CONTINUOUS FUNCTIONS

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Abstract: The aim of the present paper is to introduce the different types of $(\tau_1, \tau_2)^*\text{-}g^*r$ continuous functions and study their basic properties in bitopological spaces.

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1. Introduction

In 1963, the concept of bitopological space was introduced by Kelly [1]. Considerable effort had been expended in obtaining appropriate generalizations of standard topological properties to bitopological category. Kelly initiated the study of separation properties for bitopological spaces. This is the initiative of the study of bitopological spaces and generalized closed sets in bitopological spaces. There are several works dedicated to the investigation of pairs of topologies on the same set (i.e) bitopologies, most of them deal with the theory

Fukutake also introduced generalized closed sets and pairwise generalized closure operator [5] in bitopological spaces in 1986. For any subset \( A \subseteq X \), the intersection (resp. union) of all \( \tau_i \)-closed sets containing \( A \) (resp. \( \tau_i \)-open sets contained in \( A \)) is called the \( \tau_i \)-closure (resp. \( \tau_i \)-interior) of \( A \), denoted by \( \tau_i \-cl(A) \) {resp. \( \tau_i \-int(A) \)}. He defined a set \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) to be \( \tau_i \tau_j \)-generalized closed set (briefly \( \tau_i \tau_j \)-g closed) if \( \tau_j \-cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-open in \( X \), \( i,j = 1, 2 \) and \( i \neq j \). Also, he defined a new closure operator and strongly pairwise \( T_1^2 \)-space. Semi generalized closed sets and generalized semi closed sets are extended to bitopological settings by Khedr and Al-saadi. In 2002, Fukutake, Sundaram and Sheik John [6] introduced the concept of \( w \)-closed sets, \( w \)-open sets and \( w \)-continuous maps in bitopological spaces. The \( g^* \)-closed sets in bitopological spaces were introduced by Sheik and Sundaram [7] in 2004. Lellis Thivagar and Ravi [8] introduced a new type of generalized sets called \((1,2)^*\)-semi generalized closed sets and a new class of generalized functions called \((1,2)^*\)-semi generalized continuous maps in 2006.

On the other hand, Kannan et. al. [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] introduced the various concepts on closed sets in a bitopological space.

In the present paper, we introduce the concepts of \((\tau_1, \tau_2)^*\)-generalized star regular continuous functions, \((\tau_1 \tau_2)^*-g^*r\) homeomorphisms, \((\tau_1, \tau_2)^*\)-GRO compact spaces and study their basic properties in bitopological spaces.

2. Preliminaries

Let \((X, \tau_1, \tau_2)\) or simply \( X \) denote a bitopological space. A subset \( A \) of \( X \) is \( \tau_1 \tau_2 \)-open if \( A = S \cup T \) where \( S \in \tau_1, T \in \tau_2 \) and \( A \) is \( \tau_1 \tau_2 \)-closed if \( A^C \) is \( \tau_1 \tau_2 \)-open. For any subset \( A \subseteq X \), the intersection (resp. union) of all \( \tau_1 \tau_2 \)-closed sets containing \( A \) (resp. \( \tau_1 \tau_2 \)-open sets contained in \( A \)) is called the \( \tau_1 \tau_2 \)-closure (resp. \( \tau_1 \tau_2 \)-interior) of \( A \), denoted by \( \tau_1 \tau_2 \-cl(A) \) {resp. \( \tau_1 \tau_2 \-int(A) \)}.

**Definition 2.1.** A set \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is called

(a) \( \tau_1 \tau_2 \)-semi open [4] if \( A \subseteq \tau_2 \-cl[\tau_1 \-int(A)] \).

(b) \( \tau_1 \tau_2 \)-semi closed [4] if \( X - A \) is \( \tau_1 \tau_2 \)-semi open.

Equivalently, a set \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is called \( \tau_1 \tau_2 \)-semi closed if \( \tau_2 \-int[\tau_1 \-cl(A)] \subseteq A \).
(c) $(\tau_1, \tau_2)^*$-semi open [8] if $A \subseteq \tau_1\tau_2-cl[\tau_1\tau_2-int(A)]$.

(d) $(\tau_1, \tau_2)^*$-semi closed [8] if $X - A$ is $(\tau_1, \tau_2)^*$-semi open.

Equivalently, a set $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $(\tau_1, \tau_2)^*$-semi closed if $\tau_1\tau_2-int[\tau_1\tau_2-cl(A)] \subseteq A$

(e) $(\tau_1, \tau_2)^*$-generalized closed $\{(\tau_1, \tau_2)^*-g \text{ closed } \}$ if $\tau_1\tau_2-cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1\tau_2$-open in $X$.

(f) $(\tau_1, \tau_2)^*$-generalized open $\{(\tau_1, \tau_2)^*-g \text{ open } \}$ if $X - A$ is $(\tau_1, \tau_2)^*$-g closed.

