International Journal of Pure and Applied Mathematics
Volume 94 No. 4 2014, 501-513
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v94i4.5
ijpam.eu

EXISTENCE OF WEAK SOLUTION FOR NONLINEAR ELLIPTIC SYSTEMS

Tahar Bouali[1 §], Rafik Guefaifia[2]
[1,2]Department of Mathematics
Tebessa University
Tebessa, 12000, ALGERIA

Abstract: This paper is concerned with some nonlinear elliptic systems,Under suitable conditions on the nonlinearities f and g, we obtain weak solution in sobolev space $H = H_0^1(\Omega) \times H_0^1(\Omega)$ by applying the Banach fixed point theorem.

AMS Subject Classification: 35D05, 35J60, 35J70, 15A18
Key Words: weak solution, nonlinear elliptic systems, Laplace operator, Riesz theorem, Banach fixed point theorem

1. Introduction

We study the nonlinear elliptic systems of the form

$$\begin{cases} -\triangle u - \operatorname{div}\left(\left(1+|\nabla u|^2\right)^{\frac{P(x)-2}{2}} \nabla u\right) = \lambda\ f(x,u,v) \ \text{ in } \Omega \\ -\triangle v - \operatorname{div}\left(\left(1+|\nabla v|^2\right)^{\frac{P(x)-2}{2}} \nabla v\right) = \lambda\ g(x,u,v) \ \text{ in } \Omega \\ u = v = 0 \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded smooth open set in $\mathbb{R}^N. (N \geq 3), p : \overline{\Omega} \to (1,2)$ is con-

Received: February 9, 2014

§Correspondence author

tinuous function, λ is positive constant and $\triangle u = \operatorname{div}(\nabla u)$ is the Laplacian of u.

Theorems concerning the existence and properties of fixed point are known as fixed point theorems.Such theorems are the most important tools for proving the existence and uniquencess of the solutions to various mathematical models (differential,integral and partial differential equation,ect...).

In recent years,many publications have appeared concerning quasilinear elliptic systems which have been used in a great variety of applications,we refer the readers to $[5, 6, 7, 8, 9]$ used variational methods to obtain weak solution of semlinear elliptic system and quasilinear elliptic system.

Motivated by [4] in this paper,we will discuss problem $(1, 1)$,Under the suitable condition on the nonlinearities $f(x, u, v)$ and $g(x, u, v)$,using Banach fixed point theorem (see [10], we show that system $(1, 1)$ has a unique weak solutions.

Throughout this paper for $(u, v) \in \mathbb{R}^2$,denote $|(u, v)|^2 = |u|^2 + |v|^2$.We assume that f and g are L^2 functions which are Lipschitz continuous with respect to the secand variable,i.e,there exist constants c_1,$c_2 \succ 0$ such that for a.e.$x \in \Omega$ and for any $(u, v), (u_1, v_1) \in \mathbb{R}^2$

$$|f(x, u, v) - f(x, u_1, v_1)| \leq c_1 |(u, v) - (u_1, v_1)| \tag{1.2}$$

$$|g(x, u, v) - g(x, u_1, v_1)| \leq c_2 |(u, v) - (u_1, v_1)| \tag{1.3}$$

Let $\lambda_1$be the first eigenvalue of Dirichlet problem $-\triangle u = \lambda u$.The main result of this paper is a follows:

Definition 1. We say that $(u, v) \in H$ is a weak solution of $(1, 1)$ if

$$\int_\Omega \left[\nabla u \nabla \xi + \nabla v \nabla \eta + \left(1 + |\nabla u|^2\right)^{\frac{P(x)-2}{2}} \nabla u \nabla \xi + \left(1 + |\nabla v|^2\right)^{\frac{P(x)-2}{2}} \nabla v \nabla \eta\right] dx$$

$$= \lambda \int_\Omega [f(x, u, v)\xi + g(x, u, v)\ \eta]\, dx. \quad \text{for all } (\xi, \eta) \in H.$$

Theorem 2. *Suppose that conditions* (1.2) *and* (1.3) *hold. For any* $\lambda \in \left(0, \frac{\lambda_1}{c_1 + c_2}\right)$ *there existe a unique weak solution of* $(1, 1)$.

This paper is organized as follows. In Section 2, we present some relevant lemmas. We reserve the Section 3 for the proof of the main result.

