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PRIME RADICAL IN TERNARY HEMIRINGS

R.D. Giri[1], B.R. Chide[2 §]

[1] 128A, Aakar Nagar, Katol Road, Nagpur-440013, INDIA
[2]Department of Mathematics
Shri Ramdeobaba College of Engineering and Management
Nagpur, 440013, INDIA

Abstract: A hemiring is already defined by Golan. We generalize this concept by changing multiplication operation from binary to ternary one. This article studies in detail all basic notions like prime and semiprime ideal, primary ideal, prime radical of an ideal in a ternary hemiring and prime radical of ternary hemiring.

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1. Introduction

Lister [4] introduced the notion of ternary rings and studied some of their properties and radical theory of such rings. Golan [3] introduced the notion of semiring. Dutta and Kar [1], [2] studied prime ideal, semiprime ideal, irreducible ideal and prime radical of ternary semiring. A ternary hemiring is the generalization of ternary semiring defined by Dutta et al and also a generalization of ternary ring defined by Lister [4].

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§Correspondence author

Recalling Golan a ring without additive inverse and multiplicative identity is known as hemiring. We wish to study the concepts involved in the context of ternary hemiring. In $\xi 2$, we give some basic definitions and examples. In $\xi 3$, we introduce and study prime ideals of a ternary hemirings. In $\xi 4$, we inroduce and study semi prime ideals of a ternary hemiring. In $\xi 5$, we study prime radical of an ideal of a ternary hemiring. In $\xi 6$, we introduce and study irreducible ideals, strongly irreducible ideals and primary ideals of a ternary hemiring. In $\xi 7$, we study prime radical of a ternary hemiring.

2. Definitions and Examples

Definition 2.1. A non-empty set H together with binary operation $+$, called addition and ternary multiplication, denoted by juxtaposition, is said to be ternary hemiring if it is additive commutative monoid with 0 as additive identity and multiplication is ternary along with distributivity (left, right and lateral).

That is:

(i) $(H, +)$ is a commutative monoid with identity element 0,

(ii) $(abc)de = a(bcd)e = ab(cde)$,

(iii) $(a + b)cd = acd + bcd$,

(iv) $a(b + c)d = abd + acd$,

(v) $ab(c + d) = abc + abd$,

(vi) $0ab = a0b = ab0 = 0$ for all $a, b, c, d, e \in H$.

Example 2.2. Let $H = \{0, -1\}$, with $0, -1$ in ring of integers. We define binary addition and ternary multiplication on H as follows:

$$0 + 0 = 0, \quad 0 + (-1) = (-1) + 0 = -1, \quad (-1) + (-1) = -1,$$

$$0(-1)(-1) = (-1)0(-1) = (-1)(-1)0 = 0, \quad (-1)(-1)(-1) = -1.$$

Then H is a ternary hemiring with binary addition and ternary multiplication but it is not a hemiring.

Definition 2.3. A ternary hemiring H is said to be

(1) commutative if $abc = bac = bca$ for all $a, b, c \in H$,

(2) laterally commutative if $abc = cba$ for all $a, b, c \in H$.

Example 2.4. Let $H = \{0, -1\}$. We define binary addition and ternary multiplication on H as defined in example 2.2.

Let $S = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} | a, b \in H \right\}$.

Now S is a ternary hemiring with the binary addition of matrices and ternary multiplication of matrices.

Note that this ternary hemiring is not commutative.

Definition 2.5. A non-empty subset T of H is called ternary subhemiring if

(i) $a + b \in T$, for all $a, b \in T$,

(ii) $abc \in T$, for all $a, b, c \in T$.

Definition 2.6. An additive submonoid I of H is called a left (right, lateral) ideal of H if $h_1 h_2 i$ (respectively $i h_1 h_2, h_1 i h_2$) $\in I$ for all $h_1, h_2 \in H$, and $i \in I$.If I is a left, a right and a lateral ideal of H, then I is called an ideal of H.

Definition 2.7. Let H be a ternary hemiring and $a \in H$.

