International Journal of Pure and Applied Mathematics
Volume 94 No. 4 2014, 515-524
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v94i4.6
ijpam.eu

ON THE MOMENTS OF STOCHASTIC ANNUITIES

A.R. Mugdadi[1 §], Ibrahim A. Ahmad[2]

[1]Department of Mathematics and Statistics
Jordan University of Science and Technology
Irbid, JORDAN

[2]Department of Statistics
Oklahoma State University.
Stillwater, OK 74078, USA

Abstract: The interest (discount) rate of annuity is a random variable, which is called stochastic. For stochastic annuities, also known as annuities under random rates of interest, we derive some moments, moments inequalities, and moments limits. When the discount rates are bounded, a strong convergence result is given showing that a stochastic annuity converges to its mean value as the number of years tend to infinity. Hence, stochastic annuities satisfy a type of strong law of large numbers.

AMS Subject Classification: 62P05, 60H30
Key Words: stochastic annuity, moments inequalities, strong limit theorem, actuarial science, probability

1. Introduction

An annuity is a series of payments made at equal intervals of time for a finite (or an infinite) number of periods. When the interest rates are affect its growth are random variables, the annuity is "stochastic", or is "under random rate of interest." Let $Y_1, Y_2, ..., Y_n$ be n variables such that $Y_i \geq 0$, $i = 1, 2, ..., n$

Received: February 20, 2014

§Correspondence author

representing the random effective rates of interest over n random periods which is the life of the annuity. to find how the accumulated value of this annuity can be measured, one must decide on how the interest rates will operate; if Y_i is used for period i regardless of when this payment is made, then the accumulated value of one unit payment is given by (cf. Bowers, et. al [1]):

$$A_{\overline{n}|} = \sum_{i=1}^{n} \prod_{l=1}^{i} (1 + Y_{n-l+1}). \tag{1.1}$$

While if X_i is used for payment made at time i over all i periods, the we have

$$A^*_{\overline{n}|} = \sum_{i=1}^{n} (1 + Y_{n-i+1})^i. \tag{1.2}$$

Another function, the present value of the annuity is often of interest and is defined by

$$P_{\overline{n}|} = \sum_{i=1}^{n} \prod_{l=1}^{i} (1 + Y_l)^{-1} \tag{1.3}$$

in the first situation, while the second is given by

$$P^*_{\overline{n}|} = \sum_{i=1}^{n} (1 + Y_i)^{-i}. \tag{1.4}$$

Another representation for the present value is possible via the "random discount rate" X_i related to the random interest rate y_i by the relation $1 - X_i = (1 + Y_i)^{-1}$ or $Y_i = \frac{X_i}{1-X_i}$ and $0 \leq X_i \leq 1$. Thus, we get

$$P_{\overline{n}|} = \sum_{i=1}^{n} \prod_{l=1}^{i} (1 - X_l) \tag{1.5}$$

and

$$P^*_{\overline{n}|} = \sum_{i=1}^{n} (1 - X_i)^i. \tag{1.6}$$

In this note, we concentrate on (1.5) and indicate corresponding results for (1.6). We shall assume throughout that $X_1, X_2, ..., X_n$ are independent random variables such that $\alpha_i \leq X_i \leq \beta_i$, for constants $0 \leq \alpha_i \leq \beta_i$, $i = 1, 2, ..., n$. stochastic annuities is a field in actuarial science that is seeing an increasing attention recently. Several authors attempted calculating the mean and the variance of the present (or accumulated) value and found that there is appreciable difficulty in so doing, cf. Kling, Ruez, and Rub [6], Dufresne [4], Burnecki, Marciniuk and Weron [3], Panjer and Belhouse [1] and [2], Promislow [9] and Boyle [2] for some work related to this issue.

