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# APPLICATIONS OF THE EXTRAGRADIENT APPROXIMATION METHOD FOR VARIATIONAL INEQUALITY PROBLEM ON FIXED POINT PROBLEM

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**Abstract:** We apply an iterative sequence for finding the common element of the set of fixed points of a nonexpansive mapping and the solutions of the variational inequality problem for tree inverse-strongly monotone mappings. Under suitable conditions, some strong convergence theorems for approximating a common element of the above two sets are obtained. Moreover, using the above theorem, we also apply to finding solutions of a general system of variational inequality and a zero of a maximal monotone operator in a real Hilbert space. As applications, at the end of paper we utilize our results to study the zeros of the maximal monotone and some convergence problem for strictly pseudocontractive mappings.

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**Key Words:** nonexpansive mapping, fixed point problems, variational inequality, relaxed extragradient approximation method, maximal monotone

## 1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and norm $\| \cdot \|$, and

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respectively and let $C$ be a closed convex subset of $H$. Let $F$ be a bifunction of $C \times C$ into $R$, where $R$ is the set of real number. The equilibrium problem for $F : C \times C \longrightarrow R$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall x, y \in C. \tag{1.1}$$

The set of solution of (1.1) id denoted by $EP(F)$. Give a mapping $T : C \longrightarrow H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tx, y - x\rangle \geq 0$ for all $y \in C$, z is a solution of the variational inequality. Numerous problems in Physics, optimization, and economics reduce to find a solution of (1.1). In 1997 Combettes and Hirstoaga introduced an iterative scheme of finding the best approximation to initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. et $A : C \longrightarrow H$ be a mapping. The classical variational inequality, denote by $VI(A, C)$, is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^*\rangle \geq 0$$

for all $v \in C$. The variational inequality has been extensively studied in the literature. A mapping $A$ of $C$ into $H$ is called $\alpha-$inverse-strongly monotone if there exists a positive real number $\alpha$ such that

$$\langle Au - Av, u - v\rangle \geq \alpha \parallel Au - Av \parallel^2$$

for all $u, v \in C$. We denote by $F(S)$ the set of fixed point of S. For finding an element of $F(S) \cap VI(A, C)$, Takahashi and Toyoda [15] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP(x_n - \lambda_n Ax_n) \tag{1.2}$$

for every $n = 0, 1, 2, \ldots$, where $x_0 = x \in C$, $\alpha_n$ is a sequence in $(0, 1)$, and $\lambda_n$ is a sequence in $(0, 2\alpha)$. Recently, Nadezhkina and Takahashi[9] and Zeng and Yao[21] proposed some new iterative schemes for finding element in $F(S) \cap VI(A, C)$. In 2006, Yao and Yao [19] introduced the following iterative scheme.

Let $C$ be a closed convex subset of real Hilbert space $H$. Let $A$ be an $\alpha$-inverse-strongly monotone mapping of $C$ into $H$ and let $S$ be a nonexpansive mapping of $C$ into itself such that $F(S) \cap VI(A, C) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ are give by

$$\begin{cases} y_n = PC(x_n - \lambda_n Ax_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda_n Ay_n) \end{cases} \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ and $\{\lambda_n\}$ is a sequenced in $[0, 2\alpha]$. They proved that the sequence $\{x_n\}$ converges strongly to common element of the set of fixed point of a nonexpansive mapping and the set of solutions of the variational inequality for $\alpha$-inverse-strongly monotone monotone mappings under some parameters controlling condition.

Moreover, Takahashi and Takahashi[16] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solution of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. They also proved a strong convergence theorem which is connected with Combettes and Hirtoaga's result[6] and Wittmann's result.

In this paper motivated by the iterative schemes, we will introduce a new iterative process below for finding a common element of the set of fixed point of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solution set of the variational inequality problem for an $\alpha$-inverse-strongly monotone monotone mappings in a real Hilbert space. Then, we prove a strong convergence theorem which is connected with Yao and Yao's result[19] and Takahashi and Takahashi's result[16].

