A NOTE ON PERFECT FUZZY MATCHING

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Abstract: The notion of matching in a fuzzy graph could be defined using the concept of effective edges \cite{8} or by fractional matching \cite{4}. In this paper, we derive a necessary condition for a fuzzy graph on a cycle or a complete graph or a stargraph to have a perfect fuzzy matching. Also we discuss perfect fuzzy matching on strong regular fuzzy graphs.

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1. Introduction

Zadeh introduced the notion of Fuzzy sets and Fuzzy relations to deal with the problems of uncertainty in real physical world. In 1975, Rosenfeld \cite{5} introduced the concept of fuzzy graphs. Using the concept of effective edges, Somasundaram \cite{8} defined matching in a fuzzy graph. This matching is defined only for those graphs having effective edges. Ramakrishnan P.V and Vaidyanathan M \cite{4} introduced matching in a fuzzy graph using the concept of fractional matching. The notion of fractional matching given by Scheinerman \cite{1}, in 1997, involves the vertex weight, the edge weight and the incidence of

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edges. In this paper, we derive the necessary condition for the fuzzy graph on a cycle or a complete graph or a star graph to have perfect fuzzy matching. Further, we prove that a strong regular fuzzy graph will have no perfect fuzzy matching.

2. Preliminaries

**Definition 2.1.** A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ with $\mu(u, v) \leq \sigma(u) \land \sigma(v)$, $\forall u, v \in V$, where $V$ is a finite nonempty set and $\land$ denote minimum.

**Definition 2.2.** A fuzzy graph $G = (\sigma, \mu)$ is defined to be a strong fuzzy graph if $\mu(u, v) = \sigma(u) \land \sigma(v)$, $\forall (u, v) \in E$.

**Definition 2.3.** A fuzzy graph $G = (\sigma, \mu)$ is defined to be a complete fuzzy graph if $\mu(u, v) = \sigma(u) \land \sigma(v)$, $\forall u, v \in V$.

**Definition 2.4.** Let $G = (\sigma, \mu)$ be a fuzzy graph on $G^* = (V, E)$. The fuzzy degree of a node $u \in V$ is defined as $(fd)(u) = \sum_{u \neq v, v \in V} \mu(u, v)$.

$G$ is said to be regular fuzzy graph if each vertex has same fuzzy degree. If $(fd)(v) = k$, $\forall v \in V, G$ is said to be $k$–regular fuzzy graph.

**Definition 2.5.** Let $G = (\sigma, \mu)$ be a fuzzy graph on $(V, E)$, where $V$ is the vertex set and $E$ is the set of edges with non-zero weights. A subset $M$ of $E$ is called a Fuzzy matching if for each vertex $u$, we have $\sum_{v \in V} \mu(u, v) \leq \sigma(u)$.

**Example 2.6.** Let $G = (\sigma, \mu)$ be a fuzzy graph on $(V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1, e_2, e_3, e_4\}$ with $e_1 = v_1v_2, e_2 = v_2v_3, e_3 = v_3v_4$ and $e_4 = v_4v_1$

$$
\begin{align*}
\sigma(v_1) &= 1, \sigma(v_2) = 0.5, \sigma(v_3) = 0.4, \sigma(v_4) = 0.7 \\
\mu(e_1) &= 0.3, \mu(e_2) = 0.4, \mu(e_3) = 0.2, \mu(e_4) = 0.5 \\
\sum_{v_2 \in V, (v_1, v_2) \in M} \mu(v_1, v_2) &= 0.3 + 0.5 = 0.8 \leq 1 = \sigma(v_1).
\end{align*}
$$
\[
\sum_{v_3 \in V, (v_2, v_3) \in M} \mu(v_2, v_3) = 0.4 + 0.3 = 0.7 \leq 0.5 = \sigma(v_2).
\]
\[
\sum_{v_1 \in V, (v_1, v_4) \in M} \mu(v_1, v_4) = 0.5 + 0.2 = 0.7 = \sigma(v_4).
\]
Thus \(M = \{e_1, e_2, e_4\}\) is a fuzzy matching in \(G\).

**Definition 2.7.** A fuzzy matching \(M\) is called a Perfect Fuzzy Matching if for each vertex \(u, \sum_{v \in V, (u, v) \in M} \mu(u, v) = \sigma(u)\).