In the current investigation we offer bounds of the mean and variance of $P_{\overline{n}|}$. Also, when the discount rates have same mean we obtain limit expressions of the variance as well. Finally we give sufficient conditions such that $P_{\overline{n}|} - EP_{\overline{n}|} \to 0$ with probability one, as $n \to \infty$

section Moments Inequalities

In this section, we present relations of moments and moments inequalities for P_n (and P_n^* given in (1.3) and (1.4) or equivalently by (1.5) and (1.6) respectively . We shall assume that the X_i's are independent.

Theorem 1. *(i) If $E(X_i) = \mu$ for all $i = 1, 2, ..., n$, then*

$$E(P_{\overline{n}|}) = \frac{1-\mu}{\mu}[1 - (1-\mu)^n] = E_n(1-\mu), \quad say. \tag{2.1}$$

(ii) In general, if $\bar{\mu}_n = max_{1 \le i \le n} \mu_i$ and $\bar{\mu}_n = min_{1 \le i \le n} \mu_i$, and $\mu_i = E(X_i)$, $i = 1, 2, ..., n$, then

$$E_n(1-\bar{\mu}_n) \le E(P_{\overline{n}|}) \le E_n(1-\underline{\mu}_n) \tag{2.2}$$

Proof. (i) Follows directly from the fact that

$$E(P_{\overline{n}|}) = \sum_{i=1}^{n} (1-\mu)^i = \frac{1-\mu}{\mu}[1-(1-\mu)^n] \tag{2.3}$$

(ii) Follows directly from the fact that

$$\sum_{i=1}^{n} (1-\bar{\mu}_n)^i \le E(P_{\overline{n}|}) = \sum_{i=1}^{n} \prod_{l=1}^{i} (1-\mu) \le \sum_{i=1}^{n} (1-\underline{\mu}_n)^i \tag{2.4}$$

$\square$

Note that if $E(X_i) = \mu$, $\mu \in (0, 1)$, then $lim_{n\to\infty} = E(P_{\overline{n}|}) = \frac{1-\mu}{\mu}$, a result known for nonstochastic annuities, cf. Kellison [5], p69. Also, if $\underline{\mu}_n \to \underline{\mu}$ and $\bar{\mu}_n \to \bar{\mu}$, where μ and $\bar{\mu}$ are finite, then, in general,

$$b = \frac{1-\bar{\mu}}{\bar{\mu}} \le lim_{n\to\infty} E(P_{\overline{n}|}) \le \frac{1-\underline{\mu}}{\underline{\mu}} = a \tag{2.5}$$

Theorem 2. *(i) If for $i = 1, 2, ..., n$, $E(X_i) = \alpha$ and $E(X_i^2) = \alpha_{(2)}$, then as $n \to \infty$, $EP_n \to \frac{1-\alpha}{\alpha}$ and*

$$Var(P_{\overline{n}|}^2) \to \frac{\alpha_{(2)} - \alpha^2}{\alpha^2(2\alpha - \alpha_{(2)}}, \tag{2.6}$$

as $n \to \infty$, provided $\alpha \in (0, 1)$.

(ii) In general, if $\bar{\mu}_n^{(2)} = max_{1 \le i \le n} \mu_i^{(2)}$ and $\underline{\mu}_n^{(2)} = min_{1 \le i \le n} \mu_i^{(2)}$, then

$$B_n \le E(P_{\overline{n}|})^2 \le A_n, \tag{2.7}$$

where,

$$\begin{aligned} A_n &= \frac{(1 - 2\underline{\mu}_n + \bar{\mu}_n^{(2)})}{2\underline{\mu}_n - \bar{\mu}_n^{(2)}}[1 - (1 - 2\underline{\mu}_n + \bar{\mu}_n^{(2)})^n] \\ &+ \frac{2(1 - \underline{\mu}_n)}{\underline{\mu}_n}(1 - 2\underline{\mu}_n + \bar{\mu}_n^{(2)}) \\ &\Big[\frac{1 - (1 - 2\underline{\mu}_n - \bar{\mu}_n^{(2)})^n}{2\underline{\mu}_n - \bar{\mu}_n^{(2)}} - \frac{(1 - \underline{\mu}_n)^n - (1 - 2\underline{\mu}_n + \bar{\mu}_n^{(2)})^n}{(1 - \underline{\mu}_n)(\underline{\mu}_n - \bar{\mu}_n^{(2)})}\Big], \end{aligned}$$

and B_n is obtained from A_n after replacing $\underline{\mu}_n$ by $\bar{\mu}_n$ and $\bar{\mu}_n^{(2)}$ by $\underline{\mu}_n^{(2)}$.

