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OPTIMAL CONVEX COMBINATION BOUNDS OF THE CONTRAHARMONIC AND HARMONIC MEANS FOR THE SEIFFERT MEAN

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Abstract: We find the greatest value α and the least value β such that the double inequality

$$\alpha C(a,b) + (1-\alpha)H(a,b) < P(a,b) < \beta C(a,b) + (1-\beta)H(a,b)$$

holds for all $a, b > 0$ with $a \neq b$. Here $\mathrm{C}(a,b)$, $\mathrm{H}(a,b)$ and $\mathrm{P}(a,b)$ denote the contraharmonic, harmonic, and the Seiffert means of two positive numbers a and b respectively.

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Key Words: optimal convex combination bound, contraharmonic mean, harmonic mean, the Seiffert means

1. Introduction

For $a, b > 0$ with $a \neq b$ the Seiffert means $\mathrm{P}(a,b)$ was introduced by Seiffert [1, 2] as follows:

$$P(a,b) = \frac{a-b}{4\arctan\left(\sqrt{a/b}\right) - \pi}. \tag{1}$$

Recently, the inequalities for means have been the subject of intensive re-

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search [3, 8]. In particularly, many remarkable inequalities for the Seiffert mean can be found in the literature [4, 6, 8].

Let $H(a,b)=\frac{2ab}{a+b}$, $G(a,b)=\sqrt{ab}$, $\mathrm{L}(a,b)=\frac{b-a}{\ln b-\ln a}$, $P(a,b)=\frac{a-b}{4\arctan\sqrt{\frac{a}{b}}-\pi}$, $I(a,b)=\frac{1}{e}(\frac{b^{b}}{a^{a}})^{\frac{1}{b-a}}$, $A(a,b)=\frac{a+b}{2}$, $\mathrm{T}(a,b)=\frac{2(a^2+ab+b^2)}{3(a+b)}$ and $C(a,b)=\frac{a^2+b^2}{a+b}$ be the harmonic, geometric, logarithmic, Seiffert, identric, arithmetic, centroidal and contraharmonic means of two positive real numbers a and b with $a\neq b$. Then

$$\begin{aligned}&\min\{a,b\}<H(a,b)<G(a,b)<L(a,b)<\mathrm{P}(a,b)<I(a,b)<A(a,b)\\&<T(a,b)<S(a,b)<C(a,b)<max\{a,b\}.\end{aligned}\tag{2}$$

In [1], Seiffert proved

$$L(a,b)<P(a,b)<I(a,b)\tag{3}$$

for all $a,b>0$ with $a\neq b$.

The following bounds for the Seiffert mean $P(a,b)$ in terms of the power mean $M_r(a,b)=((a^r+b^r)/2)^{1/r}(r\neq 0)$ were presented by Jagers in [7]:

$$M_{1/2}(a,b)<P(a,b)<M_{2/3}(a,b)\tag{4}$$

for all $a,b>0$ with $a\neq b$.

Hästö[8] found the sharp lower bound for the Seiffert mean as follow:

$$M_{\log 2/\log\pi}(a,b)<P(a,b)\tag{5}$$

for all $a,b>0$ with $a\neq b$.

In [3], Seiffert proved

$$P(a,b)>\frac{3A(a,b)G(a,b)}{A(a,b)+2G(a,b)}\tag{6}$$

and

$$P(a,b)>\frac{2}{\pi}A(a,b)\tag{7}$$

for all $a,b>0$ with $a\neq b$.

In [4], the authors found the greatest value α and the least value β such that the double inequality

$$\alpha A(a,b)+(1-\alpha)H(a,b)<P(a,b)<\beta A(a,b)+(1-\beta)H(a,b)\tag{8}$$

for all $a,b>0$ with $a\neq b$.

The purpose of the present paper is to find the greatest value α and the least value β such that the double inequality

$$\alpha C(a,b) + (1-\alpha)H(a,b) < P(a,b) < \beta C(a,b) + (1-\beta)H(a,b) \tag{9}$$

holds for all $a, b > 0$ with $a \neq b$.

2. Main Result

2.1. Theorem

The double inequality

$$\alpha C(a,b) + (1-\alpha)H(a,b) \leq P(a,b) \leq \beta C(a,b) + (1-\beta)H(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq \frac{1}{\pi}$ and $\beta \geq \frac{5}{12}$.