Then the principal

(i) left ideal generated by a is given by

$$< a >_\ell = \left\{ \sum u_i v_i a + na | u_i, v_i \in H \text{ and } n \in Z_0^+ \right\},$$

(ii) right ideal generated by a is given by

$$< a >_r = \left\{ \sum a u_i v_i + na | u_i, v_i \in H \text{ and } n \in Z_0^+ \right\},$$

with Z_0^+ denoting the set of non negative integers,

(iii) two-sided ideal generated by a is given by

$$< a >_t = \{ \sum u_i v_i a + \sum a p_j q_j + \sum c_k d_k a e_k f_k + na | u_i, v_i, p_j, q_j, c_k, d_k, e_k, f_k \in H \text{ and } n \in Z_0^+ \},$$

(iv) lateral ideal generated by a is given by

$$< a >_m = \{\sum u_i a v_i + \sum p_j q_j a a_j b_j + na | u_i, v_i, p_j, q_j, a_j, b_j \in H \text{ and } n \in Z_0^+\},$$

(v) ideal generated by a is given by

$$< a > = \{\sum p_i q_i a + \sum a r_j s_j + \sum u_k a v_k + \sum p'_\ell q'_\ell a r'_\ell s'_\ell + na | p_i, q_i, r_j, s_j, u_k, v_k, p'_\ell, q'_\ell, r'_\ell, s'_\ell \in H \text{ and } n \in Z_0^+\}.$$

Definition 2.8. An ideal I of a ternary hemiring H is said to be semi subtractive if and only if $h \in I \cap \vee(H)$, with $\vee(H) = \{h | h + h' = 0 \text{ for some } h' \in H\}$ implies that $h' \in I \cap \vee(H)$.

That is I is semisubstractive if and only if every element of I has additive inverse.

Definition 2.9. An ideal I of a ternary hemiring H is called subtractive (k-ideal) if $a + b \in I, a \in H, b \in I$ imply that $a \in I$.

Definition 2.10. An ideal I of a ternary hemiring H is called strongly subtractive if $a + b \in I$ imply that $a \in I$ and $b \in I$.

Remark 2.11. Every strongly subtractive ideal is subtractive and every subtractive ideal is semisubtractive.

Example 2.12. The set $H = Z_0^-$, the set of non positive integers, with usual operations of addition and ternary multiplication is a ternary hemiring. The set $I = 3Z_0^-$ is a subtractive ideal of H but not strongly substractive since $(-5) + (-4) = -9 \in I$, but neither -4 nor $-5 \in I$.

Definition 2.13. Let A be an ideal of a ternary hemiring H. Then the k-closure of A, is defined as

$$\bar{A} = \{a \in H | a + b = c \text{ for some } b, c \in A\}.$$

Definition 2.14. An element a of a ternary hemiring H is said to be regular if there exists an element $x \in H$ such that $a = axa$.
A ternary hemiring H is called regular if every element of H is regular.

Definition 2.15. An element a of a ternary hemiring H is said to be left (respectively right) regular if there exists an element $x \in H$ such that $a = xaa$ (respectively $a = aax$). If all the elements of H are left (respectively right) regular, then H is called left (respectively right) regular.

Definition 2.16. A proper ideal P of a ternary hemiring H is called a prime ideal of H if and only if for any ideals A, B, C of H, $ABC \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.

Definition 2.17. A proper ideal I of a ternary hemiring H is called a semiprime ideal of H if and only if, for any ideal A of H, $A^3 \subseteq I$ implies that $A \subseteq I$.

Remark 2.18. It is easy to verify from above definition that I is semiprime if and only if for any ideal of A of H, $A^n \subseteq I$, for some odd positive integer n, implies $A \subseteq I$.

Definition 2.19. An ideal M of a ternary hemiring H is maximal if it is not properly contained in other proper ideal of H i.e. $M \subseteq M' \subseteq H$ implies that $M = M'$ or $M' = H$.

Example 2.20. Let $H = \{a, b, z\}$, define binary addition $+$, and ternary multiplication as follows:

$$a + z = z + a = a, \quad b + z = z + b = b,$$

$$z + z = z, \quad a + a = b + b = a, \quad a + b = b + a = b,$$

$$bzb = zbb = bbz = aza = zaa = aaz = abz = baz = zab = zba = azb = bza = z,$$

$$aaa = bbb = a, \quad aab = aba = baa = a = bba = bab = abb.$$

An ideal $P = \{z, a\}$ is a maximal but not prime ideal of H.
Since $bbb = a \in P$ but $b \notin P$.