2. Preliminary Lemmas

Given a bounded smooth open set $\Omega \subset \mathbb{R}^N$. Let us consider the Hilbert space $H = H_0^1(\Omega) \times H_0^1(\Omega)$ and $\langle , \rangle_{L^2}$ the inner product in $L^2(\Omega)$. The norm on H given by

$$\|(u,v)\| = \left(\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 \right) dx \right)^{\frac{1}{2}}$$

and the norm on $L^2(\Omega) \times L^2(\Omega)$ is given by

$$\|(u,v)\|_{L^2(\Omega)\times L^2(\Omega)} = \left(\int_\Omega \left(|u|^2 + |v|^2 \right) dx \right)^{\frac{1}{2}}$$

Lemma 3. *Let $F : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be on operator defined by*

$$F(x,\xi) = \left(1 + |\xi|^2\right)^{\frac{P(x)-2}{2}}, \forall x \in \Omega, \forall \xi \in \mathbb{R}^N.$$

then

$$F(x,\xi) \le 1$$

for all $\xi \in \mathbb{R}^N$ and all $x \in \Omega$.

Proof. We fix $x_0 \in \overline{\Omega}$ and denote by $q = p(x_0)$ .Then

$$F(x_0,\xi) = \frac{1}{\left(1 + |\xi|^2\right)^{\frac{2-q}{2}}}$$

where

$$0 < \frac{2-q}{2} < 1$$

We define the auxiliary function $g : [0,\infty) \to \mathbb{R}$ by

$$g(t) = \left(1 + t^2\right)^{\frac{2-q}{2}}$$

It can be easily seen that g is increasing on $[0,\infty)$ since $\frac{2-q}{2} > 0$.Thus,we obtain that $g(t) \ge g(0) = 1, \forall t \ge 0$.Taking $t = |\xi|$, the conclusion of the lemma follows. □

Lemma 4. *For any $\xi, \eta \in \mathbb{R}^N$, any $x \in \overline{\Omega}$ and any $i \in \{1, 2, ..., N\}$ the following inequalities hold true:*

$$\left[\left(1+|\xi|^2\right)^{\frac{P(x)-2}{2}}\xi-\left(1+|\eta|^2\right)^{\frac{P(x)-2}{2}}\eta\right].(\xi-\eta)\geq 0. \qquad \text{(a)}$$

$$\left[\left(1+|\xi|^2\right)^{\frac{P(x)-2}{2}}\xi_i-\left(1+|\eta|^2\right)^{\frac{P(x)-2}{2}}\eta_i\right]\leq\frac{N+3}{2}|\xi-\eta|. \qquad \text{(b)}$$

Here,the symbol "." means the inner product in $\mathbb{R}^N$.

Proof. (a) Let $x_0\in\Omega$ be arbitrary but fixed, and define $q=p(x_0)\in(1,2)$ .Our inequality is equivalent with

$$\left[\left(1+|\xi|^2\right)^{\frac{q-2}{2}}|\xi|^2+\left(1+|\eta|^2\right)^{\frac{q-2}{2}}|\eta|^2\right]\geq\left[\left(1+|\xi|^2\right)^{\frac{q-2}{2}}+\left(1+|\eta|^2\right)^{\frac{q-2}{2}}\right]\xi.\eta.$$

We shall prove that

$$\left(1+|\xi|^2\right)^{\frac{q-2}{2}}|\xi|^2+\left(1+|\eta|^2\right)^{\frac{q-2}{2}}|\eta|^2\geq\left[\left(1+|\xi|^2\right)^{\frac{q-2}{2}}+\left(1+|\eta|^2\right)^{\frac{q-2}{2}}\right]|\xi.\eta|,$$

or

$$\left[\left(1+|\xi|^2\right)^{\frac{q-2}{2}}|\xi|-\left(1+|\eta|^2\right)^{\frac{q-2}{2}}|\eta|\right](|\xi|-|\eta|)\geq 0 \qquad (2.1)$$

We define the auxiliary function $\phi:[0,\infty)\to\mathbb{R}$ by $\phi(t)=\left(1+t^2\right)^{\frac{q-2}{2}}t$. Obviously ϕ is derivable and we have:

$$\phi'(t)=\left(1+t^2\right)^{\frac{q-4}{2}}\left[1+(q-1)t^2\right]\geq 0$$

Hence ϕ is increasing on $[0,\infty)$ which implies

$$[\phi(t_1)-\phi(t_2)](t_1-t_2)\geq 0,\forall t_1,t_2\in[0,\infty].$$

Taking $t_1=|\xi|$, $t_2=|\eta|$ the inequality (1.1) follows.