A mapping $A : C \longrightarrow H$ is called $\alpha$-inverse-strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C$$

(see Browder and Petryshyn [2], Liu and Nashed [8]). It is obvious that every $\alpha$-inverse-strongly monotone mapping $A$ is monotone and Lipschitz continuous. A mapping $S : C \longrightarrow C$ is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We denote by $F(S)$ the set of fixed points of $S$ and by $P_C$ the metric projection of $H$ onto $C$. Recall that the classical variational inequality, denoted by $VI(A, C)$, is to find an $x^* \in C$ such that

$$\langle Ax^*, v - x^* \rangle \geq 0, \quad \forall x \in C.$$

The set of solutions of $VI(A, C)$ is denoted by $\Gamma$. The variational inequality has been widely studied in the literature; See, e.g. [1], [7], [18], [20], [21] and the references therein.

For finding an element of $F(S) \cap \Gamma$,Takahashi and Toyoda [15] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (-\alpha_n) SP_C(x_n + \lambda_n Ax_n), \tag{1.4}$$

for every $n = 0, 1, 2, ...$, where $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $0, 2\alpha$. On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbf{R}^n$, Korpelevich introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = P_C(x_n - \lambda_n A y_n) \end{cases} \tag{1.5}$$

for every $n = 0, 1, 2, ...$, where $\lambda_n \in (0, \frac{1}{k})$. Recently, Nadezhkina and Takahashi [9] and Zeng and Yao [21] proposed some iterative schemes for finding elements in $F(S) \cap \Gamma$ by combining [15]. Further, these iterative schemes are extended in Yao and Yao [19] to develop a new iterative scheme for finding elements in $F(S) \cap \Gamma$.

Consider the following problem of finding $(x^*, y^*) \in C \times C$ such that (see [5])

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu B x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.6}$$

which is called a general system of variational inequalities where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then problem (1.6) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu A x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.7}$$

which is defined by Verma (1999)and Verma (2001) , and is called the new system of variational inequalities. Further, if $x^* = y^*$, then problem (1.7) reduces to the classical variational inequality $VI(C, A)$.

In 2008 Ceng [5], introduced a relaxed extragradient method for finding solutions of problem (1.6). Let the mappings $A, B : C \longrightarrow H$ be $\alpha$-inverse-strongly monotone and $\beta$-inverse-strongly monotone, respectively. Let $S : C \longrightarrow C$ be a nonexpansive mapping. Suppose $x_1 = u \in C$ and $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n + \lambda_n A y_n), \end{cases} \tag{1.8}$$

where $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta)$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequence in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1$. First, problem (1.6) is proven to be equivalent to a fixed point problem of nonexpansive mapping.

## 2. Preliminaries

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. For every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \qquad \forall y \in C.$$

$P_C$ is call the metric projection of $H$ onto $C$.

**Lemma 2.1.** (see [23]) *The metric projection $P_C$ has the following properties:*

*(i)* $P_C : H \longrightarrow C$ *is nonexpansive;*

*(ii)* $P_C : H \longrightarrow C$ *is firmly nonexpansive i.e.,* $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H$;

*(iii) for each* $x \in H$, $z = P_C(x) \Leftrightarrow \langle x - z, z - y\rangle \geq 0, \quad \forall y \in C$.

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $A, B, C : C \longrightarrow H$ be three mappings. We consider the following problem of finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle \lambda A z^* + x^* - z^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu B y^* + z^* - y^*, x - z^* \rangle \geq 0, & \forall x \in C, \\ \langle \tau C x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{2.1}$$

which is called a general system of variational inequalities where $\lambda > 0$, $\mu > 0$ and $\tau > 0$ are three constants.

In particular, if $A = B = C$, then problem (2.1) reduces to finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle \lambda A z^* + x^* - z^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu A y^* + z^* - y^*, x - z^* \rangle \geq 0, & \forall x \in C, \\ \langle \tau A x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{2.2}$$

**Lemma 2.2.** *For given $x^*, y^*, z^* \in C \times C \times C$, $(x^*, y^*, z^*)$ is a solution of problem (2.1) if and only if $x^*$ is a fixed point of the mapping $G : C \longrightarrow C$ defined by*

$$\begin{aligned} G(x) = P_C\{&P_C[P_C(z - \lambda A z) - \mu B P_C(z - \lambda A z)] \\ &- \tau C P_C[P_C(z - \lambda A z) - \mu B P_C(z - \lambda A z)]\}, \quad \forall x \in C, \end{aligned}$$

*where* $y^* = P_C(z^* - \lambda A z^*)$.