Definition 2.21. A non-empty subset T of a hemiring H is called a multiplicative system if $a, b, c \in T$ implies that $abc \in T$.

Definition 2.22. A non-empty subset A of a ternary hemiring H is called an m-system if for elements $a, b, c \in A$ there exist elements x_1, x_2, x_3, x_4 of H such that $ax_1bx_2c \in A$ or $ax_1x_2bx_3x_4c \in A$ or $ax_1x_2bx_3cx_4 \in A$ or $x_1ax_2bx_3x_4c \in A$.

Definition 2.23. A non-empty subset A of ternary hemiring H is called a p-system if for each $a \in A$ there exist elements x_1, x_2, x_3, x_4 of H such that $ax_1ax_2a \in A$ or $ax_1x_2ax_3x_4a \in A$ or $ax_1x_2ax_3ax_4 \in A$ or $x_1ax_2ax_3x_4a \in A$.

Every multiplicative system is an m-system and every m-system is a p-system.

Definition 2.24. A non-empty subset A of a ternary hemiring H is called an i-system if $a, b \in A$ implies that

$$< a > \cap < b > \cap A \neq \phi.$$

Remark 2.25. Every multiplicative system is an m-system and every m-system is a p-system. Further every p-system is an i-system. However all reverse implications are untrue.

Definition 2.26. A proper ideal I of a ternary hemiring H is said to be irreducible if for ideals A, B of H, $A \cap B = I$ implies that $A = I$ or $B = I$.

Definition 2.27. A proper ideal I of a ternary hemiring H is said to be strongly irreducible if for ideals A, B of H, $A \cap B \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$.

Remarks 2.28. Every strongly irreducible ideal is an irreducible ideal.

3. Prime Ideals in Ternary Hemirings

We give equivalent definitions of prime ideal through the following characterization theorem.

Theorem 3.1. *Let H be a ternary hemiring. Then the following conditions in H are equivalent.*

(i.) *P is a prime ideal of H,*

(ii.) *$aHbHc \subseteq P$, $aHHbHHc \subseteq P$, $aHHbHcH \subseteq P$, $HaHbHHc \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$,*

(iii.) *$< a >< b >< c > \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$.*

Proof. The proof is analogous to that of Theorem 3.2 in [1].

Corollary 3.2. *An ideal P of a commutative ternary hemiring H is prime if and only if $abc \in P$ implies that $a \in P$ or $b \in P$ or $c \in P$ for all elements a, b, c of P.*

Proof. The proof is analogous to that of Corollary 3.3 in [1].

Theorem 3.3. *If I is an ideal of a ternary hemiring H and P is a prime ideal of H, then $I \cap P$ is a prime ideal of I, considering I as a ternary hemiring.*

Proof. The proof is analogous to that of Theorem 3.5 in [1].

Theorem 3.4. *An ideal P of a ternary hemiring H is prime if and only if $H - P$ is an m-system.*

Proof. The proof is analogous to that of Theorem 3.7 in [1].

Proposition 3.5. *Let A be an m-system and N be an ideal of a ternary hemiring H such that $N \cap A = \phi$. Then there exists a maximal prime ideal M of H containing N such that $M \cap A = \phi$.*

Proof. Let $\{N_i, i \in \Omega\}$ be a chain of ideals of H containing N such that $N_i \cap A = \phi$. Since $N \subseteq N$, this chain is non-empty. Now $\bigcup_{i \in \Omega} N_i$ is an ideal of H containing N.

Therefore by Zorn's Lemma there exists a maximal ideal M of H containing N such that $M \cap A = \phi$.

Now we show that M is prime. Let A_1, B_1, C_1 be ideals of H such that $A_1B_1C_1 \subseteq M$.

$M \cap A = \phi$ implies that $M \subseteq A^c$. Since A is an m-system, A^c is prime by theorem 3.4.