For any subset $A \subseteq X$, the intersection (resp. union) of all $(\tau_1, \tau_2)^*$-semi closed sets containing $A$ (resp. $(\tau_1, \tau_2)^*$-semi open sets contained in $A$) is called the $(\tau_1, \tau_2)^*$-semi closure (resp. $(\tau_1, \tau_2)^*$-semi interior) of $A$, denoted by $(\tau_1, \tau_2)^*$-cl$(A)$ (resp. $(\tau_1, \tau_2)^*$-int$(A)$).

**Definition 2.2.** A set $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called

(a) $(\tau_1, \tau_2)^*$-semi generalized closed $\{(\tau_1, \tau_2)^*-sg \text{ closed } \}$ if $(\tau_1, \tau_2)^*$-scl$(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $(\tau_1, \tau_2)^*$-semi open in $X$.

(b) $(\tau_1, \tau_2)^*$-semi generalized open $\{(\tau_1, \tau_2)^*-sg \text{ open } \}$ if $X - A$ is $(\tau_1, \tau_2)^*$-sg closed.

(c) $\tau_1\tau_2$-regular closed [8] if $\tau_1-cl[\tau_2-int(A)] = A$.

(d) $\tau_1\tau_2$-regular open [8] if $\tau_1-int[\tau_2-cl(A)] = A$.

(e) $(\tau_1, \tau_2)^*$-regular closed [8] if $\tau_1\tau_2- cl[\tau_1\tau_2- \text{int}(A)] = A$.

(f) $(\tau_1, \tau_2)^*$-regular open [8] if $X - A$ is $(\tau_1, \tau_2)^*$-regular closed.

The intersection (resp. union) of all $(\tau_1, \tau_2)^*$-regular closed sets containing $A$ (resp. $(\tau_1, \tau_2)^*$-regular open sets contained in $A$) is called the $(\tau_1, \tau_2)^*$-regular closure (resp. $(\tau_1, \tau_2)^*$-regular interior) of $A$, denoted by $(\tau_1, \tau_2)^*$-rcl$(A)$ (resp. $(\tau_1, \tau_2)^*$-rint$(A)$). The set of all $(\tau_1, \tau_2)^*$-regular closed sets in $X$ is denoted by $(\tau_1, \tau_2)^*$-R.C $(X, \tau_1, \tau_2)$. The set of all $(\tau_1, \tau_2)^*$-regular open sets in $X$ is denoted by $(\tau_1, \tau_2)^*$-R.O $(X, \tau_1, \tau_2)$. $A^C$ denotes the complement of $A$ in $X$ unless explicitly stated.

**Definition 2.3.** A set $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called
(a) \((\tau_1, \tau_2)^*\)-regular generalized closed \(((\tau_1, \tau_2)^*\text{-rg} \text{ closed})\) [21] in \(X\) if \((\tau_1, \tau_2)^*\text{-cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \((\tau_1, \tau_2)^*\)-regular open in \(X\).

(b) \((\tau_1, \tau_2)^*\)-regular generalized open \(((\tau_1, \tau_2)^*\text{-rg} \text{ open})\) [21] in \(X\) if \(F \subseteq (\tau_1, \tau_2)^*\text{-int}(A)\) whenever \(F \subseteq A\) and \(F\) is \((\tau_1, \tau_2)^*\)-regular closed in \(X\).

(c) \((\tau_1, \tau_2)^*\)-regular generalized star closed \(((\tau_1, \tau_2)^*\text{-rg}^* \text{ closed})\) [22] in \(X\) if \((\tau_1, \tau_2)^*\text{-rcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \((\tau_1, \tau_2)^*\)-regular open in \(X\).

(d) \((\tau_1, \tau_2)^*\)-regular generalized star open \(((\tau_1, \tau_2)^*\text{-rg}^* \text{ open})\) [22] in \(X\) if \(F \subseteq (\tau_1, \tau_2)^*\text{-rint}(A)\) whenever \(F \subseteq A\) and \(F\) is \((\tau_1, \tau_2)^*\)-regular closed in \(X\).

(e) \((\tau_1, \tau_2)^*\)-generalized star regular closed \(((\tau_1, \tau_2)^*\text{-g}^* \text{ r} \text{ closed})\) [23] in \(X\) if \((\tau_1, \tau_2)^*\text{-rcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\tau_1\tau_2\)-open in \(X\).

(f) \((\tau_1, \tau_2)^*\)-generalized star regular open \(((\tau_1, \tau_2)^*\text{-g}^* \text{ r} \text{ open})\) [23] in \(X\) if \(F \subseteq (\tau_1, \tau_2)^*\text{-rint}(A)\) whenever \(F \subseteq A\) and \(F\) is \(\tau_1\tau_2\)-closed in \(X\).

**Definition 2.4.** A function \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is called

(a) \((\tau_1, \tau_2)^*\text{-g} \text{ continuous} [8]\) in \(X\) if the inverse image of each \(\sigma_1\sigma_2\)-closed set is \((\tau_1, \tau_2)^*\text{-g} \text{ closed in } X\).

(b) \((\tau_1, \tau_2)^*\text{-gs} \text{ continuous} [24]\) in \(X\) if the inverse image of each \(\sigma_1\sigma_2\)-closed set is \((\tau_1, \tau_2)^*\text{-gs} \text{ closed in } X\).