Proof. We start by proving (ii). Let $T_{n,i} = \prod_{j=1}^{i}(1 - X_j)$, $i = 1, 2, ..., n$. Thus

$$\begin{aligned} E(P_{\overline{n}|})^2 &= E(\sum_{i=1}^{n} T_{n,i})^2 \\ &= \sum_{i=1}^{n} ET_{n,i}^2 + 2\sum\sum_{i<i^*} ET_{n,i}T_{n,i^*} \\ &= \sum_{i=1}^{n} E\prod_{i=1}^{n}(1 - X_j)^2 + 2\sum\sum_{i<i^*} E\prod_{j=1}^{i}(1 - X_j)^2 \prod_{l=i+1}^{i^*}(1 - X_l) \\ &= \sum_{i=1}^{n}\prod_{j=l}^{i}(1 - 2\mu_{n-j} + \mu_{n-j}^{(2)}) \\ &\quad +2\sum\sum_{i<i^*}\prod_{j=1}^{i}(1 - \mu_{n-j} + \mu_{n-j}^{(2)}) \prod_{l=i+1}^{i^*}(1 - \underline{\mu}_{n-l}) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=l}^{n}(1-2\underline{\mu}_n+\bar{\mu}_n^{(2)})^i + 2\sum\sum_{i<i^*}(1-2\underline{\mu}_n+2\bar{\mu}_n^{(2)})^i(1-\underline{\mu}_n)^{i^*-i} \\
&= I_{1n}+I_{2n}, \quad say.
\end{aligned}$$

Now,

$$I_{1n} = \frac{(1-2\underline{\mu}_n+\bar{\mu}_n^{(2)})}{2\underline{\mu}_n-\bar{\mu}_n^{(2)}}[1-(1-2\underline{\mu}_n+\bar{\mu}_n^{(2)})^2], \tag{2.8}$$

and

$$\begin{aligned}
I_{2n} &= 2\sum_{i=1}^{n}\sum_{i^*=i+1}^{n}(1-2\underline{\mu}_n+\bar{\mu}_n^{(2)})^i(1-\underline{\mu}_n)^{i^*-i} \\
&= 2\sum_{i=1}^{n}(1-2\underline{\mu}_n+\bar{\mu}_n^{(2)})^i\frac{(1-\underline{\mu}_n)(1-(1-\underline{\mu}_n)^{n-i^*}}{\underline{\mu}_n} \\
&= \frac{2(1-\mu_n)}{\mu_n}\sum_{i=1}^{n}(1-2\underline{\mu}_n+\bar{\mu}_n^{(2)})^i - \frac{2(1-\underline{\mu}_n)^n}{\underline{\mu}_n}\sum_{i=l}^{n}(\frac{1-2\underline{\mu}_n+\underline{\mu}_n^{(2)}}{1-\underline{\mu}_n})^i \\
&= \frac{(1-\underline{\mu}_n)(1-2\underline{\mu}_n+\bar{\mu}_n^{(2)})}{\underline{\mu}_n(2\underline{\mu}_n-\bar{\mu}_n^{(2)})}[1-(1-2\underline{\mu}_n+\bar{\mu}_n^{(2)})^n]-(1-\underline{\mu}_n)^n. \\
&\quad \frac{1-2\underline{\mu}_n+\bar{\mu}_n^{(2)}(1-\underline{\mu}_n)}{1-\underline{\mu}_n}\Big[\frac{(1-\underline{\mu}_n)^n-(1-2\underline{\mu}_n+\bar{\mu}_n^{(2)})^n}{(1-\underline{\mu}_n)^2[\underline{\mu}_n-\bar{\mu}_n^{(2)}]}\Big] \\
&= \frac{2(1-\underline{\mu}_n)}{\underline{\mu}_n}(1-2\underline{\mu}_n+\bar{\mu}_n^{(2)}). \\
&\quad \Big[\frac{1-(1-2\underline{\mu}_n+\bar{\mu}_n^{(2)})^n}{2\underline{\mu}_n-\bar{\mu}_n^{(2)}}-\frac{(1-\underline{\mu}_n)^n-(1-2\underline{\mu}_n+\bar{\mu}_n^{(2)})^n}{(1-\underline{\mu}_n)^2(\underline{\mu}_n-\bar{\mu}_n^{(2)})}\Big].
\end{aligned} \tag{2.9}$$