**Lemma 2.3.** (see [10]) *Let $(E, \langle ., . \rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2.$$

**Lemma 2.4.** (see [14]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\longrightarrow\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|) \leq 0$. Then, $\lim_{n\longrightarrow\infty} \|y_n-x_n\| = 0$.*

**Lemma 2.5.** (see [17]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n,\ n \geq 0$$

*where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbf{R}$ such that*

*(i)* $\sum_{n=1}^{\infty} \alpha_n = \infty$

*(ii)* $\limsup_{n\longrightarrow\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

*Then* $\lim_{n\longrightarrow\infty} a_n = 0$.

**Lemma 2.6.** ([[4]) *Demi-closedness Principle. Assume that $T$ is a nonexpansive self-mapping of a nonempty closed convex subset $C$ of a real Hilbert space $H$. If $T$ has a fixed point, then $I-T$ is demi-closed; that is, whenever $\{x_n\}$ is a sequence in $C$ converging weakly to some $x \in C$ (for short, $x_n \rightharpoonup x \in C$), and the sequence $\{(I-T)x_n\}$ converges strongly to some $y$ (for short, $(I-T)x_n \longrightarrow y$), it follows that $(I-T)x = y$. Here $I$ is the identity operator of $H$.*

The following lemma is an immediate consequence of an inner product.

**Lemma 2.7.** *In a real Hilbert space $H$, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

*Remark* We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I-\lambda A)u-(I-\lambda A)v\|^2 &= \|(u-v)-\lambda(Au-Av)\|^2 \\ &= \|u-v\|^2 - 2\lambda\langle u-v, Au-Av\rangle \\ &\quad +\lambda^2\|Au-Av\|^2 \\ &\leq \|u-v\|^2 + \lambda(\lambda-2\alpha)\|Au-Av\|^2. \quad (2.3) \end{aligned}$$

So, if $\lambda \leq 2\alpha$, then $I-\lambda A$ is a nonexpansive mapping from $C$ to $H$.

Next, we introduce an iterative precess by the relaxed extragradient approximation method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem, and the solution set of the variational inequality problem for two inverse-strongly monotone mappings in a real Hilbert space. We prove that the iterative sequences converges strongly to a common element of the above three sets.

**Theorem 2.8.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let the mapping $A, B, C : C \longrightarrow H$ be $\alpha$-inverse-strongly monotone, $\beta$-inverse-strongly monotone and $\gamma$-inverse-strongly monotone, respectively. Let $S$ be a nonexpansive mapping of $C$ into itself such that $F(S) \cap \Omega \neq \emptyset$. Let $f$ be a contraction of $H$ into itself and given $x_0 \in H$ arbitrarily and $\{x_n\}$ is generated by*

$$\begin{cases} F(u_n, x) + \frac{1}{r_n}\langle x - u_n, u_n - x_n\rangle \geq 0 \\ y_n = (1-\gamma_n)u_n + \gamma_n P_c(u_n - \lambda_n A u_n) \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n f(y_n) + \beta_n S P_c(x_n - \lambda_n A y_n) \end{cases} \tag{2.4}$$

*where $\lambda_n \in (0, 2\alpha), \mu_n \in (0, 2\beta), \tau_n \in (0, 2\gamma)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ such that*

*(i)* $\alpha_n + \beta_n + \gamma_n = 1$,

*(ii)* $\lim_{n\longrightarrow\infty} \alpha_n = 0, \lim_{n\longrightarrow\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

*(iii)* $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

*Then $\{x_n\}$ converges strongly to $\overline{x} \in F(S) \cap \Omega$, where $\overline{x} = P_{F(S)\cap\Omega} f(x)$.*

## 3. Main Results

**Theorem 3.1.** *Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $A, B$ be $\alpha$-inverse-strongly monotone and $\beta$-inverse-strongly monotone mapping of $C$ into $H$, respectively and let $S$ be a nonexpansive mapping of $C$ into itself such that $F(S) \cap \Omega \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}$ are given by*

$$\begin{cases} y_n = P_C(x_n - \mu_n B x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda_n A y_n), \end{cases}$$

*where $\lambda_n \in (0, 2\alpha), \mu_n \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ such that*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

*Then $\{x_n\}$ converges strongly to $\overline{x} = P_{F(S)\cap\Omega}u$. Moreover, we also have $(\overline{x}, \overline{y})$ is a solution of problem, where $\overline{y} = P_C(\overline{x} - \mu B\overline{x})$.*

