OPTIMAL CONVEX COMBINATION BOUNDS OF THE CONTRAHARMONIC AND HARMONIC MEANS FOR THE SEIFFERT MEAN

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Abstract: We find the greatest value $\alpha$ and the least value $\beta$ such that the double inequality

$$\alpha C(a, b) + (1 - \alpha) H(a, b) < P(a, b) < \beta C(a, b) + (1 - \beta) H(a, b)$$

holds for all $a, b > 0$ with $a \neq b$. Here $C(a, b)$, $H(a, b)$ and $P(a, b)$ denote the contraharmonic, harmonic, and the Seiffert means of two positive numbers $a$ and $b$ respectively.

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1. Introduction

For $a, b > 0$ with $a \neq b$ the Seiffert means $P(a, b)$ was introduced by Seiffert [1, 2] as follows:

$$P(a, b) = \frac{a - b}{4 \arctan \left( \sqrt{\frac{a}{b}} \right) - \pi}. \quad (1)$$

Recently, the inequalities for means have been the subject of intensive re-
search \[3, 8\]. In particularly, many remarkable inequalities for the Seiffert mean can be found in the literature \[4, 6, 8\].

Let $H(a, b) = \frac{2ab}{a+b}$, $G(a, b) = \sqrt{ab}$, $L(a, b) = \frac{b-a}{\ln b - \ln a}$, $P(a, b) = \frac{a-b}{4 \arctan \sqrt{\frac{b}{a}-1}}$, $I(a, b) = \frac{1}{e} \left( \frac{b}{a} \right)^{\frac{b-a}{b-a}}$, $A(a, b) = \frac{a+b}{2}$, $T(a, b) = \frac{2(a^2+ab+b^2)}{3(a+b)}$ and $C(a, b) = \frac{a^2+b^2}{a+b}$ be the harmonic, geometric, logarithmic, Seiffert, identric, arithmetic, centroidal and contraharmonic means of two positive real numbers $a$ and $b$ with $a \neq b$. Then

$$\min\{a, b\} < H(a, b) < G(a, b) < L(a, b) < P(a, b) < I(a, b) < A(a, b) < T(a, b) < S(a, b) < C(a, b) < \max\{a, b\}. \quad (2)$$

In \[1\], Seiffert proved

$$L(a, b) < P(a, b) < I(a, b) \quad (3)$$

for all $a, b > 0$ with $a \neq b$.

The following bounds for the Seiffert mean $P(a, b)$ in terms of the power mean $M_r(a, b) = ((a^r + b^r)/2)^{1/r}$ ($r \neq 0$) were presented by Jagers in \[7\]:

$$M_{1/2}(a, b) < P(a, b) < M_{2/3}(a, b). \quad (4)$$

for all $a, b > 0$ with $a \neq b$.

Hästö\[8\] found the sharp lower bound for the Seiffert mean as follow:

$$M_{\log 2/\log \pi}(a, b) < P(a, b) \quad (5)$$

for all $a, b > 0$ with $a \neq b$.

In \[3\], Seiffert proved

$$P(a, b) > \frac{3A(a, b)G(a, b)}{A(a, b) + 2G(a, b)} \quad (6)$$

and

$$P(a, b) > \frac{2}{\pi} A(a, b) \quad (7)$$

for all $a, b > 0$ with $a \neq b$.

In \[4\], the authors found the greatest value $\alpha$ and the least value $\beta$ such that the double inequality

$$\alpha A(a, b) + (1 - \alpha)H(a, b) < P(a, b) < \beta A(a, b) + (1 - \beta)H(a, b) \quad (8)$$

for all $a, b > 0$ with $a \neq b$. 

The purpose of the present paper is to find the greatest value $\alpha$ and the least value $\beta$ such that the double inequality

$$\alpha C(a, b) + (1 - \alpha) H(a, b) < P(a, b) < \beta C(a, b) + (1 - \beta) H(a, b)$$

holds for all $a, b > 0$ with $a \neq b$.