From (2.8) and (2.9) it follows that $E(P_{\overline{n}|})^2 \leq A_n$. Next, let $\bar{\mu}_n$ and $\underline{\mu}_n^{(2)}$ replace $\underline{\mu}_n$ and $\bar{\mu}_n^{(2)}$, respectively, we obtain exactly as above that

$$E(P_{\overline{n}|})^2 \geq L_{1n}+L_{2n} = B_n,$$

where,

$$L_{1n} = \frac{1-2\bar{\mu}_n+\underline{\mu}_n^{(2)}}{2\bar{\mu}_n-\underline{\mu}_n^{(2)}}[1-(1-2\bar{\mu}_n+\underline{\mu}_n^{(2)})^n], \tag{2.10}$$

and

$$L_{2n} = \frac{2(1-\bar{\mu}_n)}{\bar{\mu}_n}(1-2\bar{\mu}_n+\underline{\mu}_n^{(2)})[1-(1-2\bar{\mu}_n+\underline{\mu}_n^{(2)})^2] - \frac{(1-\bar{\mu}_n)^n-(1-2\bar{\mu}_n+\underline{\mu}_n^{(2)})^n}{(1-\bar{\mu}_n)(\bar{\mu}_n-\underline{\mu}_n^{(2)})}. \quad (2.11)$$

We now prove part (i)

$$E(P_{\overline{n|}}) = \sum_{i=1}^{n}\prod_{j=1}^{i}(1-\mu_j) = \sum_{i=1}^{n}(1-\alpha)^i = \frac{(1-\alpha)[1-(1-\alpha)^n]}{\alpha}. \quad (2.12)$$

Thus, we have

$$lim_{n\to\infty}[E(P_{\overline{n|}})]^2 = (\frac{1-\alpha}{\alpha})^2. \quad (2.13)$$

Next, let

$$\begin{aligned} J_{1n}+J_{2n} &= \sum_{i=1}^{n}\prod_{j=1}^{i}E(1-X_j)^2 + 2\sum\sum_{i<i^*}\prod_{j=1}^{i}(1-X_j)^2\prod_{l=i+1}^{i^*}E(1-X_j)^{i^*-1} \\ &= \sum_{i=1}^{n-1}(1-2\alpha+\alpha_{(2)})^i + 2\sum\sum_{i<i^*}(1-2\alpha+\alpha_{(2)})^i(1-\alpha)^{i^*-1}. \end{aligned} \quad (2.14)$$

Now, as $n \to \infty$

$$J_{1n} = \frac{(1-2\alpha+\alpha_{(2)})[1-(1-2\alpha+\alpha_{(2)})^n]}{2\alpha-\alpha_{(2)}} \to \frac{1-2\alpha+\alpha_{(2)}}{2\alpha-\alpha_{(2)}}, \quad (2.15)$$

and, after some calculations, we get that

$$J_{2n} = \frac{2(1-\alpha)}{\alpha}\sum_{i=1}^{n-1}(1-2\alpha+\alpha_{(2)})^i - \frac{2(1-\alpha)}{\alpha}\sum_{i=1}^{n-1}(\frac{1-2\alpha+\alpha_{(2)}}{1-\alpha})^i\frac{2(1-\alpha)(1-2\alpha+\alpha_{(2)})}{\alpha(2\alpha-\alpha_{(2)})}, \quad (2.16)$$