Proof. Firstly, we prove that

$$P(a,b) < \frac{5}{12}C(a,b) + \frac{7}{12}H(a,b), \tag{10}$$

$$P(a,b) > \frac{1}{\pi}C(a,b) + (1-\frac{1}{\pi})H(a,b), \tag{11}$$

for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume $a > b > 0$. Let $t = \sqrt{\frac{a}{b}} > 1$ and $q \in \{\frac{1}{\pi}, \frac{5}{12}\}$. Then (1) leads to

$$\begin{aligned} &\tfrac{1}{b}\left[qC(a,b) + (1-q)H(a,b) - P(a,b)\right] \\ &= \tfrac{qt^4+2(1-q)t^2+q}{(t^2+1)(4\text{arctant}-\pi)} f(t), \end{aligned} \tag{12}$$

where

$$f(t) = 4\arctan t - \pi - \frac{t^4-1}{qt^4+2(1-q)t^2+q}. \tag{13}$$

Simple computations lead to

$$\lim_{t\to 1+} f(t) = 0, \tag{14}$$

$$\lim_{t\to+\infty} f(t) = \pi - \frac{1}{q}. \tag{15}$$

$$f'(t) = \frac{4}{(t^2+1)\left[qt^4+2(1-q)t^2+q\right]^2} g_1(t), \tag{16}$$

where

$$\begin{aligned} g_1(t) = {} & q^2t^8 - (1-q)t^7 + 4q(1-q)t^6 - (1+q)t^5 \\ & + 2(3q^2-4q+2)t^4 - (1+q)t^3 \\ & + 4q(1-q)t^2 - (1-q)t + q^2. \end{aligned} \tag{17}$$

Now we divide the proof into two cases:

Case 1. If $q = \frac{5}{12}$. (17) leads to

$$g_1(t) = \frac{(t-1)^4}{144}(25t^4 + 16t^3 + 54t^2 + 16t + 25) > 0 \tag{18}$$

for $t > 1$. (18) and (16) imply $f'(t) > 0$, thus $f(t)$ is strictly increasing for $t > 1$. Then inequality (10) follows from (12)-(14).

Case 2. If $q = \frac{1}{\pi}$, then from (17) we get

$$\lim_{t\to1+} g_1(t) = 0, \tag{19}$$

$$\lim_{t\to+\infty} g_1(t) = +\infty. \tag{20}$$

$$\begin{aligned} {g_1}'(t) = {} & 8q^2t^7 - 7(1-q)t^6 + 24q(1-q)t^5 - 5(1+q)t^4 \\ & + 8(3q^2-4q+2)t^3 - 3(1+q)t^2 + 8q(1-q)t - (1-q), \end{aligned} \tag{21}$$

$$\lim_{t\to1+} {g_1}'(t) = 0, \tag{22}$$

$$\lim_{t\to+\infty} {g_1}'(t) = +\infty. \tag{23}$$

$$g_1''(t) = 56q^2t^6 - 42(1-q)t^5 + 120q(1-q)t^4 - 20(1+q)t^3$$
$$+24(3q^2 - 4q + 2)t^2 - 6(1+q)t + 8q(1-q), \tag{24}$$

$$\lim_{t\to 1+} g_1''(t) = 48q - 20 = \frac{48}{\pi} - 20 < 0, \tag{25}$$

$$\lim_{t\to +\infty} g_1''(t) = +\infty. \tag{26}$$

$$g_1'''(t) = 6[\frac{56}{\pi^2}t^5 - 35(1-\frac{1}{\pi})t^4 + \frac{80}{\pi}(1-\frac{1}{\pi})t^3 - 10(1+\frac{1}{\pi})t^2$$
$$+ (\frac{3}{2\pi^2} - \frac{2}{\pi} + 1)t - (1+\frac{1}{\pi})] \tag{27}$$

Let

$$g_2(t) = \frac{56}{\pi^2}t^5 - 35(1-\frac{1}{\pi})t^4 + \frac{80}{\pi}(1-\frac{1}{\pi})t^3 - 10(1+\frac{1}{\pi})t^2$$
$$+ (\frac{3}{2\pi^2} - \frac{2}{\pi} + 1)t - (1+\frac{1}{\pi}). \tag{28}$$