(c) \((\tau_1, \tau_2)^*\text{-rg} \text{ continuous} [21]\) in \(X\) if the inverse image of each \(\sigma_1\sigma_2\)-closed set is \((\tau_1, \tau_2)^*\text{-rg} \text{ closed in } X\).

(d) almost \((\tau_1, \tau_2)^*\text{-g} \text{ continuous} [21]\) in \(X\) if the inverse image of each \((\sigma_1, \sigma_2)^*\)-regular closed set is \((\tau_1, \tau_2)^*\text{-g} \text{ closed in } X\).

(e) almost \((\tau_1, \tau_2)^*\text{-gs} \text{ continuous in } X\) if the inverse image of each \((\sigma_1, \sigma_2)^*\)-regular closed set is \((\tau_1, \tau_2)^*\text{-gs} \text{ closed in } X\).

(f) almost \((\tau_1, \tau_2)^*\text{-rg} \text{ continuous} [21]\) in \(X\) if the inverse image of each \((\sigma_1, \sigma_2)^*\)-regular closed set is \((\tau_1, \tau_2)^*\text{-rg} \text{ closed in } X\).

**Definition 2.5.** A function \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is called

(a) \((\tau_1, \tau_2)^*\text{-g} \text{ irresolute} [8]\) in \(X\) if the inverse image of each \((\sigma_1, \sigma_2)^*\text{-g} \text{ closed set is } (\tau_1, \tau_2)^*\text{-g} \text{ closed in } X\).
(b) \((\tau_1, \tau_2)^*g^*r\) irresolute [24] in \(X\) if the inverse image of each \((\sigma_1, \sigma_2)^*g^*s\) closed set is \((\tau_1, \tau_2)^*g^*s\) closed in \(X\).

(c) \((\tau_1, \tau_2)^*r^g\) irresolute [21] in \(X\) if the inverse image of each \((\sigma_1, \sigma_2)^*r^g\) closed set is \((\tau_1, \tau_2)^*r^g\) closed in \(X\).

3. \((\tau_1, \tau_2)^*g^*r\) Continuous Functions

Definition 3.1. A function \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is called

(a) \((\tau_1, \tau_2)^*g^*r\) irresolute

(b) \((\tau_1, \tau_2)^*g^*r\) irresolute in \(X\) if the inverse image of each \((\sigma_1, \sigma_2)^*g^*r\) continuous in \(X\) if the inverse image of each \((\sigma_1, \sigma_2)^*g^*r\) closed set is \((\tau_1, \tau_2)^*g^*r\) continuous in \(X\).

Example 3.2. Let \(X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}, \tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}, Y = \{a, b, c\}, \sigma_1 = \{\phi, Y, \{a\}\}, \sigma_2 = \{\phi, Y, \{b\}\}. Let f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) be the identity function. Then, \(f\) is \((\tau_1, \tau_2)^*r^g\) continuous and \((\tau_1, \tau_2)^*g^*r\) irresolute.

Theorem 3.3. Every \((\tau_1, \tau_2)^*g^*r\) continuous function is \((\tau_1, \tau_2)^*g^*r\) continuous.

Proof. Let \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) be \((\tau_1, \tau_2)^*g^*r\) continuous and \(F\) be \(\sigma_1, \sigma_2\)-closed in \(Y\). Since \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) be \((\tau_1, \tau_2)^*g^*r\) continuous, \(f^{-1}(F)\) is \((\tau_1, \tau_2)^*g^*r\) closed in \(X\). Since every \((\tau_1, \tau_2)^*g^*r\) closed set is \((\tau_1, \tau_2)^*g^*r\) closed, \(f^{-1}(F)\) is \((\tau_1, \tau_2)^*g^*r\) closed in \(X\). Therefore, \(f\) is \((\tau_1, \tau_2)^*g^*r\) continuous.

Corollary 3.4. Every \((\tau_1, \tau_2)^*g^*r\) continuous function is \((\tau_1, \tau_2)^*g^*s\) continuous \{resp. \((\tau_1, \tau_2)^*r^g\) continuous, almost \((\tau_1, \tau_2)^*g^*r\) continuous, almost \((\tau_1, \tau_2)^*g^*s\) continuous, almost \((\tau_1, \tau_2)^*r^g\) continuous\}.

Remark 3.5. The converse of the above theorem is not true in general. The following example supports our claim

Example 3.6. Let \(X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}, \{a, b\}\}, \tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}, Y = \{a, b, c\}, \sigma_1 = \{\phi, Y, \{a\}\}, \sigma_2 = \{\phi, Y, \{a\}\}. Let f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) be the identity function. Then, \(f\) is \((\tau_1, \tau_2)^*g^*r\) continuous, \((\tau_1, \tau_2)^*g^*s\) continuous, \((\tau_1, \tau_2)^*r^g\) continuous, almost \((\tau_1, \tau_2)^*g^*r\) continuous, almost \((\tau_1, \tau_2)^*g^*s\) continuous, almost \((\tau_1, \tau_2)^*r^g\) continuous, almost \((\tau_1, \tau_2)^*g^*r\) continuous, almost \((\tau_1, \tau_2)^*g^*s\) continuous, almost \((\tau_1, \tau_2)^*r^g\) continuous. 
(τ₁, τ₂)*-gs continuous, almost (τ₁, τ₂)*-rg continuous but not a (τ₁, τ₂)*-g* r continuous function.