(b) Let $x\in\overline{\Omega}$ be fixed and $h_i:\mathbb{R}^N\to\mathbb{R}$ be defined by

$h_i(\xi)=\left(1+|\xi|^2\right)^{\frac{P(x)-2}{2}}\xi_i,\forall\xi\in\mathbb{R}^N,\forall i\in\{1,2,...,N\}$.

Using the mean value theorem we deduce that

$$|h_i(\xi)-h_i(\eta)|\leq|\xi-\eta|\sup_{\theta\in[\xi,\eta]}|\nabla h_i(\theta)| \qquad (2.2)$$

where $[\xi,\eta]$ is the line segment in $\mathbb{R}^N$ between the points ξ and η, i.e.,

$$[\xi,\eta]=\{t\xi+(1-t)\eta:t\in[0,1]\}$$

but

$$|\nabla h_i(\theta)| = \left(\sum_{j=1}^{N} \left(\frac{\partial h_i(\theta)}{\partial \theta_j}\right)^2\right)^{\frac{1}{2}} \leq \sum_{j=1}^{N} \left|\frac{\partial h_i(\theta)}{\partial \theta_j}\right| \tag{2.3}$$

For $j \neq i$

$$\frac{\partial h_i(\theta)}{\partial \theta_j} = (p(x) - 2)\left(1 + |\theta|^2\right)^{\frac{P(x)-2}{2}} \theta_i \theta_j,$$

and for $j = i$

$$\frac{\partial h_i(\theta)}{\partial \theta_j} = \left(1 + |\theta|^2\right)^{\frac{P(x)-2}{2}} + (p(x) - 2)\left(1 + |\theta|^2\right)^{\frac{P(x)-2}{2}} \theta_i^2.$$

Thus, by $(2,3)$ and lemma 3 we find

$$\begin{aligned} |\nabla h_i(\theta)| &\leq \textstyle\sum_{j=1}^{N} \left|\frac{\partial h_i(\theta)}{\partial \theta_j}\right| \\ &\leq \left(1 + |\theta|^2\right)^{\frac{P(x)-2}{2}} + \textstyle\sum_{j=1}^{N} \left|(p(x) - 2)\left(1 + |\theta|^2\right)^{\frac{p(x)-4}{2}} \theta_i \theta_j\right| \\ &= \left(1 + |\theta|^2\right)^{\frac{P(x)-2}{2}} + (2 - p(x))\left(1 + |\theta|^2\right)^{\frac{p(x)-4}{2}} \textstyle\sum_{j=1}^{N} |\theta_i \theta_j| \\ &\leq 1 + (2 - p(x))\left(1 + |\theta|^2\right)^{\frac{p(x)-4}{2}} \textstyle\sum_{j=1}^{N} \frac{\theta_i^2 + \theta_j^2}{2} \\ &\leq 1 + (2 - p(x))\left(1 + |\theta|^2\right)^{\frac{p(x)-4}{2}} \frac{N+1}{2} |\theta|^2 \\ &\leq 1 + (2 - p(x))\left(1 + |\theta|^2\right)^{\frac{p(x)-2}{2}} \frac{N+1}{2} \\ &\leq 1 + \frac{N+1}{2} = \frac{N+3}{2} \end{aligned}$$

Combining the above estimates with relation (2.2) we obtain

$$\left|\left(1 + |\xi|^2\right)^{\frac{P(x)-2}{2}} \xi_i - \left(1 + |\eta|^2\right)^{\frac{P(x)-2}{2}} \eta_i\right| \leq \frac{N+3}{2} |\xi - \eta|.$$

The proof of lemma 4 is now complet. □

Proof of the main result. In order to prove theorem 2 we use a methode borrowed from the proof of a nonlinear version of the Lax-Milgram theorem

(see Zeidler [10], section 2.15).Our proof will use as main tool the Banach fixed point theorem (see Zeidler [10], section 1.6)

First,we define the operators $a : H \times H \to \mathbb{R}$ by

$$a\left(\left(u,v\right),\left(\xi,\eta\right)\right) = \int\limits_{\Omega} \nabla u \nabla \xi dx + \int\limits_{\Omega} \nabla v \nabla \eta dx + \int\limits_{\Omega} \left(1+\left|\nabla u\right|^2\right)^{\frac{P(x)-2}{2}} \nabla u \nabla \xi dx + \int\limits_{\Omega} \left(1+\left|\nabla v\right|^2\right)^{\frac{P(x)-2}{2}} \nabla v \nabla \eta dx,$$

respectively $b_\lambda : H \times H \to \mathbb{R}$ by

$$b_\lambda\left(\left(u,v\right),\left(\xi,\eta\right)\right) = \lambda \left[\int\limits_{\Omega} f\left(x,u,v\right)\xi dx + \int\limits_{\Omega} g\left(x,u,v\right)\eta dx \right].$$