as $n \to \infty$. Hence as $n \to \infty$

$Var(P_{\overline{n|}}) \to$

$$\frac{\alpha^2(1-2\alpha+\alpha_{(2)})+2\alpha(1-\alpha)(1-2\alpha+\alpha_{(2)})-(1-\alpha)^2(2\alpha-\alpha_{(2)})}{\alpha^2(2\alpha-\alpha_{(2)})}$$
$$=\frac{\alpha_{(2)}-\alpha^2}{\alpha^2(2\alpha-\alpha_{(2)})}. \quad (2.17)$$

Theorem 2 is now proved. □

Corollary 3. *If $lim_{n\to\infty}\underline{\mu}_n=\alpha$, $lim_{n\to\infty}\bar{\mu}^n=\beta$, and, if α and β are both positive, then $lim_{n\to\infty}E(P_{\overline{n}|})\in[\frac{1-\beta}{\beta},\frac{1-\alpha}{\alpha}]$. Furthermore, if $lim_{n\to\infty}\underline{\mu}^{(2)}=\alpha^{(2)}$ and $lim_{n\to\infty}\bar{\mu}_n^{(2)}=\beta^{(2)}$, then*

$$E(P_{\overline{n}|})^2\in\Big[\frac{1-2\beta+\alpha^{(2)}}{2\beta-\alpha^{(2)}}+\frac{2(1-\beta)(1-2\beta+\alpha^{(2)})}{\beta(2\beta-\alpha^{(2)})},$$
$$\frac{1-2\alpha+\beta^{(2)}}{2\alpha-\beta^{(2)}}+\frac{2(1-\alpha)(1-2\alpha+\beta^{(2)})}{\alpha(2\alpha-\beta^{(2)})}\Big].$$

Note that if $\underline{\mu}_n\to\bar{\mu}$, $\bar{\mu}_n\to\bar{\mu}$, $\underline{\mu}_n^{(2)}\to\underline{\mu}^{(2)}$ and $\bar{\mu}_n^{(2)}\to\bar{\mu}^{(2)}$ as $n\to\infty$ and provided that $1-2\underline{\mu}_n+\bar{\mu}_n^{(2)}<1$ and $1-2\bar{\mu}_n+\underline{\mu}_n^{(2)}<1$ for all n, then we easily see from Theorem 2 Part (ii) that

$$B\le lim_{n\to\infty}E(P_{\overline{n}|})^2\le A, \quad (2.18)$$

where $A=\frac{1-2\underline{\mu}+\bar{\mu}^{(2)}}{2\underline{\mu}-\bar{\mu}^{(2)}}+\frac{2(1-\underline{\mu})}{\underline{\mu}}(1-2\underline{\mu}+\bar{\mu}^{(2)})(\frac{1}{2\underline{\mu}-\bar{\mu}^{(2)}})$. B is obtained from A by replacing $\underline{\mu}$ with $\bar{\mu}$ and $\bar{\mu}^{(2)}$ by $\underline{\mu}^{(2)}$.

Thus combining (2.5) and (2.16) we get that

$$B-a^2\le Var(P_{\overline{n}|})\le A-b^2 \quad (2.19)$$

Hence, in view of Chebychev inequality $n^{-\epsilon}(P_{\overline{n}|}-E(P_{\overline{n}|})$ converges in probability to 0 as $n\to\infty$ for any $\epsilon>0$. Further, in view of the Borel-Cantelli lemma $n^{-\epsilon}(P_{\overline{n}|}-E(P_{\overline{n}|})$ converges to 0 with probability 1 for all $\epsilon>\frac{1}{2}$. A more refined result in this direction is given in the next section.

2. Strong Convergence Results

In this section, we address the question: when does $P_{\overline{n}|}-E(P_{\overline{n}|})$ converge to 0 with probability 1. We provide an answer in the case when X_i's are bounded above and below (away from zero). In particular, we prove the following result.