**Theorem 3.7.** If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \((\tau_1, \tau_2)*-g*r\) irresolute and \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \mu_1, \mu_2) \) is \((\sigma_1, \sigma_2)*-g*r\) continuous, then \(gof : (X, \tau_1, \tau_2) \rightarrow (Z, \mu_1, \mu_2)\) is \((\tau_1, \tau_2)*-g*r\) continuous.

**Proof.** Let \( F \) be \( \mu_1\mu_2 \)-closed set in \( Z \). Since \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \mu_1, \mu_2) \) is \((\sigma_1, \sigma_2)*-g*r\) continuous, \( g^{-1}(F) \) is \((\sigma_1, \sigma_2)*-g*r\) closed in \( Y \). Since \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \((\tau_1, \tau_2)*-g*r\) irresolute, \((gof)^{-1}(F) = f^{-1}[g^{-1}(F)]\) is \((\tau_1, \tau_2)*-g*r\) closed in \( X \). Hence \(gof : (X, \tau_1, \tau_2) \rightarrow (Z, \mu_1, \mu_2)\) is \((\tau_1, \tau_2)*-g*r\) continuous. \(\square\)

**Theorem 3.8.** If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \((\tau_1, \tau_2)*-g*r\) continuous and \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \mu_1, \mu_2) \) is \((\sigma_1, \sigma_2)*\)-continuous, then \(gof : (X, \tau_1, \tau_2) \rightarrow (Z, \mu_1, \mu_2)\) is \((\tau_1, \tau_2)*-g*r\) continuous.

**Proof.** Let \( F \) be \( \mu_1\mu_2 \)-closed set in \( Z \). Since \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \mu_1, \mu_2) \) is \((\sigma_1, \sigma_2)*\)-continuous, \( g^{-1}(F) \) is \(\sigma_1\sigma_2\)-closed in \( Y \). Since \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \((\tau_1, \tau_2)*-g*r\) irresolute, \((gof)^{-1}(F) = f^{-1}[g^{-1}(F)]\) is \((\tau_1, \tau_2)*-g*r\) closed in \( X \). Hence \(gof : (X, \tau_1, \tau_2) \rightarrow (Z, \mu_1, \mu_2)\) is \((\tau_1, \tau_2)*-g*r\) continuous. \(\square\)

**Remark 3.9.** If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \((\tau_1, \tau_2)*-g*r\) continuous and \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \mu_1, \mu_2) \) is \((\sigma_1, \sigma_2)*-g*r\) continuous, then \(gof : (X, \tau_1, \tau_2) \rightarrow (Z, \mu_1, \mu_2)\) need not be \((\tau_1, \tau_2)*-g*r\) continuous. The following example supports our claim.

**Example 3.10.** Let \( X = \{a, b, c\} \), \( \tau_1 = \{\phi, X, \{a\}, \{b, c\}\} \), \( \tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\} \), \( Y = \{a, b, c\} \), \( \sigma_1 = \{\phi, Y, \{a\}\} \), \( \sigma_2 = \{\phi, Y, \{b, c\}\} \), \( Z = \{a, b, c\} \), \( \mu_1 = \{\phi, Z, \{a\}\} \), \( \mu_2 = \{\phi, Z, \{a, b\}\} \). Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be the identity function and \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \mu_1, \mu_2) \) be the function defined by \(g(a) = b, g(b) = c, g(c) = a\). Then \( f \) and \( g \) are \((\tau_1, \tau_2)*-g*r\) continuous and \((\sigma_1, \sigma_2)*-g*r\) continuous functions, but \(gof\) is not \((\tau_1, \tau_2)*-g*r\) continuous.

**Theorem 3.11.** If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \((\tau_1, \tau_2)*-g*r\) irresolute and \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \mu_1, \mu_2) \) is \((\sigma_1, \sigma_2)*-g*r\) irresolute, then \(gof : (X, \tau_1, \tau_2) \rightarrow (Z, \mu_1, \mu_2)\) is \((\tau_1, \tau_2)*-g*r\) irresolute.

**Proof.** Let \( F \) be \((\mu_1, \mu_2)*-g*r\) closed set in \( Z \). Since \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \mu_1, \mu_2) \) is \((\sigma_1, \sigma_2)*-g*r\) irresolute, \( g^{-1}(F) \) is \((\sigma_1, \sigma_2)*-g*r\) closed in \( Y \). Since \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \((\tau_1, \tau_2)*-g*r\) irresolute, \((gof)^{-1}(F) = f^{-1}[g^{-1}(F)]\) is \((\tau_1, \tau_2)*-g*r\) closed in \( X \). Hence \(gof : (X, \tau_1, \tau_2) \rightarrow (Z, \mu_1, \mu_2)\) is \((\tau_1, \tau_2)*-g*r\) irresolute. \(\square\)
Definition 3.12. Recall that a function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is called

(a) \((\tau_1, \tau_2)^*g\) closed \([24]\) in \(X\) if the image of each \(\tau_1\tau_2\)-closed set is \((\sigma_1, \sigma_2)^*g\) closed in \(Y\).