□

Lemma 5. *The operators a and b_λ satisfy the following properties:*

(A_1) *for each $(u,v) \in H$,the application $(\xi,\eta) \longmapsto a\left(\left(u,v\right),\left(\xi,\eta\right)\right)$ is linear and continuous.*

(A_2)

$$a\left(\left(u,v\right),\left(u,v\right)-\left(u_1,v_1\right)\right) - a\left(\left(u_1,v_1\right),\left(u,v\right)-\left(u_1,v_1\right)\right) \geq \left\|\left(u,v\right)-\left(u_1,v_1\right)\right\|^2$$

for all $(u,v),(u_1,v_1) \in H$.

(A_3) *there exist $M > 0$ such that*

$$\left|a\left(\left(u,v\right),\left(\xi,\eta\right)\right) - a\left(\left(u_1,v_1\right),\left(\xi,\eta\right)\right)\right| \leq M\left\|\left(u,v\right)-\left(u_1,v_1\right)\right\|\left\|\left(\xi,\eta\right)\right\|$$

for all $(u,v),(u_1,v_1),(\xi,\eta) \in H$.

(B_1) *for each $(u,v) \in H$,the application $(\xi,\eta) \longmapsto b_\lambda\left(\left(u,v\right),\left(\xi,\eta\right)\right)$ is linear and continuous.*

(B_2)

$$\begin{aligned} &b_\lambda\left(\left(u,v\right),\left(u,v\right)-\left(u_1,v_1\right)\right) - b_\lambda\left(\left(u_1,v_1\right),\left(u,v\right)-\left(u_1,v_1\right)\right) \\ &\leq \tfrac{\lambda(c_1+c_2)}{\lambda_1}\left\|\left(u,v\right)-\left(u_1,v_1\right)\right\|^2 \end{aligned}$$

for all $(u,v),(u_1,v_1) \in H$. (B_3) there exist $N = N(\lambda) > 0$ such that

$$\left|b_\lambda\left(\left(u,v\right),\left(\xi,\eta\right)\right) - b_\lambda\left(\left(u_1,v_1\right),\left(\xi,\eta\right)\right)\right| \leq N\left\|\left(u,v\right)-\left(u_1,v_1\right)\right\|\left\|\left(\xi,\eta\right)\right\|$$

for all $(u,v),(u_1,v_1),(\xi,\eta) \in H$.

Proof. (A_1) We fixe $(u,v)\in H$. It is clear that $(\xi,\eta)\longmapsto a((u,v),(\xi,\eta))$ is linear. On the other hand, using Lemma 3 and Holder's inequality we have

$$|a((u,v),(\xi,\eta))| = \left| \int_\Omega \nabla u \nabla \xi dx + \int_\Omega \nabla v \nabla \eta dx + \int_\Omega \left(1+|\nabla u|^2\right)^{\frac{P(x)-2}{2}} \nabla u \nabla \xi dx \right.$$
$$\left. + \int_\Omega \left(1+|\nabla v|^2\right)^{\frac{P(x)-2}{2}} \nabla v \nabla \eta dx \right| \le 4 \|(u,v)\| \, \|(\xi,\eta)\|$$

it follows that $(\xi,\eta)\longmapsto a((u,v),(\xi,\eta))$ is continuous.