**Example 2.8.** Let \(G = (\sigma, \mu)\) be a fuzzy graph on \((V, E)\) where \(V = \{v_1, v_2, v_3, v_4\}\) and \(E = \{e_1, e_2, e_3, e_4, e_5\}\) with \(e_1 = v_1 v_2, e_2 = v_2 v_3, e_3 = v_3, e_4 = v_4 v_1\) and \(e_5 = v_4 v_2\)

\[
\begin{align*}
\sigma(v_1) &= 1, \sigma(v_2) = 0.9, \sigma(v_3) = 0.8, \sigma(v_4) = 0.9 \\
\mu(e_1) &= 0.4, \mu(e_2) = 0.5, \mu(e_3) = 0.3, \mu(e_4) = 0.6, \mu(e_5) = 0.8
\end{align*}
\]

Here \(\sum_{v_2 \in V, (v_1, v_2) \in M} \mu(v_1, v_2) = 0.4 + 0.6 = 1 = \sigma(v_1)\).
\(\sum_{v_3 \in V, (v_2, v_3) \in M} \mu(v_2, v_3) = 0.4 + 0.5 = 0.9 = \sigma(v_2)\).
\(\sum_{v_3 \in V, (v_3, v_4) \in M} \mu(v_3, v_4) = 0.5 + 0.3 = 0.8 = \sigma(v_3)\).
\(\sum_{v_4 \in V, (v_1, v_4) \in M} \mu(v_1, v_4) = 0.3 + 0.6 = 0.9 = \sigma(v_4)\).

Thus \(M = \{v_1 v_2, v_2 v_3, v_3 v_4, v_4 v_1\}\) is a perfect fuzzy matching in \(G\).

**Definition 2.9.** Let \(G = (\sigma, \mu)\) be a fuzzy graph and \(M\) be a fuzzy matching. Then fuzzy matching number \(\Gamma(G)\) is defined to be \(\Gamma(G) = \sum_{(u, v) \in M} \mu(u, v)\).

**Example 2.10.** In example 2.6, \(\Gamma(G) = 1.2\). In example 2.8, \(\Gamma(G) = 1.8\).

**Theorem 2.11.** [6] Let \(G = (\sigma, \mu)\) be a fuzzy graph on \(K_{n,n}\) with bipartition \((X, Y)\) where \(X = \{u_1, u_2, \cdots, u_n\}\) and \(Y = \{v_1, v_2, \cdots, v_n\}\). Then \(G\) is strong regular if and only if

1. \(\sigma(v_j) = k\)
2. \(\sigma(u_i) = k\).
3. \(\mu(u_i, v_j) = k\), for \(i, j = 1, 2, \cdots n\) and for some \(k\).

### 3. Perfect Fuzzy Matching

In this section we discuss some necessary conditions for some of the fuzzy graphs to have a perfect matching.

**Theorem 3.1.** Let \(G = (\sigma, \mu)\) be a regular fuzzy graph on the cycle \((V, E)\). If \(\sigma(u) = k\), which is constant for all \(u \in V\) and \(\mu(u, v) = \frac{k}{2}\) for all \((u, v) \in E\), then \(E\) is a perfect fuzzy matching for \(G\).
Proof
Since only two edges are incident with each vertex for cycles, for any vertex \( v \in V \)
\[
\sum_{v \in V, (u,v) \in E} \mu(u,v) \text{ where } v, w \in V = \mu(u,v) + \mu(u,w)
\]
\[
= \frac{k}{2} + \frac{k}{2} = k = \sigma(u)
\]

Then \( E \) is a perfect fuzzy matching in \( G \).

The converse of the above theorem need not be true. This can be seen using the following example.

**Example 3.2.** Let \( G = (\sigma, \mu) \) be a fuzzy graph on \((V,E)\) where \( V = \{v_1, v_2, v_3, v_4\} \) and \( E = \{e_1, e_2, e_3, e_4\} \) where \( e_1 = v_1v_2, e_2 = v_2v_3, e_3 = v_3, v_4 \) and \( e_4 = v_4v_1 \)
Define \( \sigma(v_1) = 0.6, \sigma(v_2) = 0.6, \sigma(v_3) = 0.5, \sigma(v_4) = 0.5 \) and
\( \mu(e_1) = 0.4, \mu(e_2) = 0.2, \mu(e_3) = 0.3, \mu(e_4) = 0.2. \)
Then the graph \( G \) has perfect fuzzy matching but the conditions of the above theorem are not satisfied.

**Theorem 3.3.** Let \( G = (\sigma, \mu) \) be a complete fuzzy graph \( K_n \) on \((V,E)\). If \( \sigma(u) = k \) which is constant for all \( u \in V \) and \( \mu(u,v) = \left[ \frac{k}{n} \right] = k_1 \), for all \((u,v)\) on the cycle \( C_n \) and \( \mu(u,v) = \frac{k - 2k_1}{n-3} \) for all the interior edges \((u,v) \in E\), then \( E \) is a perfect fuzzy matching for \( G \).

Proof
For any complete fuzzy graph \( K_n \), two edges are incident with each vertex of the cycle and remaining \((n-3)\) edges are incident with interior vertices. Hence
\[
\sum_{v \in V, (u,v) \in E} \mu(u,v) = 2k_1 + (n-3)(\frac{k - 2k_1}{n-3})
\]
\[
= 2k_1 + k - 2k_1
\]
\[
= k
\]
\[
= \sigma(u), \text{ for each vertex } u.
\]

Therefore \( E \) is a perfect fuzzy matching for \( G \).