### 2. Main Result

#### 2.1. Theorem

The double inequality

$$\alpha C(a, b) + (1 - \alpha) H(a, b) \leq P(a, b) \leq \beta C(a, b) + (1 - \beta) H(a, b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq \frac{1}{\pi}$ and $\beta \geq \frac{5}{12}$.

**Proof.** Firstly, we prove that

$$P(a, b) < \frac{5}{12} C(a, b) + \frac{7}{12} H(a, b),$$

$$P(a, b) > \frac{1}{\pi} C(a, b) + (1 - \frac{1}{\pi}) H(a, b),$$

for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume $a > b > 0$. Let $t = \sqrt{\frac{a}{b}} > 1$ and $q \in \{\frac{1}{\pi}, \frac{5}{12}\}$. Then (1) leads to

$$\frac{1}{b} [qC(a, b) + (1 - q)H(a, b) - P(a, b)]$$

$$= \frac{qt^4 + 2(1-q)t^2 + q}{(t^2+1)(\arctan - \pi)} f(t),$$

where

$$f(t) = 4 \arctan t - \pi - \frac{t^4 - 1}{qt^4 + 2(1-q)t^2 + q}.$$  

Simple computations lead to

$$\lim_{t \to 1^+} f(t) = 0,$$  

$$\lim_{t \to 1^+} f(t) = 0.$$
\[ \lim_{t \to +\infty} f(t) = \pi - \frac{1}{q}. \]  

(15)

\[ f'(t) = \frac{4}{(t^2 + 1) [qt^4 + 2(1-q)t^2 + q]^2} g_1(t), \]  

(16)

where

\[ g_1(t) = q^2 t^8 - (1-q)t^7 + 4q(1-q)t^6 - (1+q)t^5 + 2(3q^2 - 4q + 2)t^4 - (1+q)t^3 + 4q(1-q)t^2 - (1-q)t + q^2. \]  

(17)

Now we divide the proof into two cases:

Case 1. If \( q = \frac{5}{12} \), (17) leads to

\[ g_1(t) = \frac{(t - 1)^4}{144} (25t^4 + 16t^3 + 54t^2 + 16t + 25) > 0 \]  

(18)

for \( t > 1 \). (18) and (16) imply \( f'(t) > 0 \), thus \( f(t) \) is strictly increasing for \( t > 1 \). Then inequality (10) follows from (12)-(14).

Case 2. If \( q = \frac{1}{\pi} \), then from (17) we get

\[ \lim_{t \to 1^+} g_1(t) = 0, \]  

(19)

\[ \lim_{t \to +\infty} g_1(t) = +\infty. \]  

(20)

\[ g_1'(t) = 8q^2 t^7 - 7(1-q)t^6 + 24q(1-q)t^5 - 5(1+q)t^4 + 8(3q^2 - 4q + 2)t^3 - 3(1+q)t^2 + 8q(1-q)t - (1-q), \]  

(21)

\[ \lim_{t \to 1^+} g_1'(t) = 0, \]  

(22)

\[ \lim_{t \to +\infty} g_1'(t) = +\infty. \]  

(23)
\[ g_1''(t) = 56q^2t^6 - 42(1 - q)t^5 + 120q(1 - q)t^4 - 20(1 + q)t^3 + 24(3q^2 - 4q + 2)t^2 - 6(1 + q)t + 8q(1 - q), \] (24)

\[ \lim_{t \to 1^+} g_1''(t) = 48q - 20 = \frac{48}{\pi} - 20 < 0, \] (25)

\[ \lim_{t \to +\infty} g_1''(t) = +\infty. \] (26)

\[ g_1'''(t) = 6\left[ \frac{56}{\pi^2}t^5 - 35(1 - \frac{1}{\pi})t^4 + \frac{80}{\pi}(1 - \frac{1}{\pi})t^3 - 10(1 + \frac{1}{\pi})t^2 \right. \]
\[ + \left. \left( \frac{3}{2\pi^2} - \frac{2}{\pi} + 1 \right)t - (1 + \frac{1}{\pi}) \right] \] (27)