(b) \((\tau_1, \tau_2)^*gs\) closed \([24]\) in \(X\) if the image of each \(\tau_1\tau_2\)-closed set is \((\sigma_1, \sigma_2)^*gs\) closed in \(Y\).

(c) \((\tau_1, \tau_2)^*rg\) closed \([21]\) in \(X\) if the image of each \(\tau_1\tau_2\)-closed set is \((\sigma_1, \sigma_2)^*rg\) closed in \(Y\).

(d) almost \((\tau_1, \tau_2)^*g\) closed \([21]\) in \(X\) if the image of each \((\tau_1, \tau_2)^*\)-regular closed set is \((\sigma_1, \sigma_2)^*g\) closed in \(Y\).

(e) almost \((\tau_1, \tau_2)^*gs\) closed in \(X\) if the image of each \((\tau_1, \tau_2)^*\)-regular closed set is \((\sigma_1, \sigma_2)^*gs\) closed in \(Y\).

(f) almost \((\tau_1, \tau_2)^*rg\) closed \([21]\) in \(X\) if the image of each \((\tau_1, \tau_2)^*\)-regular closed set is \((\sigma_1, \sigma_2)^*rg\) closed in \(Y\).

Definition 3.13. A function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is called \((\tau_1, \tau_2)^*g^*r\) closed in \(X\) if the image of each \(\tau_1\tau_2\)-closed set is \((\sigma_1, \sigma_2)^*\)-generalized star regular closed in \(Y\).

Example 3.14. Let \(X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}\}, \tau_2 = \{\phi, X, \{b\}\}, Y = \{a, b, c\}, \sigma_1 = \{\phi, Y, \{a\}, \{b, c\}\}, \sigma_2 = \{\phi, Y, \{a\}, \{b\}, \{a, b\}\}\). Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be the identity function. Then, \( f \) is a \((\tau_1, \tau_2)^*g^*r\) closed function.

Theorem 3.15. Every \((\tau_1, \tau_2)^*g^*r\) closed function is \((\tau_1, \tau_2)^*g\) closed.

Proof. Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be \((\tau_1, \tau_2)^*g^*r\) closed and \( F \) be \(\tau_1\tau_2\)-closed in \(X\). Since \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \((\tau_1, \tau_2)^*g^*r\) closed, \( f(F) \) is \((\sigma_1, \sigma_2)^*g^*r\) closed in \(Y\). Since every \((\sigma_1, \sigma_2)^*g^*r\) closed set is \((\sigma_1, \sigma_2)^*g\) closed, \( f(F) \) is \((\sigma_1, \sigma_2)^*g\) closed in \(Y\). Therefore, \( f \) is \((\tau_1, \tau_2)^*g\) closed.

Corollary 3.16. Every \((\tau_1, \tau_2)^*g^*r\) closed function is \((\tau_1, \tau_2)^*gs\) closed \(\text{resp.} (\tau_1, \tau_2)^*rg\) closed, almost \((\tau_1, \tau_2)^*g\) closed, almost \((\tau_1, \tau_2)^*gs\) closed, almost \((\tau_1, \tau_2)^*rg\) closed.

Remark 3.17. The converse of the above theorem is not true in general. The following example supports our claim.
Example 3.18. Let \( X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}\}, \tau_2 = \{\phi, X, \{a, b\}\} \), \( Y = \{a, b, c\}, \sigma_1 = \{\phi, Y, \{a\}\}, \sigma_2 = \{\phi, Y, \{a, c\}\} \). Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be the identity function. Then, \( f \) is \((\tau_1, \tau_2)^*\)-closed, \((\tau_1, \tau_2)^*\)-gs closed, \((\tau_1, \tau_2)^*\)-rg closed, almost \((\tau_1, \tau_2)^*\)-g closed, almost \((\tau_1, \tau_2)^*\)-gs closed, almost \((\tau_1, \tau_2)^*\)-rg closed but not a \((\tau_1, \tau_2)^*\)-g*rg closed function.

Theorem 3.19. If \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \((\tau_1, \tau_2)^*\)-closed and \( g : (Y, \sigma_1, \sigma_2) \to (Z, \mu_1, \mu_2) \) is \((\sigma_1, \sigma_2)^*\)-g*r closed, then \( gof : (X, \tau_1, \tau_2) \to (Z, \mu_1, \mu_2) \) is \((\tau_1, \tau_2)^*\)-g*r closed.

