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DELAY-RANGE-DEPENDENT MEAN SQUARE STABILITY OF STOCHASTIC SYSTEMS WITH INTERVAL TIME-VARYING DELAYS

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Abstract: This paper is concerned with mean square exponential stability of stochastic systems with interval time-varying delays. The time delay is any continuous function belonging to a given interval. By constructing a suitable augmented Lyapunov-Krasovskii functional combined with Leibniz-Newton's formula, new delay-dependent sufficient conditions for the mean square exponential stability of the stochastic systems are first established in terms of LMIs.

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1. Introduction

The analysis of stochastic systems with respect to mean square stability of their equilibria has attracted many researchers. Such systems occur in a large number of applications as in Physics, Optics or Mechanical Engineering. Often, these systems can generally be written as systems of stochastic differential equations (SDEs). There stability examinations play an essential role in judgement on qualitative behaviour of natural processes.

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The concept of mean square stability is one of the most attractive and feasible ones within the large branch of stability analysis. Due to facilities of modern computers and progress in numerical analysis of stochastic differential equations (SDEs), the interest in mean square stability analysis has come up once again. The basic questions for any numerical algorithm are accuracy and stability. The question of accuracy has been worked out well. However, the question of stability is fairly underdeveloped and still in its very beginning, despite of a number of recent contributions. These contributions exclusively deal with numerical stability analysis with respect to linear test equations in one dimension. The problem of the stability of dynamic systems is one of the basic problems in the control theory. To describe the uncertain parameters and excitations appearing in real dynamic systems in corresponding models usually the stochastic differential equations are used. Developments on the stability of stochastic dynamic systems can be found. An important class of nonlinear control systems, called bilinear control systems, are systems described by stochastic differential equations containing terms of products of state and control variables. The study of bilinear systems began in the late 1960s and has continued from its need in applications (many real world systems appearing in economy, biology, chemistry, biochemistry, physics and engineering can be approximated by bilinear models). Stability analysis of linear systems with time-varying delays $\dot{x}(t) = Ax(t) + Dx(t-h(t))$ is fundamental to many practical problems and has received considerable attention [1–15]. Most of the known results on this problem are derived assuming only that the time-varying delay $h(t)$ is a continuously differentiable function, satisfying some boundedness condition on its derivative: $\dot{h}(t) \leq \delta < 1$. In delay–dependent stability criteria, the main concerns is to enlarge the feasible region of stability criteria in given time-delay interval. Interval time-varying delay means that a time delay varies in an interval in which the lower bound is not restricted to be zero. By constructing a suitable augmented Lyapunov functional and utilizing free weight matrices, some less conservative conditions for asymptotic stability are derived in [16–21] for systems with time delay varying in an interval. However, the shortcoming of the method used in these works is that the delay function is assumed to be differential and its derivative is still bounded: $\dot{h}(t) \leq \delta$. To the best of our knowledge, interval time-varying delay and mean square exponential stability of stochastic systems, non-differentiable time-varying delays have not been fully studied yet (see, e.g., [21–26] and the references therein), which are important in both theories and applications. This motivates our research.

This paper gives the improved results for the mean square exponential stability of stochastic systems with interval time-varying delay. The time delay is assumed to be a time-varying continuous function belonging to a given interval, but not necessary to be differentiable. By constructing augmented Lyapunov functional combined with LMI technique, we propose new criteria for the mean square exponential stability of stochastic systems with interval time-varying delay. The delay-dependent mean square exponential stability of stochastic systems with interval time-varying delay conditions are formulated in terms of LMIs.

The outline of the paper is as follows. Section 2 presents definitions and some well-known technical propositions needed for the proof of the main result. LMI delay-dependent mean square exponential stability of stochastic systems with interval time-varying delay criteria showing the effectiveness of the result are presented in Section 3. The paper ends with conclusions and cited references.