*Proof.* Put $F(x, y) = 0$ for all $x, y \in C$, $\lambda_n = \lambda, \mu_n = \mu$ and $r_n = 1$ for all $n \in \mathbb{N}$. Then, we have $u_n = P_C x_n = x_n$. So, the sequence $\{x_n\}$ converges strongly to $x^* = P_{F(S)\cap\Omega} f(x^*)$. □

**Theorem 3.2.** *Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $A, B$ be $\alpha$-inverse-strongly monotone and $\beta$-inverse-strongly monotone mapping of $C$ into $H$, respectively and let $S$ be a nonexpansive mapping of $C$ into itself such that $F(S) \cap \Omega \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ are given by*

$$\begin{cases} y_n = P_C(x_n - \mu B x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n), \end{cases}$$

*where $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$. If $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{r_n\} \subset (0, \infty)$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

*then $\{x_n\}$ converges strongly to $u = P_{F(S)\cap\Omega}u$ and $(x^*, y^*)$ is a solution of problem, where $y^* = P_C(x^* - \mu B x^*)$.*

**Theorem 3.3.** *Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $A$ be an $\alpha-$inverse-strongly monotone mapping of $C$ into $H$ and let $S$ be a nonexpansive mapping of $C$ into itself such that $F(S) \cap VI(A, C) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ are given by*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda_n A y_n), \end{cases}$$

*where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and If $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some $a, b$ with $0 < a < b < 2\alpha$ satisfying the following conditions:*

*(i)* $\alpha_n + \beta_n + \gamma_n = 1$,

*(ii)* $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

*(iii)* $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

*(iv)* $\lim_{n\longrightarrow\infty}(\lambda_{n+1} - \lambda_n) = 0$,

*then* $\{x_n\}$ *converges strongly to* $P_{F(S)\cap\Omega}u$. *Moreover, we also have* $(x^*, y^*)$ *is a solution of problem (1.7), where* $y^* = P_C(x^* - \lambda Ax^*)$.

*Proof.* Taking $A = B$ and $\lambda_n = \mu_n$, we can get the desired conclusion easily. □

A mapping $T : C \longrightarrow C$ is called strictly pseudocontractive on $C$ if there exists $k$ with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x + (I - T)y\|^2, \text{ for all } x, y \in C.$$

If $k = 0$, then $T$ is nonexpansive. Put $A = I - T$, where $T : C \longrightarrow C$ is a strictly pseudocontractive mapping with $k$. Then we have, for all $x, y \in C$,

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + k\|Ax - Ay\|^2.$$

On the other hand, we have

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 - 2\langle x - y, Ax - Ay\rangle + \|Ax - Ay\|^2.$$

Hence we have

$$\langle x - y, Ax - Ay\rangle \geq \frac{1-k}{2}\|Ax - Ay\|^2.$$

Then, $A$ is $\frac{1-k}{2}-$ inverse strongly monotone.

Now, we state a strong convergence theorem for a pair of nonexpansive mappings and strictly pseudocontractive mappings.

**Theorem 3.4.** *Let* $C$ *be a closed convex subset of a real Hilbert space* $H$. *Let* $f$ *be a bifunction from* $C\times C \longrightarrow \mathbf{R}$ *satisfying (A1)-(A4) and let* $S$ *be a nonexpansive mapping of* $C$ *into itself and let* $T, V$ *be a strictly pseudocontractive mapping with constant* $k$ *of* $C$ *into itself such that* $F(S) \cap F(T) \cap EP(F) \neq \emptyset$. *Suppose* $x_1 = u \in C$ *and* $\{x_n\}, \{y_n\}$ *and* $\{u_n\}$ *are given by*

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C; \\ y_n = (1 - \mu_n)u_n + \mu_n V u_n \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S((1 - \lambda_n)y_n + \lambda_n T y_n), \end{cases}$$

*for all* $n \in \mathbf{N}$, *where* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are three sequences in* $[0,1]$ *and* $\{\lambda_n\}$ *is a sequence in* $[0, 1-k]$. *If* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\lambda_n\}$ *are chosen so that* $\lambda_n \in [a,b]$ *for some* $a, b$ *with* $0 < a < b < 1-k$ *and* $\{r_n\} \subset (0,\infty)$ *satisfying*

*(i)* $\alpha_n + \beta_n + \gamma_n = 1$,

*(ii)* $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

*(iii)* $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

*(iv)* $\liminf_{n\longrightarrow\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

*then* $\{x_n\}$ *converges strongly to* $z = P_{F(S)\cap F(T)\cap EP(f)}u$.