Proof. Let \( F \) be \( \tau_1 \tau_2 \)-closed set in \( X \). Since \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \((\tau_1, \tau_2)^*\)-closed, \( f(F) \) is \( \sigma_1 \sigma_2 \)-closed in \( Y \). Since \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \((\tau_1, \tau_2)^*\)-g*r closed, \( (gof)(F) = g[f(F)] \) is \((\mu_1, \mu_2)^*\)-g*r closed in \( Z \). Hence \( gof : (X, \tau_1, \tau_2) \to (Z, \mu_1, \mu_2) \) is \((\tau_1, \tau_2)^*\)-g*r closed. \( \square \)

Corollary 3.20. If \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \((\tau_1, \tau_2)^*\)-closed and \( g : (Y, \sigma_1, \sigma_2) \to (Z, \mu_1, \mu_2) \) is \((\sigma_1, \sigma_2)^*\)-g*r closed, then \( gof : (X, \tau_1, \tau_2) \to (Z, \mu_1, \mu_2) \) is \((\tau_1, \tau_2)^*\)-rg closed \( \)resp. \((\tau_1, \tau_2)^*\)-gs closed, almost \((\tau_1, \tau_2)^*\)-g closed, almost \((\tau_1, \tau_2)^*\)-rg closed, almost \((\tau_1, \tau_2)^*\)-gs closed.

Definition 3.21. Recall that a function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is called

(a) \((\tau_1, \tau_2)^*\)-g open [24] in \( X \) if the image of each \( \tau_1 \tau_2 \)-open set is \((\sigma_1, \sigma_2)^*\)-g open in \( Y \).

(b) \((\tau_1, \tau_2)^*\)-gs open [24] in \( X \) if the image of each \( \tau_1 \tau_2 \)-open set is \((\sigma_1, \sigma_2)^*\)-gs open in \( Y \).

(c) \((\tau_1, \tau_2)^*\)-rg open in \( X \) if the image of each \( \tau_1 \tau_2 \)-open set is \((\sigma_1, \sigma_2)^*\)-rg open in \( Y \).

(d) almost \((\tau_1, \tau_2)^*\)-g open in \( X \) if the image of each \( \tau_1 \tau_2 \)-regular open set is \((\sigma_1, \sigma_2)^*\)-g open in \( Y \).

(e) almost \((\tau_1, \tau_2)^*\)-gs open in \( X \) if the image of each \( \tau_1 \tau_2 \)-regular open set is \((\sigma_1, \sigma_2)^*\)-gs open in \( Y \).

(f) almost \((\tau_1, \tau_2)^*\)-rg open in \( X \) if the image of each \( \tau_1 \tau_2 \)-regular open set is \((\sigma_1, \sigma_2)^*\)-rg open in \( Y \).

Definition 3.22. A function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is called \((\tau_1, \tau_2)^*\)-g*r open in \( X \) if and only if image of each \( \tau_1 \tau_2 \)-open set is \((\sigma_1, \sigma_2)^*\)-g*r open in \( Y \).
Example 3.23. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{b\}\}$, $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, Y, \{a\}, \{b, c\}\}$, $\sigma_2 = \{\phi, Y, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity function. Then, $f$ is a $(\tau_1, \tau_2)^\star-g^\star r$ open function.

Theorem 3.24. Every $(\tau_1, \tau_2)^\star-g^\star r$ open function is $(\tau_1, \tau_2)^\star-g$ open.

Proof. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be $(\tau_1, \tau_2)^\star-g^\star r$ open and $F$ be $\tau_1\tau_2$-open in $X$. Since $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $(\tau_1, \tau_2)^\star-g^\star r$ open, $f(F)$ is $(\sigma_1, \sigma_2)^\star-g^\star r$ open in $Y$. Since every $(\sigma_1, \sigma_2)^\star-g^\star r$ open set is $(\sigma_1, \sigma_2)^\star-g$ open, $f(F)$ is $(\sigma_1, \sigma_2)^\star-g$ open in $Y$. Therefore, $f$ is $(\tau_1, \tau_2)^\star-g$ open. \qed

Corollary 3.25. Every $(\tau_1, \tau_2)^\star-g^\star r$ open function is $(\tau_1, \tau_2)^\star-rg$ open {resp. $(\tau_1, \tau_2)^\star-g$s open, almost $(\tau_1, \tau_2)^\star-g$ open, almost $(\tau_1, \tau_2)^\star-rg$ open, almost $(\tau_1, \tau_2)^\star-g$s open}.

Remark 3.26. The converse of the above theorem is not true in general. The following example supports our claim

Example 3.27. In Example 3.18, $f$ is $(\tau_1, \tau_2)^\star-g$ open, $(\tau_1, \tau_2)^\star-g$s open, $(\tau_1, \tau_2)^\star-rg$ open, almost $(\tau_1, \tau_2)^\star-g$ open, almost $(\tau_1, \tau_2)^\star-rg$ open, almost $(\tau_1, \tau_2)^\star-g$s open but not a $(\tau_1, \tau_2)^\star-g^\star r$-open function.

4. $(\tau_1, \tau_2)^\star-g^\star r$ Homeomorphisms

Definition 4.1. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $(\tau_1, \tau_2)^\star$-generalized star regular homeomorphism ($(\tau_1, \tau_2)^\star-g^\star r$ homeomorphism) if and only if both $f$ is $(\tau_1, \tau_2)^\star-g^\star r$ continuous and $(\tau_1, \tau_2)^\star-g^\star r$ open.

Example 4.2. In Example 3.10, $f$ is $(\tau_1, \tau_2)^\star-g^\star r$ homeomorphism.