Theorem 4. *Let $\alpha_i \leq X_i \leq \beta_i$ for $i = 1, 2, ..., n$ with probability 1. Set $\underline{\alpha}_n = min_{1\leq i\leq n}\alpha_i$ and $\bar{\beta}_n = max_{1\leq \leq n}\beta_i$. If for any $\epsilon > 0$, $\sum_{n=1}^{\infty} exp[\frac{-\epsilon\bar{\beta}_n}{\underline{\alpha}_n^2}] < \infty$, then $P_{\overline{n}|} - E(P_{\overline{n}|}) \to 0$ with probability 1 as $n \to \infty$.*

Proof. Note that as above $P_{\overline{n}|} = \sum_{i=1}^{n}\prod_{l=1}^{i}(1 - X_l)$, thus

$$\beta_n^* = \frac{1-\bar{\beta}_n}{\bar{\beta}_n}[1-(1-\bar{\beta}_n)^n] \leq P_{\overline{n}|} \leq \frac{1-\underline{\alpha}_n}{\underline{\alpha}_n}[1-(1-\underline{\alpha}_n)^n] = \alpha_n^*. \tag{3.1}$$

Set $\alpha^* = \alpha_n^*$ and $\beta^* = \beta_n^*$. Thus, for any $t > 0$ and any $\xi > 0$

$$P[|P_{\overline{n}|} - EP_{\overline{n}|}| > t] \leq e^{-t\xi - \xi EP_{\overline{n}|}}Ee^{\xi P_{\overline{n}|}}. \tag{3.2}$$

But, since $\beta^* \leq P_{\overline{n}|} \leq \alpha^*$, then for any $\xi > 0$,

$$E^{\xi P_{\overline{n}|}} \leq \frac{\alpha^- EP_{\overline{n}|}}{\alpha^* - \beta^*}e^{\xi\beta^*} + \frac{EP_{\overline{n}|} - \beta^*}{\alpha^* - \beta^*}e^{\xi\alpha^*}. \tag{3.3}$$

Set

$$G(\xi) = -\xi EP_{\overline{n}|} + ln[\frac{\alpha^* - EP_{\overline{n}|}}{\alpha^* - \beta^*}e^{\xi\beta^*} + \frac{EP_{\overline{n}|} - \beta^*}{\alpha^* - \beta^*}e^{\xi\alpha^*}].$$

Further let
$T_n = \frac{EP_{\overline{n}|} - \beta^*}{\alpha^* - \beta^*}$ and $\xi_n = \xi(\alpha^* - \beta^*)$, for all $n \geq 1$, then

$$G(\xi_n) = -\xi_n T_n + ln[1 - T_n + T_n e^{\xi_n}]. \tag{3.4}$$

Using Taylor expansion, we get that
$G(\xi_n) = G(0) + G'(0)\xi_n + G^n(\theta)\xi_n^2$, where $0 \leq \theta \leq \xi_n$.
But,

$$G'(\xi_n) = -T_n + \frac{T_n}{(1-T_n)e^{-\xi_n} + T_n},$$

and

$$G"(\xi_n) = \frac{T_n(1-T_n)e^{-\xi_n}}{[(1-T_n)e^{-\xi_n} + T_n]^2}.$$

Thus

$$G"(\xi_n) = \Big[\frac{T_n}{(1-T_n)e^{-\xi_n} + T_n}][1 - \frac{T_n}{(1-T_n)e^{-\xi_n} + T_n}\Big].$$

Hence $G"(\xi_n) \leq \frac{1}{4}$ for any ξ_n and therefore,

$$G(\xi_n) \leq G(0) + G'(0)\xi_n + \frac{\xi_n^2}{8} = \frac{\xi_n^2}{8}. \tag{3.5}$$

Hence,

$$Ee^{\xi(P_{\overline{n}|} - EP_{\overline{n}|})} \leq e^{\xi^2({\alpha^*}^2 - {\beta^*}^2)/8}. \tag{3.6}$$