$A_1B_1C_1 \subseteq M \subseteq A^c$. This implies that $A_1 \subseteq A^c$ or $B_1 \subseteq A^c$ or $C_1 \subseteq A^c$. But $A^c \subseteq M$.

Therefore $A_1 \subseteq M$ or $B_1 \subseteq M$ or $C_1 \subseteq M$. This proves that M is *a* prime ideal of H.

Definition 3.6. *An ideal P of a ternary hemiring H is called a completely prime ideal of H if $abc \in P$ implies that $a \in P$ or $b \in P$ or $c \in P$ for any three elements a, b, c of H.*

Remark 3.7. Every completely prime ideal of H is prime ideal of H but not conversely. If ternary hemiring H is commutative, then both concepts are identical.

Definition 3.8. An ideal P of a ternary hemirng H is completely prime if and only if $H - P$ is a ternary subhemiring of H.
Proof: The proof is straight forward and hence omitted.

4. Semiprime Ideal of a Ternary Hemiring

In this section we prove some results on semiprime ideals of a ternary hemiring.

Proposition 4.1. *A necessary and sufficient condition for an element x of a ternary hemiring H to belong to a semiprime ideal Q of H is that $HxH \subseteq Q$.*

Proof. The proof is analogous to that of Theorem 3.3 in [2].

Proposition 4.2. *If Q is an ideal in a ternary hemiring H, then the following conditions are equivalent.*

(1) Q *is a semiprime ideal of* H,

(2) $xHxHx \subseteq Q$, $xHHxHHx \subseteq Q$, $xHHxHxH \subseteq Q$ *and* $HxHxHHx \subseteq Q$ *implies* $x \in Q$,

(3) $<x><x><x> \subseteq Q$ *implies that* $x \in Q$.

Proof. The proof is analogous to that of Theorem 3.4 in [2].

Corollary 4.3. *A proper ideal Q of a commutative ternary hemiring H is semiprime if and only if $x^3 \in Q$ implies that $x \in Q$ for any element x of H.*

Proof. The proof is straight forward.

Proposition 4.4. *A proper ideal Q of a ternary hemiring H is semiprime if and only if $H - Q$ is a p-system.*

Proof. The proof is straight forward.

Proposition 4.5. *Let B be a p-system and I be an ideal of a ternary hemiring H such that $B \cap I = \phi$.*

Then there exists a maximal ideal Q of H containing I such that $B \cap Q = \phi$.

Moreover, Q is also a semiprime ideal of H.

Proof. The proof is similar to that of Proposition 3.5.

Definition 4.6. A proper ideal Q of a ternary hemiring H is called completely semiprime ideal if $x^3 \in Q$ implies that $x \in Q$ for any $x \in H$.

Remark 4.7. Every completely semiprime ideal of H is semiprime ideal of H but not conversely. If ternary hemiring H is commutative, then both concepts are identical.

Proposition 4.8. *A proper ideal Q of a ternary hemiring H is completely semiprime if and only if $H - Q$ is a ternary subhemiring of H.*

Proposition 4.9. *A non-empty subset B of a ternary hemiring H is a p-system if and only if it is the union of m-systems.*

Proof. The proof is analogous to that Theorem 3.17 in [2].

Proposition 4.10. *A proper ideal Q of a ternary hemiring H is semiprime if and only if $H - Q$ is the union of m-systems of H.*

Proof. The proof is straight forward.

5. Prime Radical of an ideal of a Ternary Hemiring

In this section we prove some results on prime radical of an ideal of a ternary hemiring H.

Definition 5.1. Let A be an ideal of a ternary hemiring H. Prime Radical of an ideal A, denoted by $\sqrt{A}$, is defined as

$$\sqrt{A} = \{a \in H | M(a) \cap A \neq \phi \text{ with all m-systems } M(a) \text{ containing } a\}.$$

Proposition 5.2. *Let A be an ideal of a ternary hemiring H. If $a \in \sqrt{A}$, then $a^{2n+1} \in A$ for some $n \in Z_0^+$.*

Proof. If $a \in \sqrt{A}$, then $M(a) \cap A \neq \phi$, for every m-system $M(a)$ containing a.

