ON THE DIOPHANTINE EQUATION

\[(2^k - 1)^x + (2^k)^y = z^2\]

WHERE k IS AN EVEN POSITIVE INTEGER

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Abstract: In this paper, we will apply the Catalan’s conjecture in order to give the solution in non-negative integer of the Diophantine equation \((2^k - 1)^x + (2^k)^y = z^2\) where \(k\) is an even positive integer.

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1. Introduction

In 2012 and 2013, Sroysang (see [3],[4]) studied respectively the Diophantine equations \(31^x + 32^y = z^2\) and \(7^x + 8^y = z^2\) where \(x, y, z\) are non-negative integers. He has used the Catalan’s conjecture [2] to solve such equations above. The former equation, he showed that there is no non-negative integer solution and the later one, there is an only one solution \((0, 1, 3)\).

Firstly, we have proved the equation \(15^x + 16^y = z^2\) by applying the same method as Sroysang did and it turns out that \((1, 0, 4)\) is an unique solution for this equation. It is naturally to be interested in the Diophantine equation of type \((2^k - 1)^x + (2^k)^y = z^2\) in general and eventually we are currently able to...
answer the solution of such an equation in case $k$ is an even positive integer. More details are provided in the Theorem 1 and Theorem 2.

2. Preliminaries

**Proposition 2.1.** [2](The Catalan’s conjecture) $(3, 2, 2, 3)$ is a unique solution $(a, b, x, y)$ for the Diophantine equation $a^x - b^y = 1$ where $a, b, x, y$ are integers with $\min\{a, b, x, y\} > 1$

**Lemma 2.2.** The Diophantine equation $1 + (2^k)y = z^2$ has no non-negative integer solution when $k$ is an even positive integer.

**Proof.** Let $x, y$ and $z$ be non-negative integers such that $1 + (2^k)y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. If $y = 1$, then we get $(z + 2^{\frac{k}{2}})(z - 2^{\frac{k}{2}}) = 1$. Thus $z + 2^{\frac{k}{2}} = 1$ and $z - 2^{\frac{k}{2}} = 1$. It follows that $z = 1$, which contradicts to the assumption. Now we are going to apply the Catalan’s conjecture in case of $y \geq 2$. Thus we have $(2^k)^y \geq (2^k)^2$. This yields $z > 4$. Since $\min\{z, 2^k, 2, y\} > 1$, we obtain that $(3, 2, 2, 3)$ is an only one solution of $z^2 - (2^k)^y = 1$ by Proposition 2.1. Then we have $k = 1$ which leads to a contradiction.

**Lemma 2.3.** The only solution $(x, z)$ in non-negative integers to the Diophantine equation $(2^k - 1)x + 1 = z^2$ is $(1, 2^{\frac{k}{2}})$ when $k$ is an even positive integer.

**Proof.** Let $x$ and $z$ be non-negative integers. If $x = 0$ then $z^2 = 2$ which is impossible. It is clear that $(1, 2^{\frac{k}{2}})$ is the solution of $(2^k - 1)x + 1 = z^2$ as $k$ is even. In case of $x \geq 2$. Then we have $(2^k - 1)^x \geq (2^k - 1)^2$. This implies that $z > 3$. Since $\min\{z, (2^k - 1), 2, x\} > 1$, so by the Proposition 2.1 we obtain $(3, 2, 2, 3)$ is the only one solution of $z^2 - (2^k - 1)^x = 1$. Then we have $2^k = 3$ which is impossible.

3. Result

**Theorem 3.1.** Let $k$ be an even integer at least 4. Then $(1, 0, 2^{\frac{k}{2}})$ is an only one solution in non-negative integers of the Diophantine equation $(2^k - 1)^x + (2^k)^y = z^2$. 
Proof. Let $k$ be an even integer at least 4 and $x, y, z$ be non-negative integers. We divide $y$ into two cases as follows.

**Case 1:** $y = 0$. By Lemma 2.3, we have $(1, 2^k)$ is the solution of the equation $(2^k - 1)x + 1 = z^2$. It follows that $(1, 0, 2^k)$ is the solution of the equation $(2^k - 1)x + (2^k)y = z^2$.

**Case 2:** $y \geq 1$.

Subcase 2.1 $x = 0$. It follows by Lemma 2.2 that the equation $1 + (2^k)y = z^2$ has no non-negative solution.

Subcase 2.2 $x \geq 1$. Since $2^k - 1$ is odd, $(2^k - 1)x + (2^k)y$ is odd. This implies that $z^2$ is odd and finally we get $z$ is odd. Then $z^2 \equiv 1 \pmod{4}$. Since $z^2 - (2^k)y \equiv 1 \pmod{4}$ it follows that $(2^k - 1)x \equiv 1 \pmod{4}$. But $(2^k - 1)x \equiv 3^x \pmod{4}$. Thus $3^x \equiv 1 \pmod{4}$ and it is not hard to see that $x$ is even.