Definition 4.3. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $(\tau_1, \tau_2)^\star-g^\star r_{rc}$ homeomorphism) if and only if $f$ is $(\tau_1, \tau_2)^\star-g^\star r$ irresolute and $f^{-1}$ is $(\sigma_1, \sigma_2)^\star-g^\star r$ irresolute.

Example 4.4. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{a, b\}\}$, $Y = \{a, b, c\}$, $\sigma_1 = \{\phi, Y, \{a\}\}$, $\sigma_2 = \{\phi, Y, \{b\}\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity function. Then, $f$ is a $(\tau_1, \tau_2)^\star-g^\star r_{rc}$ homeomorphism.

Theorem 4.5. For any bijection $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following statements are equivalent.

(a) $f^{-1} : (Y, \sigma_1, \sigma_2) \to (X, \tau_1, \tau_2)$ is $(\sigma_1, \sigma_2)^\star-g^\star r$ continuous.
(b) \( f \) is \((\tau_1, \tau_2)^*g^*r\) open.

(c) \( f \) is \((\tau_1, \tau_2)^*g^*r\) closed.

**Proof.** \((a) \Rightarrow (b)\) : Let \( F \) be \( \tau_1\tau_2 \)-open in \( X \). Then \( X - F \) is \( \tau_1\tau_2 \)-closed in \( X \). Since \( f^{-1} \) is \((\sigma_1, \sigma_2)^*g^*r\) continuous, \([f^{-1}]^{-1}(X - F) = f(X - F) = Y - f(F)\) is \((\sigma_1, \sigma_2)^*g^*r\) closed in \( Y \). Then \( f(F) \) is \((\sigma_1, \sigma_2)^*g^*r\) open in \( Y \). Hence \( f \) is \((\tau_1, \tau_2)^*g^*r\) open.

\((b) \Rightarrow (c)\) : Let \( F \) be \( \tau_1\tau_2 \)-closed in \( X \). Then \( X - F \) is \( \tau_1\tau_2 \)-open in \( X \). Since \( f \) is \((\tau_1, \tau_2)^*g^*r\) open, \( f(X - F) = Y - f(F) \) is \((\sigma_1, \sigma_2)^*g^*r\) open in \( Y \). Then \( f(F) \) is \((\sigma_1, \sigma_2)^*g^*r\) closed in \( Y \). Hence \( f \) is \((\tau_1, \tau_2)^*g^*r\) closed.

\((c) \Rightarrow (a)\) : Let \( F \) be \( \tau_1\tau_2 \)-closed in \( X \). Since \( f \) is \((\tau_1, \tau_2)^*g^*r\) closed, \([f^{-1}]^{-1}(F) = f(F) \) is \((\sigma_1, \sigma_2)^*g^*r\) closed in \( Y \). Hence \( f \) is \((\tau_1, \tau_2)^*g^*r\) continuous.

\[ \]

**Theorem 4.6.** Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a bijective and \((\tau_1, \tau_2)^*g^*r\) continuous function. Then the following statements are equivalent.

(a) \( f \) is \((\tau_1, \tau_2)^*g^*r\) open.

(b) \( f \) is \((\tau_1, \tau_2)^*g^*r\) homeomorphism.

(c) \( f \) is \((\tau_1, \tau_2)^*g^*r\) closed.

**Proof.** \((a) \Rightarrow (b)\) : obvious from the definitions.

\((b) \Rightarrow (c)\) : Let \( F \) be \( \tau_1\tau_2 \)-closed in \( X \). Then \( X - F \) is \( \tau_1\tau_2 \)-open in \( X \). Since \( f \) is \((\tau_1, \tau_2)^*g^*r\) homeomorphism, \( f \) is \((\tau_1, \tau_2)^*g^*r\) open. Hence by above theorem, \( f \) is \((\tau_1, \tau_2)^*g^*r\) closed.

\((c) \Rightarrow (a)\) : Follows from above Theorem.

**Remark 4.7.** If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \((\tau_1, \tau_2)^*g^*r\) homeomorphism and \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \mu_1, \mu_2) \) is \((\sigma_1, \sigma_2)^*g^*r\) homeomorphism, then \( g f : (X, \tau_1, \tau_2) \rightarrow (Z, \mu_1, \mu_2) \) need not be a \((\tau_1, \tau_2)^*g^*r\) homeomorphism.

**Example 4.8.** In Example 3.10, \( f \) and \( g \) are \((\tau_1, \tau_2)^*g^*r\) homeomorphism and \((\sigma_1, \sigma_2)^*g^*r\) homeomorphism, but \( g f \) is not \((\tau_1, \tau_2)^*g^*r\) homeomorphism.
5. \((\tau_1, \tau_2)^*G^*RO\) Compact Space

**Definition 5.1.** A subset \(B\) of a bitopological space \((X, \tau_1, \tau_2)\) is said to be \((\tau_1, \tau_2)^*G^*RO\) compact relative to \(X\) if for every cover \(\{A_i : i \in \Lambda\}\) of \(B\) by \((\tau_1, \tau_2)^*\)-G-\(r\) open subsets of \((X, \tau_1, \tau_2)\), i.e. \(B \subseteq \bigcup_{i \in \Lambda} A_i\) where \(A_i, i \in \Lambda\) are \((\tau_1, \tau_2)^*\)-G-\(r\) open subsets of \((X, \tau_1, \tau_2)\), there exists a finite subset \(\Lambda_0\) of \(\Lambda\) such that \(B \subseteq \bigcup_{i \in \Lambda_0} A_i\).