(A_2) Using Lemma 4, part (a), we deduce that

$$\begin{aligned}
& a((u,v),(u,v)-(u_1,v_1)) - a((u_1,v_1),(u,v)-(u_1,v_1)) \\
&= \|(u,v)-(u_1,v_1)\|^2 \\
&+ \int_\Omega \left(1+|\nabla u|^2\right)^{\frac{P(x)-2}{2}} |\nabla u|^2 + \left(1+|\nabla u_1|^2\right)^{\frac{P(x)-2}{2}} |\nabla u_1|^2 \\
&- \left(1+|\nabla u|^2\right)^{\frac{P(x)-2}{2}} \nabla u \nabla u_1 - \left(1+|\nabla u_1|^2\right)^{\frac{P(x)-2}{2}} \nabla u \nabla u_1) dx \\
&+ \int_\Omega \left(1+|\nabla v|^2\right)^{\frac{P(x)-2}{2}} |\nabla v|^2 + \left(1+|\nabla v_1|^2\right)^{\frac{P(x)-2}{2}} |\nabla v_1|^2 \\
&- \left(1+|\nabla v|^2\right)^{\frac{P(x)-2}{2}} \nabla v \nabla v_1 - \left(1+|\nabla v_1|^2\right)^{\frac{P(x)-2}{2}} \nabla v \nabla v_1) dx \\
&= \|(u,v)-(u_1,v_1)\|^2 \\
&+ \int_\Omega \left[\left(1+|\nabla u|^2\right)^{\frac{P(x)-2}{2}} \nabla u - \left(1+|\nabla u_1|^2\right)^{\frac{P(x)-2}{2}} \nabla u_1\right] . (\nabla u - \nabla u_1) dx \\
&+ \int_\Omega \left[\left(1+|\nabla v|^2\right)^{\frac{P(x)-2}{2}} \nabla v - \left(1+|\nabla v_1|^2\right)^{\frac{P(x)-2}{2}} \nabla v_1\right] . (\nabla v - \nabla v_1) dx \\
&\ge \|(u,v)-(u_1,v_1)\|^2
\end{aligned}$$

(A_3) Using Holder's inequality and Lemma 4 part (b) we get

$$\begin{aligned}
& |a((u,v),(\xi,\eta)) - a((u_1,v_1),(\xi,\eta))| \\
&= \int_\Omega (\nabla u - \nabla u_1) \nabla \xi dx + \int_\Omega (\nabla v - \nabla v_1) \nabla \eta dx
\end{aligned}$$

$$+\int_\Omega \left[\left(1+|\nabla u|^2\right)^{\frac{P(x)-2}{2}}\nabla u-\left(1+|\nabla u_1|^2\right)^{\frac{P(x)-2}{2}}\nabla u_1\right]\nabla\xi dx$$

$$+\int_\Omega \left[\left(1+|\nabla v|^2\right)^{\frac{P(x)-2}{2}}\nabla v-\left(1+|\nabla v_1|^2\right)^{\frac{P(x)-2}{2}}\nabla v_1\right]\nabla\eta dx$$

$$\leq 2\left\|(u,v)-(u_1,v_1)\right\|\left\|(\xi,\eta)\right\|$$

$$+\int_\Omega\sum_{j=1}^N\left|\left(1+|\nabla u|^2\right)^{\frac{P(x)-2}{2}}\frac{\partial u}{\partial x_j}-\left(1+|\nabla u_1|^2\right)^{\frac{P(x)-2}{2}}\frac{\partial u_1}{\partial x_j}\right|\left|\frac{\partial\xi}{\partial x_j}\right|dx$$

$$+\int_\Omega\sum_{j=1}^N\left|\left(1+|\nabla v|^2\right)^{\frac{P(x)-2}{2}}\frac{\partial v}{\partial x_j}-\left(1+|\nabla v_1|^2\right)^{\frac{P(x)-2}{2}}\frac{\partial v_1}{\partial x_j}\right|\left|\frac{\partial\eta}{\partial x_j}\right|dx$$

$$\leq 2\left\|(u,v)-(u_1,v_1)\right\|\left\|(\xi,\eta)\right\|+\frac{N+3}{2}\int_\Omega\sum_{j=1}^N\left|(\nabla u-\nabla u_1)\right|\left|\frac{\partial\xi}{\partial x_j}\right|dx$$

$$+\frac{N+3}{2}\int_\Omega\sum_{j=1}^N\left|(\nabla v-\nabla v_1)\right|\left|\frac{\partial\eta}{\partial x_j}\right|dx$$

$$\leq 2\left\|(u,v)-(u_1,v_1)\right\|\left\|(\xi,\eta)\right\|+\frac{N+3}{2}\int_\Omega\left|(\nabla u-\nabla u_1)\right|\sum_{j=1}^N\left|\frac{\partial\xi}{\partial x_j}\right|dx$$

$$+\frac{N+3}{2}\int_\Omega\left|(\nabla v-\nabla v_1)\right|\sum_{j=1}^N\left|\frac{\partial\eta}{\partial x_j}\right|dx$$

$$\leq 2\left\|(u,v)-(u_1,v_1)\right\|\left\|(\xi,\eta)\right\|+(N+3)\sqrt{N}\left\|(u,v)-(u_1,v_1)\right\|\left\|(\xi,\eta)\right\|$$

$$\leq\left(2+(N+3)\sqrt{N}\right)\left\|(u,v)-(u_1,v_1)\right\|\left\|(\xi,\eta)\right\|=M\left\|(u,v)-(u_1,v_1)\right\|\left\|(\xi,\eta)\right\|$$

where $M=2+(N+3)\sqrt{N}$.