2. Preliminaries

The following notations will be used in this paper. R^+ denotes the set of all real non-negative numbers; R^n denotes the $n-$dimensional space with the scalar product $\langle .,. \rangle$ and the vector norm $\| . \|$; $M^{n\times r}$ denotes the space of all matrices of $(n \times r)-$dimensions; A^T denotes the transpose of matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\min/\max}(A) = \min/\max\{\text{Re}\lambda; \lambda \in \lambda(A)\}$; $x_t := \{x(t+s) : s \in [-h, 0]\}, \|x_t\| = \sup_{s\in[-h,0]} \| x(t+s) \|$; $C([0,t], R^n)$ denotes the set of all R^n-valued continuous functions on $[0,t]$; Matrix A is called semi-positive definite $(A \geq 0)$ if $\langle Ax, x\rangle \geq 0$, for all $x \in R^n$; A is positive definite $(A > 0)$ if $\langle Ax, x\rangle > 0$ for all $x \neq 0; A > B$ means $A - B > 0$. $*$ denotes the symmetric term in a matrix.

Consider a stochastic system with interval time-varying delay of the form

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Dx(t-h(t)) + \sigma(x(t), x(k-h(t)), t)\omega(t), \quad t \in R^+, \\ &x(t) = \phi(t), t \in [-h_2, 0], \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state; $A, D \in M^{n\times n}$, and $\phi(t) \in C([-h_2, 0], R^n)$ is the initial function with the norm $\| \phi \| = \sup_{s\in[-h_2,0]} \| \phi(s) \|$; The time-varying delay function $h(t)$ satisfies

$$0 \leq h_1 \leq h(t) \leq h_2, \quad t \in R^+.$$

$\omega(k)$ is a scalar Wiener process (Brownian Motion) on $(\Omega, \mathcal{F}, \mathcal{P})$ with

$$E[\omega(t)] = 0, \quad E[\omega^2(t)] = 1, \quad E[\omega(i)\omega(j)] = 0 (i \neq j), \tag{2}$$

and σ: $R^n \times R^n \times R \to R^n$ is the continuous function, and is assumed to satisfy that

$$\begin{aligned}&\sigma^T(x(t), x(t-h(t)), t)\sigma(x(t), x(t-h(t)), t) \leq \rho_1 x^T(t)x(t)\\ &+ \rho_2 x^T(t-h(t))x(t-h(t)), \qquad x(t), x(t-h(t)) \in R^n,\end{aligned} \tag{3}$$

where $\rho_1 > 0$ and $\rho_2 > 0$ are known constant scalars. For simplicity, we denote $\sigma(x(t), x(t-h(t)), t)$ by σ, respectively.

The stability problem for stochastic system (1) is to makes the system mean square exponential stability.

Definition 1. Given $\alpha > 0$. The stochastic system (1) is $\alpha-$exponentially stable in the mean square if there exists a positive number $N > 0$ such that every solution $x(t, \phi)$ of the system satisfies the following condition:

$$\exists N > 0: \quad E\{\| x(t,\phi) \|\} \leq E\left\{Ne^{-\alpha t} \| \phi \|\right\}, \quad \forall t \in R^+.$$

We end this section with the following technical well-known propositions, which will be used in the proof of the main results.

Proposition 1. (Cauchy inequality) *For any symmetric positive definite matrix* $N \in M^{n\times n}$ *and* $a, b \in R^n$ *we have*

$$\pm a^T b \leq a^T N a + b^T N^{-1} b.$$

Proposition 2. [27] *For any symmetric positive definite matrix* $M \in M^{n\times n}$*, scalar* $\gamma > 0$ *and vector function* $\omega : [0, \gamma] \to R^n$ *such that the integrations concerned are well defined, the following inequality holds*

$$\left(\int_0^\gamma \omega(s)\, ds\right)^T M \left(\int_0^\gamma \omega(s)\, ds\right) \leq \gamma \left(\int_0^\gamma \omega^T(s) M \omega(s)\, ds\right).$$