The converse of the above theorem is not true. This can be seen using the following example.
Example 3.4. Let $G = (\sigma, \mu)$ be a fuzzy graph on $(V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

where $e_1 = v_1v_2, e_2 = v_2v_3, e_3 = v_3, e_4 = v_4v_1, e_5 = v_4v_2$ and $e_6 = v_1v_3$

Define $\sigma(v_1) = 0.9, \sigma(v_2) = 0.8, \sigma(v_3) = 0.7, \sigma(v_4) = 0.8$ and $\mu(e_1) = 0.2, \mu(e_2) = 0.3, \mu(e_3) = 0.1, \mu(e_4) = 0.4, \mu(e_5) = 0.3, \mu(e_6) = 0.3$.

Then $E$ is a perfect matching for $G$ but the conditions of the above theorem are not satisfied.

Theorem 3.5. Let $G = (\sigma, \mu)$ be a strong fuzzy graph on the star graph $S_n = (V, E)$ with $V = \{v, v_1, v_2, \ldots, v_{n-1}\}$. If $\sigma(v_i) = k, \forall i = 1, 2, \cdots, (n - 1)$ and if $\sigma(v) = (n - 1)k$, then $E$ is a perfect fuzzy matching in $G$.

Proof
Since $G$ is strong,

$$\mu(v, v_i) = \sigma(v) \land \sigma(v_i), \forall v_i \in V$$

$$= (n - 1)k \land k$$

$$= k$$

Now, for any $v \in V$,

$$\sum_{v_i \in V, (v, v_i) \in E} \mu(v, v_i) = \sum_{(v, v_i) \in E} k$$

$$= (n - 1)k, \text{ since } (n - 1) \text{ edges are incident with } v.$$  

$$= \sigma(v)$$

Therefore, $E$ is a perfect fuzzy matching for $G$.

The converse of the above theorem is also not true. This can be seen from the following example.

Example 3.6. Let $G = (\sigma, \mu)$ be a fuzzy graph on $(V, E)$ where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$ where $e_1 = vv_1, e_2 = vv_2, e_3 = vv_3, e_4 = vv_4$ and $e_5 = vv_5$.

Define $\sigma(v_1) = 0.2, \sigma(v_2) = 0.1, \sigma(v_3) = 0.2, \sigma(v_4) = 0.1, \sigma(v_5) = 0.2, \sigma(v) = 0.8$

$\mu(e_1) = 0.2, \mu(e_2) = 0.1, \mu(e_3) = 0.2, \mu(e_4) = 0.1, \mu(e_5) = 0.2$.

Then $E$ is a perfect matching for $G$ but the conditions of the above theorem are not satisfied.

The following theorem establishes that a strong regular fuzzy graph need not have a perfect fuzzy matching.
Theorem 3.7. If $G = (\sigma, \mu)$ is a strong regular fuzzy graph on $(V, E)$ with each vertex is of degree at least two, then $E$ is not a perfect fuzzy matching in $G$.

Proof If possible, let $E$ be a perfect fuzzy matching for $G$. Then
\[ \sum_{v \in V, (u,v) \in E} \mu(u, v) = \sigma(u), \forall u \in V, \quad (1) \]
by definition.

Since $G$ is regular, $(fd)(u) = constant = k$(say), $\forall u \in V$.

Therefore, $\sum_{v \in V, (u,v) \in E} \mu(u, v) = k, \forall u \in V$, which imply
\[ \sigma(u) = k, \forall u \in V. \] (Using (1).)

Now from (1) $\sum_{v \in V, (u,v) \in E} \mu(u, v) = k$ $\Rightarrow$ $\mu(u, v) < k$, since at least two edges incident with $u$, for some $(u, v)$.

Since $G$ is strong, $\sigma(u) \land \sigma(v) < k$ $\Rightarrow$ $k \land k < k$ $\Rightarrow$ $k < k$, which is a contradiction.

Therefore, $E$ is not perfect fuzzy matching for $G$.

In particular, we have the following

Corollary 3.8. If $G = (\sigma, \mu)$ is strong regular fuzzy graph on $K_{n,n}$, then $E$ is not a perfect fuzzy matching.

Proof Suppose $E$ is a perfect fuzzy matching for $G$.

Then $\sum_{v \in V, (u,v) \in E} \mu(u, v) = \sigma(u)$, for each $u \in V$.

Let $\sigma(u) = k; \forall u \in X$.

Therefore, $\sum_{v \in V} \mu(u, v) = k, \forall u \in X$.

Hence $\mu(u, v) < k$, for some $v \in V$, which is a contradiction to theorem 2.11.

Therefore, $E$ is not a perfect fuzzy matching for $G$.

References