(B_1) We fixe $(u,v)\in H$.obviously, the application $(\xi,\eta)\longmapsto b_\lambda\left((u,v),(\xi,\eta)\right)$ is linear.

Using Holder's inequality,we have

$$\left|b_\lambda\left((u,v),(\xi,\eta)\right)\right|=\left|\lambda\left[\int_\Omega f(x,u,v)\,\xi dx+\int_\Omega g(x,u,v)\ \eta\mathbf{dx}\right]\right|$$

$$\leq \lambda \left(\int_{\Omega} |f(x,u,v)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\xi|^2 dx \right)^{\frac{1}{2}}$$
$$+ \lambda \left(\int_{\Omega} |g(x,u,v)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\eta|^2 dx \right)^{\frac{1}{2}} \leq C \|(\xi,\eta)\|,$$

where C is positive constant.

(B_2)

$$\begin{aligned}
& b_\lambda ((u,v),(u,v)-(u_1,v_1)) - b_\lambda ((u_1,v_1),(u,v)-(u_1,v_1)) \\
=& \lambda \int_{\Omega} [f(x,u,v) - f(x,u_1,v_1)](u-u_1)\,dx \\
& + \lambda \int_{\Omega} [g(x,u,v) - g(x,u_1,v_1)](v-v_1)\,dx \\
\leq& \lambda c_1 \left(\left(\int_{\Omega} |u-u_1|^2 + |v-v_1|^2 \right) dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |u-u_1|^2 dx \right)^{\frac{1}{2}} \\
& + \lambda c_2 \left(\left(\int_{\Omega} |u-u_1|^2 + |v-v_1|^2 \right) dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v-v_1|^2 dx \right)^{\frac{1}{2}} \\
\leq& \lambda c_1 \|(u,v)-(u_1,v_1)\|^2_{L^2(\Omega)\times L^2(\Omega)} + \lambda c_2 \|(u,v)-(u_1,v_1)\|^2_{L^2(\Omega)\times L^2(\Omega)} \\
\leq& \lambda (c_1+c_2) \|(u,v)-(u_1,v_1)\|^2_{L^2(\Omega)\times L^2(\Omega)} \\
\leq& \frac{\lambda (c_1+c_2)}{\lambda_1} \|(u,v)-(u_1,v_1)\|^2 .
\end{aligned}$$

(B_3) Using Holder's inequality and $(1.2),(1.3)$ we obtain

$$\begin{aligned}
& |b_\lambda ((u,v),(\xi,\eta)) - b_\lambda ((u_1,v_1),(\xi,\eta))| \\
=& \left| \lambda \int_{\Omega} [f(x,u,v) - f(x,u_1,v_1)]\,\xi dx + \lambda \int_{\Omega} [g(x,u,v) - g(x,u_1,v_1)]\,\eta dx \right| \\
\leq& \lambda c_1 \|(u,v)-(u_1,v_1)\|^2_{L^2(\Omega)\times L^2(\Omega)} \left(\int_{\Omega} |\xi|^2 dx \right)^{\frac{1}{2}}
\end{aligned}$$

$$+\lambda c_2 \left\|(u,v)-(u_1,v_1)\right\|^2_{L^2(\Omega)\times L^2(\Omega)} \left(\int_\Omega |\eta|^2 \,dx\right)$$
$$\leq \lambda\left(c_1+c_2\right)\left\|(u,v)-(u_1,v_1)\right\|^2_{L^2(\Omega)\times L^2(\Omega)} \left\|(\xi,\eta)\right\|_{L^2(\Omega)\times L^2(\Omega)}$$
$$\leq \frac{\lambda\left(c_1+c_2\right)}{\lambda_1}\left\|(u,v)-(u_1,v_1)\right\|\left\|(\xi,\eta)\right\|$$
$$\leq N\left\|(u,v)-(u_1,v_1)\right\|\left\|(\xi,\eta)\right\|,$$

where $N=\frac{\lambda(c_1+c_2)}{\lambda_1}$. □

3. Proof of Main Theorem

In this section we give the proof of theorem 2.