*Proof.* Put $A = I - T$ and $B = I - V$. Then $A$ is $\frac{1-k}{2}-$inverse-strongly monotone and $B$ is $\frac{1-l}{2}-$inverse-strongly monotone. We have that $F(T)$ is the solution set of $VI(A,C)$ and $\Omega$ i.e., $F(T) = VI(A,C)$ and

$$P_C(u_n - \mu_n B u_n) = (1-\mu_n)u_n + \mu_n V u_n \text{ and } P_C(y_n - \lambda_n A y_n) = (1-\lambda_n)y_n + \lambda_n T y_n.$$

Therefore, the conclusion follows. □

**Theorem 3.5.** *Let* $C$ *be a closed convex subset of a real Hilbert space* $H$. *Let* $S$ *be a nonexpansive mapping of* $C$ *into itself and let* $T, V$ *be a strictly pseudocontractive mapping with constant* $k$ *of* $C$ *into itself such that* $F(S) \cap F(T) \neq \emptyset$. *Suppose* $x_1 = u \in C$ *and* $\{x_n\}$ *is given by*

$$\begin{cases} y_n = (1-\mu_n)x_n + \mu_n V x_n \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S((1-\lambda_n)y_n + \lambda_n T y_n), \end{cases}$$

*for all* $n \in \mathbf{N}$, *where* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are three sequences in* $[0,1]$ *and* $\{\lambda_n\}$ *is a sequence in* $[0, 1-k]$. *If* $\{\lambda_n\}$ *are chosen so that* $\lambda_n \in [a,b]$ *for some* $a, b$ *with* $0 < a < b < 1-k$ *and* $\{\mu_n\}$ *are chosen so that* $\mu_n \in [c,d]$ *for some* $c, d$ *with* $0 < c < d < 2\beta$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *satisfying the following conditions:*

*(i)* $\alpha_n + \beta_n + \gamma_n = 1$,

*(ii)* $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

*(iii)* $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

*(iv)* $\lim_{n\longrightarrow\infty}(\lambda_{n+1} - \lambda_n) = 0$,

*then* $\{x_n\}$ *converges strongly to* $x^* = P_{F(S)\cap\Omega}u$ *and* $(x^*, y^*)$ *is a solution of problem, where* $y^* = (1-\mu_n)x^* - \mu V x^*)$.

**Theorem 3.6.** *Let* $C$ *be a closed convex subset of a real Hilbert space* $H$*. Let* $S$ *be a nonexpansive mapping of* $C$ *into itself and let* $T$ *be a strictly pseudocontractive mapping with constant* $k$ *of* $C$ *into itself such that* $F(S) \cap F(T) \neq \emptyset$. *Suppose* $x_1 = u \in C$ *and* $\{x_n\}$ *is given by*

$$\begin{cases} y_n = (1-\lambda_n)x_n + \lambda_n T x_n \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S((1-\lambda_n)y_n + \lambda_n T y_n), \end{cases}$$

*for all* $n \in \mathbf{N}$, *where* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are three sequences in* $[0,1]$ *and* $\{\lambda_n\}$ *is a sequence in* $[0, 1-k]$. *If* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\lambda_n\}$ *are chosen so that* $\lambda_n \in [a,b]$ *for some* $a, b$ *with* $0 < a < b < 1-k$ *and*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

(iv) $\lim_{n\longrightarrow\infty}(\lambda_{n+1} - \lambda_n) = 0$,

*then* $\{x_n\}$ *converges strongly to* $P_{F(S)\cap F(T)}u$.

