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SOLVING LINEAR DIOPHANTINE EQUATION $mn^2x + qm^2y = pm^2n^3$ BY A FINITE CONTINUED FRACTION

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Abstract: In this paper, we show that (x, y) is a positive integer solution under some conditions where m, n, p and q are prime numbers for the linear Diophantine equation $mn^2x + qm^2y = pm^2n^3$ by a finite continued fraction.

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1. Introduction

Acu [2] shown that $(3, 0, 3)$ and $(2, 1, 3)$ are only the integer solutions of the Diophantine equation $2^x + 5^y = z^2$ where x, y and z are nonnegative integers. Brown [4] studied the diophantine equation $ax^2 + Db^2 = y^n$ which has only the positive integer solution $(x, y, m, n) = (10, 7, 5, 3)$ with $gcd(x, y) = 1$, m odd and $n \geq 3$. Sierpiński [5] derived that the diophantine equation $3^x + 4^y = 5^z$ has only the positive integer solution $(x, y, z) = (2, 2, 2)$. For this paper, we show that the linear Diophantine equation $mn^2x + qm^2y = pm^2n^3$ has positive integer solution (x, y) by a finite continued fraction.

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2. Preliminaries

It is usual to use the term Diophantine equation to any equation in one or more unknowns which is to be solved in the integers. The simplest type of Diophantine equation that we will consider is the linear Diophantine equation in two unknowns:

$$ax + by = c, \tag{1}$$

Where a, b, c are given integers and a, b not both zero. A solution of this equation is a pair of integers x_0, y_0 which satisfy $ax_0 + by_0 = c$.

Theorem 1. *[3] Let a and b be integers, not both zero. Then a and b are relatively prime if and only if there exist integers x and y such that $1 = ax + by$.*

Theorem 2. *[3] The linear Diophantine equation $ax + by = c$ has a solution if and only if $d|c$, where $d = \gcd(a, b)$. If is any particular solution of this equation, then all other solution are given by $x = x_0 + t\frac{b}{d}$, $y = y_0 - t\frac{a}{d}$ for varying integer.*

Theorem 3. *The linear Diophantine equation $mn^2x + qm^2y = pm^2n^3$ has a positive solution under some conditions where m, n, p, q are prime numbers.*

Proof. It can be proved directly by Theorem 2. □

Definition 4. [1] A finite continued fraction is expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 \\ \cdot \\ \cdot \\ + \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}}$$

where a_0, a_1, $a_2, ..., a_n$ are real numbers with a_1, $a_2, a_3, ..., a_n$ positive. The real number a_1, $a_2, a_3, ..., a_n$ are called the *partial quotients* of the continued fraction.

3. Main Result

We found the integer solutions of the Diophantine equation $mn^2x + qm^2y = pm^2n^3$ by using the row reduced on matrix and by Euclidean's Algorithm. In this paper, we will prove the Theorem 3 by a finite continued fraction.

Proof. Suppose that $a = mn^2$, $b = qm^2$ and $c = pm^2n^3$ where m, n, p, q are prime numbers. Then, the linear Diophantine equation (1) becomes

$$mn^2x + qm^2y = pm^2n^3. \tag{2}$$

By using a continued fraction, since $\gcd(mn^2, qm^2) = (qm - n^2)m$, this equation may be replaced by the equation

$$\frac{n^2}{(qm-n^2)}x + \frac{qm}{(qm-n^2)}y = \frac{pmn^3}{(qm-n^2)}. \tag{3}$$

The first step is to find a particular solution to

$$\frac{n^2}{(qm-n^2)}x + \frac{qm}{(qm-n^2)}y = 1. \tag{4}$$

To complete this, we begin by writing $\frac{\frac{n^2}{(qm-n^2)}}{\frac{qm}{(qm-n^2)}}$ as a simple continued fraction. The sequence of equalities obtained by applying the Euclidean Algorithm to the number $\frac{n^2}{(qm-n^2)}$ and $\frac{qm}{(qm-n^2)}$ can be derived as

$$\frac{qm}{(qm-n^2)} = 1\left(\frac{n^2}{qm-n^2}\right) + 1 \tag{5}$$

$$\frac{n^2}{(qm-n^2)} = \frac{n^2}{(qm-n^2)}(1) + 0. \tag{6}$$

Equations (5) and (6) can be expressed, respectively, as

$$\frac{\frac{qm}{(qm-n^2)}}{\frac{n^2}{(qm-n^2)}} = 1 + \frac{1}{\frac{n^2}{(qm-n^2)}} \tag{7}$$

$$\frac{\frac{n^2}{(qm-n^2)}}{1} = \frac{n^2}{(qm-n^2)}. \tag{8}$$

Consequently, by Definition 2.4., we derive

$$\begin{aligned} \frac{\frac{n^2}{(qm-n^2)}}{\frac{qm}{(qm-n^2)}} &= 0 + \frac{1}{\frac{\frac{qm}{(qm-n^2)}}{\frac{n^2}{(qm-n^2)}}} \\ &= 0 + \frac{1}{1 + \frac{1}{\frac{n^2}{(qm-n^2)}}}. \end{aligned} \tag{9}$$

Therefore, we obtain

$$\left\langle 0, 1, \frac{n^2}{(qm-n^2)} \right\rangle = \langle a_0, a_1, a_2 \rangle .$$