Proof of Theorem 2. Let $\lambda \in \left(0, \frac{\lambda_1}{c_1+c_2}\right)$ be arbitrary but fixed.By lemma 5. (A_1) and the Riesz theorem (see e.g. Brezis [2], theorem V.5) we deduce that for each $(u,v)\in H$ there existe a unique element denote by $A(u,v)\in H$ such that $a\left((u,v),(\xi,\eta)\right)=\langle A(u,v),(\xi,\eta)\rangle$.

Thus we can define the operator $A: H\to H$.Using lemma 5. (A_2) it follows that

$$\langle A(u,v)-A(u_1,v_1),(u,v)-(u_1,v_1)\rangle \geq \left\|(u,v)-(u_1,v_1)\right\|^2$$

for all $(u,v),(u_1,v_1)\in H$ i.e, A is strongly monotone.

lemma 5. (A_3) yields

$$\left|\langle A(u,v),(\xi,\eta)\rangle-\langle A(u_1,v_1),(\xi,\eta)\rangle\right| \leq M\left\|(u,v)-(u_1,v_1)\right\|\left\|(\xi,\eta)\right\|$$

for all $(u,v),(u_1,v_1),(\xi,\eta)\in H$.Hence,

$$\left\|A(u,v)-A(u_1,v_1)\right\| = \sup_{\|(\xi,\eta)\|\leq 1}\left|A(u_1,v_1),(\xi,\eta)\right| \leq M\left\|(u,v)-(u_1,v_1)\right\| \tag{3.2}$$

i.e, A is Lipschitz continuous.

By lemma 5. (B_1) and the Riesz theorem we deduce that for each $(u,v)\in H$ there exists a unique element $B_\lambda(u,v)\in H$ such that

$$b_\lambda\left((u,v),(\xi,\eta)\right)=\langle B_\lambda(u,v),(\xi,\eta)\rangle, \forall(\xi,\eta)\in H.$$

Thus,we can also define the operator $B_\lambda : H \to H$ which satisfies

$$\begin{aligned} &\langle B_\lambda(u,v),(u,v)-(u_1,v_1)\rangle - \langle B_\lambda(u_1,v_1),(u,v)-(u_1,v_1)\rangle \\ &\leq \tfrac{\lambda(c_1+c_2)}{\lambda_1}\|(u,v)-(u_1,v_1)\|^2 \end{aligned} \tag{3.3}$$

Using lemma 5. (B_3) we find

$$\begin{aligned} \|B_\lambda(u,v)-B_\lambda(u_1,v_1)\| &= \sup_{\|(\xi,\eta)\|\leq 1} |\langle B_\lambda(u,v),(\xi,\eta)\rangle - \langle B_\lambda(u_1,v_1),(\xi,\eta)\rangle| \\ &= \sup_{\|(\xi,\eta)\|\leq 1} |b_\lambda((u,v),(\xi,\eta)) - b_\lambda((u_1,v_1),(\xi,\eta))| \\ &\leq N\|(u,v)-(u_1,v_1)\| \end{aligned} \tag{3.4}$$

we define the operator $S : H \to H$ by

$$S(u,v) = (u,v) - t(A(u,v) - B_\lambda(u,v))$$

where $t \in \left(0, \frac{2\left(1-\frac{\lambda(c_1+c_2)}{\lambda_1}\right)}{(N+M)^2}\right)$,The relation (3.1) – (3.4) shows that for each $(u,v),(u_1,v_1) \in H$ we have

$$\begin{gathered} \|S(u,v)-S(u_1,v_1)\|^2 \\ = \langle S(u,v)-S(u_1,v_1), S(u,v)-S(u_1,v_1)\rangle \\ = \langle (u,v)-t(A(u,v)-B_\lambda(u,v)) - (u_1,v_1) + t(A(u,v)-B_\lambda(u,v))\rangle \\ = \|(u,v)-(u_1,v_1)\|^2 - 2t\langle A(u,v)-A(u_1,v_1),(u,v)-(u_1,v_1)\rangle \\ +2t\langle B_\lambda(u,v)-B_\lambda(u_1,v_1),(u,v)-(u_1,v_1)\rangle - 2t^2\langle A(u,v) \\ -A(u_1,v_1), B_\lambda(u,v)-B_\lambda(u_1,v_1)\rangle + t^2\|A(u,v)-A(u_1,v_1)\|^2 \\ +t^2\|B_\lambda(u,v)-B_\lambda(u_1,v_1)\|^2 \\ \leq \|(u,v)-(u_1,v_1)\|^2 - 2t\|(u,v)-(u_1,v_1)\|^2 + 2t\tfrac{\lambda(c_1+c_2)}{\lambda_1}\|(u,v)-(u_1,v_1)\|^2 \\ +2t^2(M\|(u,v)-(u_1,v_1)\|)(N\|(u,v)-(u_1,v_1)\|) \\ +t^2M^2\|(u,v)-(u_1,v_1)\|^2 + t^2N^2\|(u,v)-(u_1,v_1)\|^2 \\ \leq \left[1-2t\left(1-\tfrac{\lambda(c_1+c_2)}{\lambda_1}\right) + M^2t^2 + N^2t^2 + 2NMt^2\right]\|(u,v)-(u_1,v_1)\|^2 \\ \leq \alpha\|(u,v)-(u_1,v_1)\|^2 \end{gathered}$$