Now, let us consider 

$$ (2^k)^y = z^2 - (2^k - 1)^{2n} \tag{1} $$

when $n$ is a positive integer and then factor the equation (1) as follow.

$$ (2^k)^y = (z + (2^k - 1)^n)(z - (2^k - 1)^n) \tag{2} $$

Let $z + (2^k - 1)^n = 2^\alpha$ and $z - (2^k - 1)^n = 2^\beta$ for some non-negative integers $\alpha$ and $\beta$ such that $\alpha > \beta$ and $\alpha + \beta = ky$.

Since $2(2^k - 1)^n = (z + (2^k - 1)^n) - (z - (2^k - 1)^n) = 2^\alpha - 2^\beta = 2^\beta(2^{\alpha-1} - 1)$ so $\beta = 1$ and $\alpha \geq 2$. Then we have $(2^k - 1)^n = 2^{\alpha-1} - 1$. If $\alpha = 2$ then $y = \frac{3}{k}$ which is impossible.

Next we will consider two cases which are $n \geq 2$ and $n = 1$ under the condition that $\alpha \geq 3$ so that we can reach a contradiction as follows. In case of $n \geq 2$, then we get the contradiction by applying the Proposition 2.1 to the equation $2^{\alpha-1} - (2^k - 1)^n = 1$. For the other case we have $2^k - 1 = 2^{\alpha-1} - 1$, which leads to get $k = \frac{2}{y - 1}$, impossibly.

**Theorem 3.2.** The Diophantine equation $3^x + 4^y = z^2$ has exactly two non-negative integer solutions which are $(1, 0, 2)$ and $(2, 2, 5)$.

Proof. Let $x, y$ and $z$ be non-negative integers. We are going to consider two cases depending to $y$ as follows.
Case1: $y = 0$. Then $3^x = z^2 - 1 = (z + 1)(z - 1)$. Let $z + 1 = 3^\alpha$ and $z - 1 = 3^\beta$ when $\alpha$ and $\beta$ are non-negative integers such that $\alpha > \beta$ and $\alpha + \beta = x$. Thus $2 = 3^{\alpha} - 3^{\beta} = 3^{\beta}(3^{\alpha-\beta} - 1)$. It follows that $\beta = 0$ and $\alpha = 1$. Hence $x = 1$ and eventually we get $z = 2$.

Case2: $y \geq 1$.

Subcase2.1: $x = 0$. Then we have $(z + 2^y)(z - 2^y) = 1$ and we can conclude that $z + 2^y = 1$ and $z - 2^y = 1$. Hence $z = 1$ which is impossible.

Subcase2.2: $x \geq 1$. Since $z^2 = 3^x + 4^y \equiv 1 (\text{mod } 2)$ so $z$ is odd. This follows that $3^x = z^2 - 4^y \equiv 1 (\text{mod } 4)$. Thus $x$ is even. Then we have $2^{2y} = z^2 - 3^{2n} = (z - 3^n)(z + 3^n)$ when $n$ is a positive integer. This concludes that $z + 3^n = 2^{\lambda}$ and $z - 3^n = 2^\gamma$ for some non-negative integers $\lambda, \gamma$ with the conditions that $\lambda > \gamma$ and $\lambda + \gamma = 2y$. Since $2(3^n) = (z + 3^n) - (z - 3^n) = 2^\gamma(2^{\lambda-\gamma} - 1)$ so $\gamma = 1$ and $\lambda \geq 2$. Then we have $3^n = 2^{\lambda-1} - 1$. If $\lambda = 2$ then $n = 0$ which is impossible.

Next we will consider two cases which are $n \geq 2$ and $n = 1$ under the condition that $\lambda \geq 3$. In case of $n \geq 2$ we get the contradiction by applying Proposition 2.1 to the equation $2^{\lambda-1} - 3^n = 1$. In the other case, we can finally obtain the solution $(2, 2, 5)$ under the condition that $\lambda + \gamma = 2y$.

4. Remark

In the work of Chotchaisthit [1] in 2013, he has answered the solutions of the Diophantine equation $p^x + (p + 1)^y = z^2$ where $x, y$ and $z$ are non-negative integers and $p$ is a Mersenne prime. However, he hasn’t given the solutions when $p$ is not Mersenne prime and now by elementary method it is still an open problem. From Theorem 1, we have answered some of such a problem previously. In Theorem 2, we therefore give two solutions in case of $p = 3$.

References