Let
\[ g_2(t) = \frac{56}{\pi^2}t^5 - 35(1 - \frac{1}{\pi})t^4 + \frac{80}{\pi}(1 - \frac{1}{\pi})t^3 - 10(1 + \frac{1}{\pi})t^2 \]
\[ + \left( \frac{3}{2\pi^2} - \frac{2}{\pi} + 1 \right)t - (1 + \frac{1}{\pi}). \] (28)

Then
\[ \lim_{t \to 1^+} g_2(t) = \frac{72}{\pi} - 30 < 0, \] (29)

\[ \lim_{t \to +\infty} g_2(t) = +\infty. \] (30)

\[ g_2'(t) = \frac{280}{\pi^2}t^4 - 140(1 - \frac{1}{\pi})t^3 + \frac{240}{\pi}(1 - \frac{1}{\pi})t^2 \]
\[ - 20(1 + \frac{1}{\pi})t + 16(1 + \frac{3}{2\pi^2} - \frac{2}{\pi}), \] (31)

\[ \lim_{t \to 1^+} g_2'(t) = \frac{64}{\pi^2} + \frac{328}{\pi} - 144 < 0, \] (32)

\[ \lim_{t \to +\infty} g_2'(t) = +\infty. \] (33)
\[ g_2''(t) = 20 \left[ \frac{56}{\pi^2} t^3 - 21(1 - \frac{1}{\pi})t^2 + \frac{24}{\pi} (1 - \frac{1}{\pi})t - (1 + \frac{1}{\pi}) \right] = 20g_3(t), \] (34)

where
\[ g_3(t) = \frac{56}{\pi^2} t^3 - 21(1 - \frac{1}{\pi})t^2 + \frac{24}{\pi} (1 - \frac{1}{\pi})t - (1 + \frac{1}{\pi}). \] (35)

Therefore,
\[ \lim_{t \to 1^+} g_3(t) = \frac{32}{\pi} + \frac{44}{\pi} - 22 < 0, \] (36)
\[ \lim_{t \to +\infty} g_3(t) = +\infty. \] (37)

\[ g_3'(t) = 6 \left[ \frac{28}{\pi^2} t^2 - 7(1 - \frac{1}{\pi})t + \frac{4}{\pi} (1 - \frac{1}{\pi}) \right] = 6g_4(t), \] (38)

where
\[ g_4(t) = \frac{28}{\pi^2} t^2 - 7(1 - \frac{1}{\pi})t + \frac{4}{\pi} (1 - \frac{1}{\pi}). \] (39)

By simple computations, we have
\[ \lim_{t \to 1^+} g_4(t) = \frac{24}{\pi^2} + \frac{11}{\pi} - 7 < 0, \] (40)
\[ \lim_{t \to +\infty} g_4(t) = +\infty. \] (41)

\[ g_4'(t) = \frac{56}{\pi^2} t - 7(1 - \frac{1}{\pi}), \] (42)
\[ \lim_{t \to 1^+} g_4'(t) = \frac{56}{\pi^2} - 7(1 - \frac{1}{\pi}) > 0, \] (43)
\[ g_4''(t) = \frac{56}{\pi^2} > 0. \] (44)

From (44) and (43) we clearly see that \( g_4'(t) > 0 \) for \( t > 1 \), hence \( g_4(t) \) is strictly increasing in \([1, +\infty)\). It follows from (40) and (41) together with the monotonicity of \( g_4(t) \) that there exists \( \lambda_1 > 1 \) such that \( g_4(t) < 0 \) for \( t \in [1, \lambda_1) \) and \( g_4(t) > 0 \) for \( t \in (\lambda_1, +\infty) \), hence from (38) \( g_3(t) \) is strictly decreasing in \([1, \lambda_1]\) and strictly increasing in \([\lambda_1, +\infty)\).
From (36) and (37) together with the monotonicity of \( g_3(t) \) we know that there exists \( \lambda_2 > 1 \) such that \( g_3(t) < 0 \) for \( t \in [1, \lambda_2) \) and \( g_3(t) > 0 \) for \( t \in (\lambda_2, +\infty) \), hence from (34) \( g_2'(t) \) is strictly decreasing in \([1, \lambda_2]\) and strictly increasing in \([\lambda_2, +\infty)\).