Proposition 3. [27] *Let* E, H *and* F *be any constant matrices of appropriate dimensions and* $F^T F \leq I$*. For any* $\epsilon > 0$*, we have*

$$EFH + H^T F^T E^T \leq \epsilon E E^T + \epsilon^{-1} H^T H.$$

Proposition 4. (Schur complement lemma [27]). *Given constant matrices* X, Y, Z *with appropriate dimensions satisfying* $X = X^T, Y = Y^T > 0$*. Then* $X + Z^T Y^{-1} Z < 0$ *if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0.$$

3. Main Results

Let us set

$$\mathcal{M} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ * & M_{22} & 0 & 0 & 0 \\ * & * & M_{33} & 0 & 0 \\ * & * & * & M_{44} & M_{45} \\ * & * & * & * & M_{55} \end{pmatrix},$$

$$\begin{aligned}
\lambda_1 &= \lambda_{\min}(P), \lambda_2 = \lambda_{\max}(P) + 2h_2^2\lambda_{\max}(R), \\
M_{11} &= A^T P + PA + 2\alpha P - e^{-2\alpha h_1} R - e^{-2\alpha h_2} R + 2\rho_1, \\
M_{12} &= e^{-2\alpha h_1} R, M_{13} = e^{-2\alpha h_2} R, M_{14} = PD, M_{15} = -S_2 A, \\
M_{22} &= -e^{-2\alpha h_1} R, M_{33} = -e^{-2\alpha h_2} R, M_{44} = -S_1 D + 2\rho_2, \\
M_{45} &= S_1 - S_2 D, M_{55} = S_2 + S_2^T + h_1^2 R + h_2^2 R.
\end{aligned}$$

The main result of this paper is summarized in the following theorem.

Theorem 1. *Given $\alpha > 0$. The zero solution of the stochastic system (1) is $\alpha-$exponentially stable in the mean square if there exist symmetric positive definite matrices P, R, and matrices $S_i, i = 1, 2$ such that satisfying the following conditions*

$$\mathcal{M} < 0. \tag{4}$$

Moreover, the solution $x(t, \phi)$ of the stochastic system satisfies

$$E\left[\| x(t,\phi) \|\right] \leq E\left[\sqrt{\frac{\lambda_2}{\lambda_1}} e^{-\alpha t} \| \phi \|\right], \quad \forall t \in R^+.$$

Proof. We consider the following Lyapunov-Krasovskii functional for the system (1)

$$V(t, x_t) = \sum_{i=1}^{3} V_i,$$

where

$$\begin{aligned}
V_1 &= x^T(t) P x(t), \\
V_2 &= h_1 \int_{-h_1}^{0} \int_{t+s}^{t} e^{2\alpha(\tau - t)} \dot{x}^T(\tau) R \dot{x}(\tau)\, d\tau\, ds, \\
V_3 &= h_2 \int_{-h_2}^{0} \int_{t+s}^{t} e^{2\alpha(\tau - t)} \dot{x}^T(\tau) R \dot{x}(\tau)\, d\tau\, ds.
\end{aligned}$$

It easy to check that

$$E[\lambda_1 \parallel x(t) \parallel^2] \leq E[V(t,x_t)] \leq E[\lambda_2 \parallel x_t \parallel^2], \quad \forall t \geq 0, \tag{5}$$

Taking the derivative of V_1 along the solution of system (1) and taking the mathematical expectation, we obtained

$$\begin{aligned}
E[\dot{V_1}] &= E[2x^T(t)P\dot{x}(t)] \\
&= E[x^T(t)[A^TP + AP]x(t) + 2x^T(t)PDx(t-h(t)) + 2x^T(t)P\sigma\omega(t)]; \\
E[\dot{V_2}] &= E[h_1^2\dot{x}^T(t)R\dot{x}(t) - h_1e^{-2\alpha h_1}\int_{t-h_1}^{t}\dot{x}^T(s)R\dot{x}(s)\,ds - 2\alpha V_2]; \\
E[\dot{V_3}] &= E[h_2^2\dot{x}^T(t)R\dot{x}(t) - h_2e^{-2\alpha h_2}\int_{t-h_2}^{t}\dot{x}^T(s)R\dot{x}(s)\,ds - 2\alpha V_3].
\end{aligned}$$

