

\[
B_3 = \{z^3 - yw^2, xyz - x^2 w, y^2 z - xyw, yz^2 - xzw, yzw - xw^2, y^3 - x^2 z, xz^2 - y^2 w\},
\]
and a vector space basis for \( I_4 \) is

\[
B_4 = \begin{cases} 
  xz^3 - xyw^2, & yz^3 - y^2w^2, & z^4 - xw^3, \\
  z^3w - yw^3, & x^2yz - x^3w, & y^3z - xy^2w, \\
  yzw^2 - xw^3, & y^2z^2 - z^2y^w, & y^2z^2 - x^2w^2, \\
  y^2w - xzw^2, & x^3z^2 - x^2zw, & y^3z^2 - x^2w^2, \\
  y^3w - x^2zw, & y^3x^2 - x^2y^2w, & x^2w - y^2w^2
\end{cases}
\]

The homogeneous ideal \( \langle I_3 \rangle \) has the generating set \( B_3 \) and one can show that it is also the reduced Gröbner basis for this ideal w.r.t. the above grevlex order. However, a Gröbner basis for the ideal \( I \) w.r.t. the grevlex order with \( w \succ y \succ z \succ x \) is

\[
\{ z^4 - xw^3, yw^2 - z^3, yz - xw, y^2w - xz^2, y^3 - x^2z \}.
\]

It follows that a basis for the \( \mathbb{K} \)-vector space \( I_3 \) is

\[
B_3' = \{ yw^2 - z^3, x^2y - x^3w, yz^2 - xy^2w, y^2z - xzw, \\
y^2w - xz^2, y^3 - x^2z \}.
\]

Since \( B_3' \) differs from \( B_3' \) only by scalar multiples, it is also a generating set for the ideal \( \langle I_3 \rangle \). However, it is not the reduced Gröbner basis w.r.t. the grevlex order with \( w \succ y \succ z \succ x \) since the S-polynomial

\[
S \left( yw^2 - z^3, yzw - xw^2 \right) = z(yw^2 - z^3) - w(yzw - xw^2) = -z^4 + xw^3
\]

has the leading term \( z^4 \) which is not divisible by any of the leading terms in \( B_3' \). This confirms the necessity of the condition \( t \geq \max \{ \deg(\text{lt}(g)) \mid g \in G \} \) for \( B_t \) to form a Gröbner basis for the ideal \( \langle I_t \rangle \).

4. Vector Space Bases for the Homogeneous Parts in Graded Modules

Let \( m \geq 1 \) be an integer. The graded submodules \( M \) of \( R^m \) can be characterized as follows [5]:

- The standard grading on \( R^m \) induces a graded module structure on \( M \), which is given by \( M_t = (R^m)_t \cap M \) for all \( t \in \mathbb{Z} \).
• There are elements \( f_1, \ldots, f_r \) in \( R^m \), whose components are homogeneous polynomials of the same degree, such that \( M = \langle f_1, f_2, \ldots, f_r \rangle \subset R^m \) for all \( t \in \mathbb{Z} \).

• A reduced Gröbner basis for \( M \) (w.r.t. any monomial order on \( R^m \)) consists of vectors of homogeneous polynomials whose components have the same degree.

Using these facts, we obtain the following result.

**Proposition 2.** Let \( M \subset R^m \) be a graded module over \( R \). Let \( G \) be a Gröbner basis for \( M \) w.r.t. any monomial order \( \succ \) and let

\[
\text{lm}_\succ(M_t) = \{\text{lm}_\succ(f) \mid f \in M_t\} = \{x^{a_1}e_{i_1}, \ldots, x^{a_s}e_{i_s}\}, \quad t \in \mathbb{N}_0,
\]

where \( e_{i_1}, \ldots, e_{i_s} \) are unit vectors in \( R^m \). Then a \( \mathbb{K} \)-basis for the vector space \( M_t \) is given by

\[
B_t = \{x^{a_1}e_{i_1} - r_1, \ldots, x^{a_s}e_{i_s} - r_s\},
\]

where \( r_j \in R^m \) is the remainder of \( x^{a_j}e_{i_j} \) on division by \( G \) for each \( 1 \leq j \leq s \). The vector space \( M_t \) is non-trivial if and only if \( t \geq \min\{\deg(lt_\succ(g)) \mid g \in G\} \).

The proof is the same as that of Prop. 1 since all statements used there are also applicable to submodules of \( R^m \) (see for instance [5, Chapter 5, 2]). Moreover, the construction of \( \text{lm}_\succ(M_t) \) in the module case is analogous to that in the ideal case (see the remark after the proof of Prop. 1).

**Example 2.** Let \( R = \mathbb{K}[x, y, z, w] \) and consider the submodule \( M \) of \( R^4 \) generated by the vectors

\[
\begin{pmatrix}
y^2 \\
xz \\
yw \\
z^2
\end{pmatrix}, \quad \begin{pmatrix}
z \\
w \\
-\color{red}z \\
-\color{red}w
\end{pmatrix}, \quad \begin{pmatrix}
x \\
y \\
-\color{red}z \\
y \\
-\color{red}z
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

The generators are vectors of homogeneous monomials of the same total degree and so by the above remark, the module \( M \) is graded.

The reduced Gröbner basis for the module \( M \) w.r.t. the POT-extension of the grevlex order \( \succ \) with \( x \succ y \succ z \succ w \) is given by the columns of the matrix

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & z & x & y^2 \\
0 & yz - xw & xz^2 - y^2w & y^3 - x^2z & w & y & xz \\
x & -z^2 & yzw & -y^2z & 0 & -z & yw \\
y & -zw & z^3 & -xz^2 & 0 & -w & z^2
\end{pmatrix}.
\]
Thus the leading ideal of $M$ is

$\langle \text{lt}_{\mathcal{POT}}(M) \rangle = \langle y^2e_1, xe_1, ze_1, y^3e_2, xz^2e_2, yze_2, xe_3 \rangle$

and therefore

$\text{lm}_{\mathcal{POT}}(M_1) = \{xe_1, ze_1, xe_3\}$,

$\text{lm}_{\mathcal{POT}}(M_2) = \{x^2e_1, xyze_1, xze_1, xwe_1, y^2e_1, yze_1, z^2e_1,$

$\text{ }zwe_1, yze_2, x^2e_3, xyze_3, xze_3, xwe_3\}.$

The bases for the $\mathbb{K}$-vector spaces $B_1$ and $B_2$ for $M_1$ and $M_2$, respectively, are

$$B_1 = \left\{ \begin{pmatrix} 0 \\ 0 \\ x \\ y \end{pmatrix}, \begin{pmatrix} z \\ w \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ y \\ -z \\ -w \end{pmatrix} \right\},$$

$$B_2 = \left\{ \begin{pmatrix} x^2 \\ xy \\ -xz \\ -xw \end{pmatrix}, \begin{pmatrix} xy \\ y^2 \\ -yz \\ -yw \end{pmatrix}, \begin{pmatrix} xz \\ wx \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} xw \\ yw \\ -zw \\ -w^2 \end{pmatrix}, \begin{pmatrix} y^2 \\ xz \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} yz \\ yw \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} zw \\ w^2 \\ yz - xw \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$