Then

$$\lim_{t\to 1+} g_2(t) = \frac{72}{\pi} - 30 < 0, \tag{29}$$

$$\lim_{t\to +\infty} g_2(t) = +\infty. \tag{30}$$

$$g_2'(t) = \frac{280}{\pi^2}t^4 - 140(1-\frac{1}{\pi})t^3 + \frac{240}{\pi}(1-\frac{1}{\pi})t^2$$
$$- 20(1+\frac{1}{\pi})t + 16(1 + \frac{3}{2\pi^2} - \frac{2}{\pi}), \tag{31}$$

$$\lim_{t\to 1+} g_2'(t) = \frac{64}{\pi^2} + \frac{328}{\pi} - 144 < 0, \tag{32}$$

$$\lim_{t\to +\infty} g_2'(t) = +\infty. \tag{33}$$

$$g_2{}''(t) = 20[\frac{56}{\pi^2}t^3 - 21(1-\frac{1}{\pi})t^2 + \frac{24}{\pi}(1-\frac{1}{\pi})t - (1+\frac{1}{\pi})] = 20g_3(t), \tag{34}$$

where

$$g_3(t) = \frac{56}{\pi^2}t^3 - 21(1-\frac{1}{\pi})t^2 + \frac{24}{\pi}(1-\frac{1}{\pi})t - (1+\frac{1}{\pi}). \tag{35}$$

Therefore,

$$\lim_{t\to 1+} g_3(t) = \frac{32}{\pi^2} + \frac{44}{\pi} - 22 < 0, \tag{36}$$

$$\lim_{t\to +\infty} g_3(t) = +\infty. \tag{37}$$

$$g_3{}'(t) = 6[\frac{28}{\pi^2}t^2 - 7(1-\frac{1}{\pi})t + \frac{4}{\pi}(1-\frac{1}{\pi})] = 6g_4(t), \tag{38}$$

where

$$g_4(t) = \frac{28}{\pi^2}t^2 - 7(1-\frac{1}{\pi})t + \frac{4}{\pi}(1-\frac{1}{\pi}). \tag{39}$$

By simple computations, we have

$$\lim_{t\to 1+} g_4(t) = \frac{24}{\pi^2} + \frac{11}{\pi} - 7 < 0, \tag{40}$$

$$\lim_{t\to +\infty} g_4(t) = +\infty. \tag{41}$$

$$g_4{}'(t) = \frac{56}{\pi^2}t - 7(1-\frac{1}{\pi}), \tag{42}$$

$$\lim_{t\to 1+} g_4{}'(t) = \frac{56}{\pi^2} - 7(1-\frac{1}{\pi}) > 0, \tag{43}$$

$$g_4{}''(t) = \frac{56}{\pi^2} > 0. \tag{44}$$

From (44) and (43) we clearly see that $g_4{}'(t) > 0$ for $t > 1$, hence $g_4(t)$ is strictly increasing in $[1, +\infty)$. It follows from (40)and (41) together with the monotonicity of $g_4(t)$ that there exists $\lambda_1 > 1$ such that $g_4(t) < 0$ for $t \in [1, \lambda_1)$ and $g_4(t) > 0$ for $t \in (\lambda_1, +\infty)$, hence from (38) $g_3(t)$ is strictly decreasing in $[1, \lambda_1]$ and strictly increasing in$[\lambda_1, +\infty)$.

From (36) and (37) together with the monotonicity of $g_3(t)$ we know that there exists $\lambda_2 > 1$ such that $g_3(t) < 0$ for $t \in [1, \lambda_2)$ and $g_3(t) > 0$ for $t \in (\lambda_2, +\infty)$, hence from (34) ${g_2}'(t)$ is strictly decreasing in $[1, \lambda_2]$ and strictly increasing in $[\lambda_2, +\infty)$.

From (32) and (33) together with the monotonicity of ${g_2}'(t)$ we can see that there exists $\lambda_3 > 1$ such that $g_2(t)$ is strictly decreasing in $[1, \lambda_3]$ and strictly increasing in $[\lambda_3, +\infty)$. It follows from (27)(29) and (30) together with the monotonicity of $g_2(t)$ we clearly see there exists $\lambda_4 > 1$ such that ${g_1}''(t)$ is strictly decreasing in $[1, \lambda_4]$ and strictly increasing in $[\lambda_4, +\infty)$.

From (25) and (26) together with the monotonicity of ${g_1}''(t)$ we can see that there exists $\lambda_5 > 1$ such that ${g_1}'(t)$ is strictly decreasing in $[1, \lambda_5]$ and strictly increasing in $[\lambda_5, +\infty)$. From (22) and (23) together with the monotonicity of ${g_1}'(t)$ we clearly see there exists $\lambda_6 > 1$ such that $g_1(t)$ is strictly decreasing in $[1, \lambda_6]$ and strictly increasing in $[\lambda_6, +\infty)$. Then (16) (19) and (20) imply that there exists $\lambda_7 > 1$ such that $f(t)$ is strictly decreasing in $[1, \lambda_7]$ and strictly increasing in $[\lambda_7, +\infty)$. Note that (15) becomes

$$\lim_{t \to +\infty} f(t) = 0 \tag{45}$$

for $p = \frac{1}{\pi}$.