Applying Proposition 2 and the Leibniz - Newton formula, we have

$$\begin{aligned}
-h_i\int_{t-h_i}^{t}\dot{x}^T(s)R\dot{x}(s)\,ds &\leq -\Big[\int_{t-h_i}^{t}\dot{x}(s)\,ds\Big]^TR\Big[\int_{t-h_i}^{t}\dot{x}(s)\,ds\Big] \\
&\leq -[x(t)-x(t-h_i)]^TR[x(t)-x(t-h_i)] \\
&= -x^T(t)Rx(t) + 2x^T(t)Rx(t-h_i) \\
&\quad - x^T(t-h_i)Rx(t-h_i);
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&E[\dot{V}(.) + 2\alpha V(.)] \\
&\leq E[x^T(t)[A^TP + PA + 2\alpha P - e^{-2\alpha h_1}R - e^{-2\alpha h_2}R + 2\rho_1 I]x(t) \\
&+ 2x^T(t)[e^{-2\alpha h_1}R]x(t-h_1) + 2x^T(t)[e^{-2\alpha h_2}R]x(t-h_2) \\
&+ x^T(t-h_1)[-e^{-2\alpha h_1}R]x(t-h_1) + x^T(t-h_2)[-e^{-2\alpha h_2}R]x(t-h_2) \\
&+ x^T(t-h(t))[-S_1D + 2\rho_2 I]x(t-h(t)) + 2x^T(t-h(t))[S_1 - S_2D]\dot{x}(t) \\
&+ \dot{x}^T(t)[S_2 + S_2^T + h_1^2R + h_2^2R]\dot{x}(t)] \\
&= E[\zeta^T(t)\mathcal{M}\zeta(t)],
\end{aligned} \tag{6}$$

where

$$\zeta(t) = [x(t), x(t-h_1), x(t-h_2), x(t-h(t)), \dot{x}(t)].$$

Therefore, we finally obtain from (6) and the condition (4) that

$$E[\dot{V}(.) + 2\alpha V(.)] < 0, \quad \forall i = 1,2,....,N, \quad t \in R^+.$$

and hence

$$E[\dot{V}(t, x_t)] \leq E[-2\alpha V(t, x_t)], \quad \forall t \in R^+. \tag{7}$$

Integrating both sides of (7) from 0 to t, we obtain

$$E[V(t, x_t)] \leq E[V(\phi)e^{-2\alpha t}], \quad \forall t \in R^+.$$

Furthermore, taking condition (5) into account, we have

$$E[\lambda_1 \parallel x(t,\phi) \parallel^2] \leq E[V(x_t)] \leq E[V(\phi)e^{-2\alpha t}] \leq E[\lambda_2 e^{-2\alpha t} \parallel \phi \parallel^2],$$

then

$$E\left[\parallel x(t,\phi) \parallel\right] \leq E\left[\sqrt{\frac{\lambda_2}{\lambda_1}} e^{-\alpha t} \parallel \phi \parallel\right], \quad t \in R^+,$$

which concludes the proof by Definition 1, concludes the proof of the theorem in the mean square. □

Remark 2. Theorem 1 provides sufficient conditions for stochastic systems with interval time-varying delays (1) in terms of the solutions of LMIs, which guarantees the closed-loop system to be exponentially stable with a prescribed decay rate α. The developed method using new inequalities for lower bounding cross terms eliminate the need for over bounding and provide larger values of the admissible delay bound. Note that the time-varying delays are non-differentiable, therefore, the methods proposed in [1–11, 17–25] are not applicable to system (1). The LMI condition (4) depends on parameters of the system under consideration as well as the delay bounds. The feasibility of the LMIs can be tested by the reliable and efficient Matlab LMI Control Toolbox [27].

4. Conclusion

In this paper, we have proposed new delay-dependent conditions for the mean square exponential stability of stochastic systems with non-differentiable interval time-varying delay. Based on the improved Lyapunov-Krasovskii functionals and linear matrix inequality technique, the conditions for the mean square exponential stability of stochastic systems with interval time-varying delay have been established in terms of LMIs.

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