where

$$\alpha = 1 - 2\left(1 - \frac{\lambda(c_1+c_2)}{\lambda_1}\right)t + (N+M)^2t^2 \geq 0$$

If $t = 0$ or $t = \frac{2\left(1-\frac{\lambda(c_1+c_2)}{\lambda_1}\right)}{(N+M)^2}$ then $\alpha = 1$.This implies that $\sqrt{\alpha} < 1$ for all $t \in \left(0, \frac{2\left(1-\frac{\lambda(c_1+c_2)}{\lambda_1}\right)}{(N+M)^2}\right)$.

Hence

$$\|S(u,v) - S(u_1,v_1)\| \leq \sqrt{\alpha}\,\|(u,v) - (u_1,v_1)\|\,, \forall\,(u,v)\,,(u_1,v_1) \in H$$

i.e., S is $\sqrt{\alpha}$ contractive with $\sqrt{\alpha} < 1$.By Banach fixed point theorem (see Zeidler [10], section 1.6) it follows that there is a unique solution $(u,v) \in H$

of problem $S(u,v) = (u,v)$ i.e.,the problem $A(u,v) = B_\lambda(u,v)$ has a unique solution $(u,v) \in H$ .It follows that

$$\langle A(u,v)\,,(\xi,\eta)\rangle = \langle B_\lambda(u,v)\,,(\xi,\eta)\rangle, \forall\,(\xi,\eta) \in H$$

i.e.,

$$a\left((u,v)\,,(\xi,\eta)\right) = b_\lambda\left((u,v)\,,(\xi,\eta)\right),$$

Thus the proof of Thoerem 2 is complet. □

References

[1] H. Amann, Fixed point equation and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Review 18 (1976), 620 − 709.dx.doi.org/10.1137/1018114.

[2] H. Brezis, Analyse Fonctionnelle: theorie et application, Masson, Paris, 1992.

[3] L. Collatz,Functional analysis and numerical analysis,Academic Press,New York,1966.

[4] N. Costea, M. Mihailescu,On an eigenvalue problems involving variable exponent growth conditions, Nonlinear Anal.71 (2009), 4271 − 4278.dx.doi.org/10.1016/j.na.2009.02.117.

[5] A. Djellit, S. Tas, Existence of solutions for a class of elliptic systems in R^N involving the $p-$Lplacian,Electronic j. Diff. Eqns. Vol.2003 (2003), No. 56, 1 − 8.

[6] A. Djellit, S. Tas, On some nonlinear elliptic systems, Nonlinear Anal. 59 (2004),695 − 706.dx.doi.org/10.1016/S0362-546X(04)00279-2

[7] A. Djellit, S. Tas, Quaslinear elliptic systems with critical sobolev exponents in R^N, Nonlinear Anal. 66 (2007), 1485 − 1497.

[8] P. Drabek, N. M. Stavrakakis, N. B. Zographopoules, Multiple non-semitrivial solutions for quasilinear systems. Differential Integral Equations 16 (12) (2003) , 1519 − 1531.

[9] D. D. Hai, H. Wang, Nontrivial solutions for $p-$Lplacian systems, j. Math. Anal. Appl 330. (2007) , 186 − 194.dx.doi.org/10.1016/j.jmaa.2006.07.072.

[10] Zeidler, Applied Functional Analysis: Applications in Mathematical Physics, Springer-Verlag, New York, 1995.

[11] J. Zhang, Z. Zhang, Existence results for some nonlinear elliptic systems, Nonlinear Anal. 71 (2009) , 2840 − 2846.dx.doi.org/10.1016/j.na.2009.01.158.