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SOLUTION OF THE DIOPHANTINE EQUATION $p^x + q^y = z^2$

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Abstract: In this paper, we found that $(p, q, x, y, z) = (3, 5, 1, 0, 2)$ is a unique solution of the Diophantine equation $p^x + q^y = z^2$ where p is an odd prime number which $q - p = 2$ and x, y and z are non-negative integers.

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1. Introduction

In 1844, Catalan [2] posed a conjecture that $(a, b, x, y) = (3, 2, 2, 3)$ is a unique solution of the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $min\{a, b, x, y\} > 1$. Then Mihailescu [3] proved the Catalan's conjecture in 2004. After that Acu [1] proved that $(3, 0, 3)$ and $(2, 1, 3)$ are only two solutions (x, y, z) for the Diophantine equation $2^x + 5^y = z^2$ where x, y and z are non-negative integers.

In 2011, Suvarnamani, Singta and Chotchaisthit [9] proved that two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution. Then Suvarnamani [5] proved that two Diophantine equations

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$4^x + 13^y = z^2$ and $4^x + 17^y = z^2$ have no non-negative integer solution. After that Suvarnamani [6] proved that the Diophantine equation $2^x + p^y = z^2$ has some non-negative integer solutions where p is a prime number.

In 2012, Suvarnamani [7] found that Diophantine equation $A^x + B^y = C^z$ has some non-negative integer solutions. Then Suvarnamani [8] found that the Diophantine equation $p^x + p^y = z^2$ has some non-negative integer solutions where p is a prime number. After that Sroysang [4] proved that $(0, 1, 3)$ is a unique non-negative integer solution of the Diophantine equation $7^x + 8^y = z^2$.

In this paper, we will use the Catalan's conjecture to solving $p^x + q^y = z^2$ where p is an odd prime number which $q-p = 2$ and x, y and z are non-negative integers.

2. Preliminaries

Lemma 2.1. $(a, b, x, y) = (3, 2, 2, 3)$ *is a unique solution of the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers with* $min\{a, b, x, y\} > 1$.

Proof. See in [4]. □

Lemma 2.2. *If* q *is an odd prime number and* y, z *are non-negative integers.Then the Diophantine equation* $1 + q^y = z^2$ *has no solution.*

Proof. Let q is an odd prime number and y, z be non-negative integers such that $1 + q^y = z^2$. We consider in 3 cases.

Case 1: $y = 0$. Then $z^2 = 2$ which is impossible.

Case 2: $y = 1$. Thus $z^2 = q + 1$. That is $z = 0$ or 2. It is impossible.

Case 3: $y > 1$. Thus $z^2 = q^y + 1 > q + 1$. Then $z > 2$. By Lemma 2.1, we have $z = 3, q = 2$ and $y = 3$. Contradiction. □

Lemma 2.3. $(p, x, z) = (3, 1, 2)$ *is a unique solution of the Diophantine equation* $p^x + 1 = z^2$ *where* p *is an odd prime number and* x, z *are non-negative integers.*

Proof. Let p be an odd prime number and x, z be non-negative integers such that $p^x + 1 = z^2$. We consider in 3 cases.

Case 1: $x = 0$. Then $z^2 = 2$ which is impossible.

Case 2: $x = 1$. Thus $z^2 = p + 1$. That is $z = 2$. Then we get $p = 3$.

Case 3: $x > 1$. Thus $z^2 = p^x + 1 > p + 1$. Then $z > 2$. By Lemma 2.1, we have $z = 3, p = 2$ and $x = 3$. Contradiction. □

3. Main Theorem

Main Theorem 3.1. $(p, q, x, y, z) = (3, 5, 1, 0, 2)$ *is a unique solution of the Diophantine equation* $p^x + q^y = z^2$ *where* p *is an odd prime number which* $q - p = 2$ *and* x, y, z *are non-negative integers.*

Proof. Let p is an odd prime number which $q - p = 2$ and x, y, z are non negative integers such that $p^x + q^y = z^2$. By Lemma 2.2, we have $x \geq 1$. Then we consider in 2 cases.

Case 1: $y = 0$.

Then $p^x + 1 = z^2$. By Lemma 2.3, we get $(p, q, x, y, z) = (3, 5, 1, 0, 2)$ is a solution.

Case 2: $y \geq 1$.

Then $z^2 = p^x + q^y$. So, z is odd. Then $z^2 \equiv 1 (mod 4)$. Next, we consider in 2 cases.

Subcase 1: x is even, i.e., $x = 2k$ where $k \in N$.

We get $q^y = z^2 - p^{2k} = (z - p^k)(z + p^k)$, where $z - p^k = q^u$ and $z + p^k = q^{x-u}, y > 2u$. Then $q^u(q^{y-2u} - 1) = 2 \cdot p^k$. That is $q^u = 2 \cdot p^v$ and $(q^{y-2u} - 1) = p^{k-v}$ where $v \in N$. But it is impossible.

Subcase 2: x is odd, i.e., $x = 2h + 1$ where $h \in N$.

If y is odd, i.e., $y = 2i + 1$ where $i \in N$. We have $z^2 = p^{2h+1} + q^{2i+1}$. We get $z = 0, 2$. But it is impossible.

If y is even, i.e., $y = 2j$ where $j \in N$. We have $z^2 = p^{2h+1} + q^{2j}$. We get $z = 0, 2$. But it is impossible. □

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