**Theorem 3.7.** *Let* $H$ *be a real Hilbert space . Let* $f$ *be a bifunction from* $H \times H \longrightarrow \mathbf{R}$ *satisfying (A1)-(A4) and let* $A$ *be an* $\alpha-$*inverse-strongly monotone mapping of* $H$ *into itself and let* $S$ *be a nonexpansive mapping of* $H$ *into itself such that* $F(S) \cap A^{-1}(0) \cap EP(F) \neq \emptyset$. *Suppose* $x_1 = u \in H$ *and* $\{x_n\}, \{y_n\}$ *and* $\{u_n\}$ *are given by*

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in H; \\ y_n = (u_n - \lambda_n A u_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S(y_n - \lambda_n A y_n), \end{cases}$$

*for all* $n \in \mathbf{N}$, *where* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are three sequences in* $[0,1]$ *and* $\{\lambda_n\}$ *is a sequence in* $[0, 2\alpha]$. *If* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\lambda_n\}$ *are chosen so that* $\lambda_n \in [a,b]$ *for some* $a, b$ *with* $0 < a < b < 2\alpha$ *and* $\{r_n\} \subset (0,\infty)$ *satisfying*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *(ii)* $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

*(iii)* $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

*(iv)* $\liminf_{n\longrightarrow\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

*then* $\{x_n\}$ *converges strongly to* $P_{F(S)\cap A^{-1}(0)\cap EP(F)}f(z)$.

*Proof.* Setting $\lambda = \mu, C = H, B = A$ and $P_H = I$. Since $A^{-1}(0)$ are the solution set of $VI(A, H)$, and $\Omega$, we can obtain the conclusion by Theorem 2.8 and by noting that $P_H$ is the identity mapping on $H$. This completes the proof. □

The following three theorems are connected with the problem of obtaining of a common element of the sets of zeroes of a maximal monotone operator and an $\alpha-$inverse-strongly monotone operator.

**Theorem 3.8.** *Let $C$ be a nonempty closed convex subset of $H$. Let $f$ be a bifunction from $C \times C$ to* **R** *satisfying $(A1) - (A4)$ and let $A$ be an $\alpha-$inverse-strongly monotone operator in $H$ and $B : H \longrightarrow 2^H$ be a maximal monotone operator such that $A^{-1}(0) \cap B^{-1}(0) \cap EP(f) \neq \varnothing$. Let $J_r^B$ be the resolvent of $B$ for each $r > 0$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = u \in H$ and*

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in H; \\ y_n = (u_n - \lambda_n A u_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_r^B(y_n - \lambda_n A y_n), \end{cases} \tag{3.1}$$

*where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the following conditions:*

*(i)* $\alpha_n + \beta_n + \gamma_n = 1$,

*(ii)* $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n \longrightarrow 0$ *and* $\{\lambda_n\} \subset [c, d]$ *for some* $[c, d] \subset (0, 2\alpha)$*;*

*(iii)* $\{r_n\} \subset (0, \infty)$, $\liminf_{n\longrightarrow\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

*(iv)* $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

*Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in A^{-1}(0) \cap B^{-1}(0) \cap EP(f)$, where $z = P_{A^{-1}(0)\cap B^{-1}(0)EP(f)}x_1$.*

*Proof.* Since $J_r^B$ is nonexpansive. We have the following

$$A^{-1}0 = V(I, A) \text{ and } F(J_r^B) = B^{-1}(0).$$

Putting $P_H = I$ then, we obtain the desired result. □ □

**Theorem 3.9.** *Let $C$ be a nonempty closed convex subset of $H$. Let $A$ be an $\alpha-$inverse-strongly monotone operator in $H$ and $B : H \longrightarrow 2^H$ be a maximal monotone operator such that $A^{-1}(0) \cap B^{-1}(0) \neq \varnothing$. Let $J_r^B$ be the resolvent of $B$ for each $r > 0$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = u \in H$ and*

$$\begin{cases} y_n = (u_n - \lambda_n A u_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_r^B(y_n - \lambda_n A y_n), \end{cases} \tag{3.2}$$

*where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:*

*(i)* $\alpha_n + \beta_n + \gamma_n = 1,;$

*(ii)* $\{\alpha_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n \longrightarrow 0$ *and* $\{\lambda_n\} \subset [c,d]$ *for some* $[c,d] \subset (0, 2\alpha)$*;*

*(iii)* $0 < \liminf_{n \longrightarrow \infty} \beta_n \leq \limsup_{n \longrightarrow \infty} \beta_n < 1.$

*Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in A^{-1}(0) \cap B^{-1}(0) \cap EP(F)$, where $z = P_{A^{-1}(0) \cap B^{-1}(0) EP(F)} x_1$.*

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