Since $M = \left\{a^{2k+1} | k \in Z_0^+\right\}$ is a multiplicative system and hence an m-system containing a. So $M \cap A \neq \phi$. Therefore, there exists $n \in Z_0^+$ such that $a^{2n+1} \in A$

Remark 5.3. If A is an ideal of a commutative ternary hemiring H, then

$$\sqrt{A} = \left\{a \in H | a^{2n+1} \in A, \text{ for some } n \in Z_0^+\right\}.$$

Proposition 5.4. If A is an ideal of a commutative ternary hemiring H, then $\sqrt{A}$ is an ideal of H containing A.

Proof. First we show that $A \subseteq \sqrt{A}$.

Let $a \in A$. Now every m-system containing a meets A. Therefore $a \in \sqrt{A}$.

Now we show that $\sqrt{A}$ is an ideal of H. Let $a, b \in \sqrt{A}$. Then these exist $m, n \in Z_0^+$ such that $a^{2m+1} \in A$ and $b^{2n+1} \in A$.

Now $= (a+b)^{2m+2n-1} = (a+b)^{2(m+n)-1} = \sum\limits_{i+j=2(m+n)-1} a^i b^j$.

In each summand on right either $i \geq 2m+1$ or $j \geq 2n+1$ i.e. either $a^i \in A$ or $b^j \in A$ and hence $(a+b)^{2(m+n)-1} \in A$. This implies that $a+b \in \sqrt{A}$.

Let $a \in \sqrt{A}$ and $h_1, h_2 \in H$. Now $a \in \sqrt{A}$ implies that $a^{2m+1} \in A$, for some $m \in Z_0^+$.

Since H is commutative $(ah_1h_2)^{2m+1} = a^{2m+1}h_1^{2m+1}.h_2^{2m+1} \in A$. This implies that $ah_1h_2 \in \sqrt{A}$. Therefore $\sqrt{A}$ is an ideal of H.

Proposition 5.5. *If A, B, C are ideals of a ternary hemiring H then:*

(1) $A \subseteq B$ implies that $\sqrt{A} \subseteq \sqrt{B}$,

(2) $\sqrt{ABC} = \sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$,

(3) $\sqrt{\sqrt{A}} = \sqrt{A}$,

(4) $\sqrt{A+B} = \sqrt{\sqrt{A} + \sqrt{B}}$.

Proof. (1) Let $a \in \sqrt{A}$. Then for every m-system $M(a)$ containing a has a non-empty intersection with A i.e. $M(a) \cap A \neq \phi$. But $A \subseteq B$ implies that $\phi \neq M(a) \cap A \subseteq M(a) \cap B$ i.e. $M(a) \cap B \neq \phi$, i.e. $a \in \sqrt{B}$.

Therefore $\sqrt{A} \subseteq \sqrt{B}$.

(2) Since A, B, C are ideals of H, $ABC \subseteq A, ABC \subseteq B$ and $ABC \subseteq C$. Therefore $ABC \subseteq A \cap B \cap C$. By (1), $\sqrt{ABC} \subseteq \sqrt{A \cap B \cap C} \ldots\ldots (i)$. Now $A \cap B \cap C \subseteq A$, $A \cap B \cap C \subseteq B$, $A \cap B \cap C \subseteq C$ $\sqrt{A \cap B \cap C} \subseteq \sqrt{A}$, $\sqrt{A \cap B \cap C} \subseteq$

$\sqrt{B}$, $\sqrt{A\cap B\cap C}\subseteq\sqrt{C}$. So, $\sqrt{A\cap B\cap C}\subseteq\sqrt{A}\cap\sqrt{B}\cap\sqrt{C}\ldots\ldots(ii)$. From (i) and (ii), we have

$$\sqrt{ABC}\subseteq\sqrt{A\cap B\cap C}\subseteq\sqrt{A}\cap\sqrt{B}\cap\sqrt{C}.$$