**Definition 5.2.** A bitopological space \((X, \tau_1, \tau_2)\) is said to be \((\tau_1, \tau_2)^*G^*RO\) compact if for every cover \(\{A_i : i \in \Lambda\}\) of \(X\) by \((\tau_1, \tau_2)^*\)-G-\(r\) open subsets of \((X, \tau_1, \tau_2)\), i.e. \(X \subseteq \bigcup_{i \in \Lambda} A_i\) where \(A_i, i \in \Lambda\) are \((\tau_1, \tau_2)^*\)-G-\(r\) open subsets of \((X, \tau_1, \tau_2)\), there exists a finite subset \(\Lambda_0\) of \(\Lambda\) such that \(X \subseteq \bigcup_{i \in \Lambda_0} A_i\).

**Theorem 5.3.** Every \((\tau_1, \tau_2)^*G^*O\) compact space \{resp. \((\tau_1, \tau_2)^*G^*SO\) compact space, \((\tau_1, \tau_2)^*G^*RO\) compact space\} is \((\tau_1, \tau_2)^*G^*RO\) compact space.

**Proposition 5.4.** A \((\tau_1, \tau_2)^*\)-G-\(r\) closed subset of a \((\tau_1, \tau_2)^*G^*RO\) compact space \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2)^*G^*RO\) compact relative to \(X\).

**Proof.** Let \((X, \tau_1, \tau_2)\) be a \((\tau_1, \tau_2)^*G^*RO\) compact space and \(Y\) be a \((\tau_1, \tau_2)^*\)-G-\(r\) closed subset of \(X\). Let \(\zeta = \{A_i : i \in \Lambda\}\) be a \((\tau_1, \tau_2)^*\)-G-\(r\) open cover of \(Y\). Since \(Y\) is \((\tau_1, \tau_2)^*\)-G-\(r\) closed subset, \(Y^C\) is \((\tau_1, \tau_2)^*\)-G-\(r\) open. Also \(\zeta \cup Y^C = Y^C \cup \left( \bigcup_{i \in \Lambda} A_i \right)\) be a \((\tau_1, \tau_2)^*\)-G-\(r\) open cover of \(X\). Since \(X\) is \((\tau_1, \tau_2)^*G^*RO\) compact, \(X = Y^C \cup \left( \bigcup_{i \in \Lambda_0} A_i \right)\) where \(\Lambda_0\) is a finite subset \(\Lambda\). Hence \(Y = \bigcup_{i \in \Lambda_0} A_i\). Therefore, \(Y\) is \((\tau_1, \tau_2)^*G^*RO\) compact.

**Proposition 5.5.** If \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is \((\tau_1, \tau_2)^*G^*r\) continuous and a subset \(B\) of \(X\) is \((\tau_1, \tau_2)^*G^*RO\) compact relative to \(X\), then \(f(B)\) is \((\sigma_1, \sigma_2)^*G^*RO\) compact in \(Y\).

**Proof.** Let \(\{A_i : i \in \Lambda\}\) be any collection of \(\sigma_1\sigma_2\)-open subsets of \(Y\) such that \(f(B) \subseteq \bigcup_{i \in \Lambda} A_i\). Then \(B \subseteq \bigcup_{i \in \Lambda} f^{-1}(A_i)\) holds and there exists a finite subset \(\Lambda_0\) of \(\Lambda\) such that \(B \subseteq \bigcup_{i \in \Lambda_0} f^{-1}(A_i)\). Therefore, we have \(f(B) \subseteq \bigcup_{i \in \Lambda_0} A_i\) which shows that \(f(B)\) is \((\sigma_1, \sigma_2)^*G^*RO\) compact in \(Y\).
Proposition 5.6. If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is resp. \((\tau_1, \tau_2)^*g^*r\) irresolute and a subset \( B \) of \( X \) is \((\tau_1, \tau_2)^*G^*RO\) compact relative to \( X \), then \( f(B) \) is \((\tau_1, \tau_2)^*G^*RO\) compact relative to \( Y \).

Proof. Let \( \{A_i : i \in \Lambda\} \) be any collection of \((\sigma_1, \sigma_2)^*g^*r\) open subsets of \( Y \) such that \( f(B) \subseteq \bigcup_{i \in \Lambda} A_i \). Then \( B \subseteq \bigcup_{i \in \Lambda} f^{-1}(A_i) \) holds and there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( B \subseteq \bigcup_{i \in \Lambda_0} f^{-1}(A_i) \). Therefore, we have \( f(B) \subseteq \bigcup_{i \in \Lambda_0} A_i \) which shows that \( f(B) \) is \((\sigma_1, \sigma_2)^*G^*RO\) compact in \( Y \). \( \square \)

References


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