Therefore,

$$P[|P_{\overline{n}|} - EP_{\overline{n}|}| > t] \leq e^{-t\xi + \xi^2({\alpha^*}^2 - {\beta^*}^2)/8}. \tag{3.7}$$

Set

$$f(\xi) = -t\xi + \xi^2({\alpha^*}^2 - {\beta^*}^2)/8, \tag{3.8}$$

then $f'(\xi) = 0$ gives $\xi = \frac{4t}{{\alpha^*}^2 - {\beta^*}^2}$, as minimum value and hence we get

$$P[|P_{\overline{n}|} - EP_{\overline{n}|}| > t] \leq e^{-2t^2/({\alpha^*}^2 - {\beta^*}^2)}. \tag{3.9}$$

But,

$$\begin{aligned}
\alpha^* - \beta^* &= \sum_{i=1}^{n} [(1 - \underline{\alpha}_n)^i - (1 - \bar{\beta}_n)^i] \\
&= \sum_{i=1}^{n} (\bar{\beta}_n - \underline{\alpha}_n)[(1 - \underline{\mu}_n)^{i-1} + (1 - \underline{\alpha}_n)^{i-2}(1 - \bar{\beta}_n) + (1 - \bar{\beta}_n)^{i-1}] \\
&\leq (\bar{\beta}_n - \underline{\alpha}_n) \sum_{i=1}^{n} i(1 - \underline{\alpha}_n)^{i-1} \\
&= \frac{\bar{\beta}_n - \underline{\alpha}_n}{\underline{\alpha}_n^2} \\
&\leq \frac{\bar{\beta}_n}{\underline{\alpha}_n^2}.
\end{aligned} \tag{3.10}$$

Finally,

$$ {\alpha^*}^2 - {\beta^*}^2 = (\alpha^* - \beta^*)(\alpha^* + \beta^*) \leq \frac{2(\alpha^* - \beta^*)}{\underline{\alpha}_n} \leq \frac{2\bar{\beta}_n}{\underline{\alpha}_n^3}.$$

The theorem is proved in view of the Borel-Cantelli lemma.

□

For example, we can apply Theorem 4 when we take for all $i = 1, 2, ..., n$, $\frac{C_1}{n} \leq X_i \leq \frac{C_2}{n}$ where $C_1 < C_2$.

References

[1] N. I. Bowers, H. U. Gerber, J. C. Hickman, D. A. Jones, D. A. and C. J. Nebitt. em Actuarial Mathematics, second edition, Society of Actuaries, schaumburg, IL. U.S.A. (1997)

[2] P. P. Boyle, Rates of return as random variables. *Journal of Risk and Insurance* **53** (4), (1976), 693-713.

[3] K. Burnecki, A. Marciniuk, and A. Weron, A. Annuities under random rates of interest-Revisited. Insurance: *Mathematics and Economics* **32**, (2003) 457-460.

[4] D. Dufresne, Stochastic life annuities. *N. Am. Actuar. Journal* **11** (1), (2007). 136-157.

[5] S. G. Kellieson, *The theory of interest*. McGraw-Hill, New York, NY. U.S.A. (1997).

[6] A. Kling, F. Ruez, and J. Rub, The impact of stochastic volatility on pricing, hedging, and hedge efficiency of withdrawal benefit guarantees in variable annuities. *Astin Bull* **41**, 2, (2011) 511-545.

[7] H. Panjer, and D. Bellhouse. Stochastic modeling of interest rates with application to life contingencies. *Journal of Risk and Insurance* **47**, (1980). 91-119.

[8] H. Panjer, and D. Bellhouse. Stochastic modeling of interest rates with application to life contingencies - Part II. *Journal of Risk and Insuarance* **48**, (1981) 728-637.

[9] S. D. Promislow. A new approach to the theory of interest. *Transactions of Society of Actuaries*. **32**, (19810) 53-92.