From (32) and (33) together with the monotonicity of \( g_2'(t) \) we can see that there exists \( \lambda_3 > 1 \) such that \( g_2(t) \) is strictly decreasing in \([1, \lambda_3]\) and strictly increasing in \([\lambda_3, +\infty)\). It follows from (27)(29) and (30) together with the monotonicity of \( g_2(t) \) we clearly see there exists \( \lambda_4 > 1 \) such that \( g_1''(t) \) is strictly decreasing in \([1, \lambda_4]\) and strictly increasing in \([\lambda_4, +\infty)\).

From (25) and (26) together with the monotonicity of \( g_1''(t) \) we can see that there exists \( \lambda_5 > 1 \) such that \( g_1'(t) \) is strictly decreasing in \([1, \lambda_5]\) and strictly increasing in \([\lambda_5, +\infty)\). From (22) and (23) together with the monotonicity of \( g_1'(t) \) we clearly see there exists \( \lambda_6 > 1 \) such that \( g_1(t) \) is strictly decreasing in \([1, \lambda_6]\) and strictly increasing in \([\lambda_6, +\infty)\). Then (16) (19) and (20) imply that there exists \( \lambda_7 > 1 \) such that \( f(t) \) is strictly decreasing in \([1, \lambda_7]\) and strictly increasing in \([\lambda_7, +\infty)\). Note that (15) becomes

\[
\lim_{t \to +\infty} f(t) = 0
\] (45)

for \( p = \frac{1}{\pi} \).

It follows from (14) and (45) together with the monotonicity of \( f(t) \) that

\[
f(t) < 0
\] (46)

for \( t > 1 \).

Therefore, inequality (11) follows from (12) and (13) together with (46).

Secondly, we prove that \( \frac{5}{12} C(a, b) + \frac{7}{12} H(a, b) \) is the best possible upper convex combination bound of contraharmonic and harmonic means for the Seiffert mean \( P(a, b) \).

For any \( t > 1 \) and \( \alpha \in R \), we have

\[
\alpha C(t^2, 1) + (1 - \alpha) H(t^2, 1) - P(t^2, 1)
\]

\[= \frac{q(t^2+1)}{t^2+1} + (1 - q) \frac{2t^2}{t^2+1} - \frac{(t^2-1)}{4\arctan - \pi} \] (47)

\[= \frac{h(t)}{(t^2+1)(4\arctan - \pi)}, \]

where

\[
h(t) = [qt^4 + q + 2(1-q)t^2](4\arctan - \pi) - (t^4 - 1). \] (48)
It follows from (2.39) that
\[ h(1) = h'(1) = h''(1) = 0, \] (49)
\[ h'''(1) = 4(12\alpha - 5). \] (50)

If \( \alpha < \frac{5}{12} \), then (2.41) leads to
\[ h'''(1) < 0. \] (51)

From (2.42) and the continuity of \( h'''(t) \) we see that there exists \( \delta = \delta(\alpha) > 0 \) such that
\[ h'''(t) < 0 \] (52)
for \( t \in [1, 1 + \delta) \). Then (2.40) and (2.43) imply that
\[ h(t) < 0 \] (53)
for \( t \in [1, 1 + \delta) \).

Therefore, \( \alpha C(t^2, 1) + (1 - \alpha) H(t^2, 1) < P(t^2, 1) \) for \( t \in (1, 1 + \delta) \) follows from (2.38) and (2.44).

Finally, we prove that \( \frac{1}{\pi} C(a, b) + (1 - \frac{1}{\pi}) H(a, b) \) is the best possible lower convex combination bound of contraharmonic and harmonic means for the Seiffert mean \( P(a, b) \). In fact, for \( \beta > \frac{1}{\pi} \), we have
\[ \lim_{x \to +\infty} \frac{\beta C(1, x) + (1 - \beta) H(1, x)}{P(1, x)} = \pi \beta > 1 \] (54)

Inequality (2.45) implies that for any \( \beta > \frac{1}{\pi} \) there exists \( X = X(\beta) > 1 \) such that \( \beta C(1, x) + (1 - \beta) H(1, x) > P(1, x) \) for \( x \in (X, +\infty) \).

\[ \square \]

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References