Then, we have

$$\begin{aligned} r_0 &= a_0 = 0 \qquad ; \ q_0 = 1. \\ r_1 &= a_1 a_0 + 1 = 1(0) + 1 = 1 \qquad ; \ q_1 = a_1 = 1. \\ r_2 &= a_2 r_1 + r_0 = \frac{n^2}{(qm-n^2)}(1) + 0 = \frac{n^2}{(qm-n^2)} \ ; \\ q_2 &= a_2 q_1 + q_0 = \frac{n^2}{(qm-n^2)}(1) + 1 = \frac{qm}{(qm-n^2)}. \end{aligned}$$

It follows that $r_2 q_1 - q_2 r_1 = (-1)^{(2-1)} = -1$. Therefore, we obtain

$$\frac{n^2}{(qm-n^2)}(-1) + \frac{qm}{(qm-n^2)}(1) = 1. \tag{10}$$

The particular solution of the Diophantine equation is

$$x_0 = -1, \qquad y_0 = 1. \tag{11}$$

Multiplying equation (8) by $\frac{pmn^3}{(qm-n^2)}$

$$\frac{n^2}{(qm-n^2)} \left(-\frac{pmn^3}{qm-n^2} \right) + \frac{qm}{(qm-n^2)} \left(\frac{pmn^3}{qm-n^2} \right) = \frac{pmn^3}{(qm-n^2)}. \tag{12}$$

We also have a particular solution of the Diophantine equation as

$$x_0 = -\frac{pmn^3}{(qm-n^2)}, \qquad y_0 = \frac{pmn^3}{(qm-n^2)}. \tag{13}$$

The general solution of the Diophantine equation is given by the equation

$$x = -\frac{pmn^3}{(qm-n^2)} + \frac{qm}{(qm-n^2)}t, \quad y = \frac{pmn^3}{(qm-n^2)} - \frac{n^2}{(qm-n^2)}t \ ; \ t \in I. \tag{14}$$

If $x > 0$, then $-\frac{pmn^3}{(qm-n^2)} + \frac{qm}{(qm-n^2)}t > 0$, and we obtain $t > \frac{pn^3}{q}$. If $y > 0$, then $\frac{pmn^3}{(qm-n^2)} - \frac{n^2}{(qm-n^2)}t > 0$, and we have $t < pmn$. Because t must be an integer, we are forced to accomplish that $\frac{pn^3}{q} < t < pmn$. Consequently, our Diophantine equation has a positive solution x and y corresponding the value of integer t. □

For this paper, we consider only 3 cases and all the prime numbers less than or equal to 13. For $m = n = p = q = 2$, the diophantine equation (1) has no solution since there does not exist x and y in equation (14). Likewise, for $m = n = p = q > 2$. For consecutive prime numbers $m < n < p < q$, If $m = 2, n = 3, p = 5, q = 7$, there are infinitely many solutions of diophantine equation (1) since there exist positive integer t such that (1) has positive integer solutions. Besides, it is true for $m > 2$. For $m < p < n < q$, there exists a positive integer t such that the solution of diophantine equation (1) exists (see example 3.1).

Conclusion : The only positive integer solution (x, y) of the diophantine equation (1) exists in the case of $m < p < n < q$.

Example 3.1. Consider the Diophantine equation $(2)(5)^2x + (13)(2)^2y = (3)(2)^2(5)^3$ which has a $\gcd((2)(5)^2, (13)(2)^2) = 2$ such that $2|(3)(2)^2(5)^3$. So, there exists a solution for which can be solved from $(5)^2x+(13)(2)y = (3)(2)(5)^3$. We find a solution of equation $(5)^2x+(13)(2)y = 1$ by using a continued fraction as follows

$$\begin{aligned}\frac{25}{26} &= 0 + \frac{1}{\frac{26}{25}}\\ &= 0 + \frac{1}{1 + \frac{1}{25}}\\ &= < 0, 1, 25 > \quad = \quad < a_0, a_1, a_2 >\end{aligned}$$

Then, we have

$$\begin{aligned}r_0 &= a_0 = 0 \qquad ; \; q_0 = 1.\\ r_1 &= a_1a_0 + 1 = 1(0) + 1 = 1 \qquad ; \; q_1 = a_1 = 1.\\ r_2 &= a_2r_1 + r_0 = 25(1) + 0 = 25 \;\; ;\\ q_2 &= a_2q_1 + q_0 = 25(1) + 1 = 26.\end{aligned}$$

Since $r_2q_1 - q_2r_1 = 1$. Therefore, $25(-1) + 26(1) = 1$. Then, we have $x_0 = -1$, $y_0 = 1$ as a particular solution of equation $(5)^2x + (13)(2)y = 1$. It follows that $25(-(3)(2)(5)^3) + 26((3)(2)(5)^3) = (3)(2)(5)^3$. Therefore, $x_0 = -(2)(3)(5)^3 = -750$, $y_0 = (2)(3)(5)^3 = 750$ is a particular solution of equation $(5)^2x + (13)(2)y = (3)(2)(5)^3$. The general solution is in the form of $x = -750 + 26t$, $y = 750 - 25t$ where t is an integer. If $x > 0$ and $y > 0$, then $28\frac{22}{26} < t < 30$. Since t must be integer, we conclude that $t = 29$. Therefore, the Diophantine equation has only a positive solution $x = 4$ and $y = 25$.

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