It follows from (14) and (45) together with the monotonicity of $f(t)$ that

$$f(t) < 0 \tag{46}$$

for $t > 1$.

Therefore, inequality (11) follows from (12) and (13) together with (46).

Secondly, we prove that $\frac{5}{12}C(a,b) + \frac{7}{12}H(a,b)$ is the best possible upper convex combination bound of contraharmonic and harmonic means for the Seiffert mean $P(a,b)$.

For any $t > 1$ and $\alpha \in R$, we have

$$\begin{aligned} &\alpha C(t^2,1) + (1-\alpha)H(t^2,1) - P(t^2,1) \\ &= \tfrac{q(t^4+1)}{t^2+1} + (1-q)\tfrac{2t^2}{t^2+1} - \tfrac{(t^2-1)}{4\text{arctant}-\pi} \\ &= \tfrac{h(t)}{(t^2+1)(4\text{arctant}-\pi)}, \end{aligned} \tag{47}$$

where

$$h(t) = [qt^4 + q + 2(1-q)t^2](4\text{arctant} - \pi) - (t^4 - 1). \tag{48}$$

It follows from (2.39) that

$$h(1) = h'(1) = h''(1) = 0, \tag{49}$$

$$h'''(1) = 4(12\alpha - 5). \tag{50}$$

If $\alpha < \frac{5}{12}$, then (2.41) leads to

$$h'''(1) < 0. \tag{51}$$

From (2.42) and the continuity of $h'''(t)$ we see that there exists $\delta = \delta(\alpha) > 0$ such that

$$h'''(t) < 0 \tag{52}$$

for $t \in [1, 1+\delta)$. Then (2.40) and (2.43) imply that

$$h(t) < 0 \tag{53}$$

for $t \in [1, 1+\delta)$.

Therefore, $\alpha C(t^2, 1) + (1-\alpha)H(t^2, 1) < P(t^2, 1)$ for $t \in (1, 1+\delta)$ follows from (2.38) and (2.44).

Finally, we prove that $\frac{1}{\pi}C(a, b) + (1 - \frac{1}{\pi})H(a, b)$ is the best possible lower convex combination bound of contraharmonic and harmonic means for the Seiffert mean $P(a, b)$. In fact, for $\beta > \frac{1}{\pi}$, we have

$$\lim_{x \to +\infty} \frac{\beta C(1, x) + (1-\beta)H(1, x)}{P(1, x)} = \pi\beta > 1 \tag{54}$$

Inequality (2.45) implies that for any $\beta > \frac{1}{\pi}$ there exists $X = X(\beta) > 1$ such that $\beta C(1, x) + (1-\beta)H(1, x) > P(1, x)$ for $x \in (X, +\infty)$. □

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References

[1] H. Seiffert,Problem 887,*Nieuw Archief voor Wiskunde*, vol.11, No.2 (1993),176-176.

[2] H. Seiffert, Aufgabe 16, *Die Wurzel*, no.29 (1995), 221-222.

[3] H. Seiffert, Ungleichungen für einen bestimmten mittelwert, *Nieuw Archief voor Wiskunde*, vol.13, No.2 (1995), 195-198.

[4] Y. Chu, Y. Qiu, M. Wang, G. Wang, The optimal convex combination bounds of arithmetic and harmonic means for the Seifferts mean, *Journal of Inequalities and Applications*, Article ID 436457, dio: 10.1155/436457,2010,7 pages .

[5] T. Hara, M. Uchiyama, S. Takahasi, A refinement of various mean inequalities, *Journal of Inequalities and Applications*, vol.2, No.4 (1998), 387-395.

[6] E. Neuman, J. Sändor, On the Schwab-Borchardt mean, *Mathematica Pannonica*, vol.17, No.1 (2006), 49-59.

[7] A. Jagers, Solution of problem 887, *Nieuw Archief voor Wiskunde*, 12(1994), 230-231.

[8] P. P. Hästö, Optimal inequalities between Seiffert′s mean and power mean,*Mathematical Inequalities and Applications*, vol.7, No.1 (2004), 47-53.