We show that

$$\sqrt{A}\cap\sqrt{B}\cap\sqrt{C}\subseteq\sqrt{ABC}$$

Let $x\in\sqrt{A}\cap\sqrt{B}\cap\sqrt{C}$. This implies that $x\in\sqrt{A}$, $x\in\sqrt{B}$ and $x\in\sqrt{C}$. So every m-system containing x meets in A,B and C respetively, i.e.

$$M(x)\cap A\neq\phi,\ M(x)\cap B\neq\phi,\ M(x)\cap C\neq\phi.$$

Let $a\in M(x)\cap A$, $b\in M(x)\cap B$ and $c\in M(x)\cap C$ i.e. $a,b,c\in M(x)$, and $M(x)$ is an m-system. So there exist $x_1,x_2,x_3,x_4\in H$. Such that $ax_1bx_2c\in M(x)$ or $ax_1x_2bx_3x_4c\in M(x)$ or $ax_1x_2bx_3cx_4\in M(x)$ or $x_1ax_2bx_3x_4c\in M(x)$, i.e. $ax_1bx_2c\in ABC$ or $ax_1x_2bx_3x_4c\in ABC$ or $ax_1x_2bx_3cx_4\in ABC$ or $x_1ax_2bx_3x_4c\in ABC$.

Therefore $ABC\cap M(x)\neq\phi$, for every m-system $M(x)$ containing x, i.e $x\in\sqrt{ABC}$. Thus, $\sqrt{A}\cap\sqrt{B}\cap\sqrt{C}\subseteq\sqrt{ABC}$ Thus we have $\sqrt{ABC}=\sqrt{A\cap B\cap C}=\sqrt{A}\cap\sqrt{B}\cap\sqrt{C}$.

(3) We have $A\subseteq\sqrt{A}$. So by (1), $\sqrt{A}\subseteq\sqrt{\sqrt{A}}\ldots\ldots(i)$. It remains to show that $\sqrt{\sqrt{A}}\subseteq\sqrt{A}$. Let $a\in\sqrt{\sqrt{A}}$. This implies that for every m-system $M(a)$ containing a has a non-empty intersection with $\sqrt{A}$ i.e. $M(a)\cap\sqrt{A}\neq\phi$. Let $b\in M(a)\cap\sqrt{A}$. This implies that $b\in M(a)$ and $b\in\sqrt{A}$. Therefore every m-system containing b meets in A. But $M(a)$ is an m-system containing b. Therefore $M(a)\cap A\neq\phi$. This implies that $a\in\sqrt{A}$. Therefore $\sqrt{\sqrt{A}}\subseteq\sqrt{A}\ldots\ldots(ii)$. Thus from (i) and (ii), $\sqrt{\sqrt{A}}=\sqrt{A}$.

(4) We have $A\subseteq\sqrt{A}$ and $B\subseteq\sqrt{B}$. Therefore $A+B\subseteq\sqrt{A}+\sqrt{B}$. By (1), $\sqrt{A+B}\subseteq\sqrt{\sqrt{A}+\sqrt{B}}\ldots\ldots(i)$. Now $A\subseteq A+B$, $B\subseteq A+B$. Thus implies that $\sqrt{A}\subseteq\sqrt{A+B}$, and $\sqrt{B}\subseteq\sqrt{A+B}$. So, $\sqrt{A}+\sqrt{B}\subseteq\sqrt{A+B}$. By (1), $\sqrt{\sqrt{A}+\sqrt{B}}\subseteq\sqrt{\sqrt{A+B}}=\sqrt{A+B}\ldots\ldots(ii)$. Therefore from (i) and (ii), $\sqrt{A+B}=\sqrt{\sqrt{A}+\sqrt{B}}$.

Proposition 5.6. *If P is a prime ideal of a ternary hemiring H,then $P=\sqrt{P}$.*

Proof. We have $P\subseteq\sqrt{P}$. Suppose this inclusion is proper i.e. there exists an element $a\in\sqrt{P}$ and $a\notin P$.

Since $a \notin P$, $a \in P^c$ But P^c is an m-system. Thus we have an m-system P^c containing a such that $P^c \cap P = \phi$. This contradicts that $a \in \sqrt{P}$.

Therefore this inclusion is not proper.

Thus, $P = \sqrt{P}$.

Corollary 5.7. *If P is a prime ideal of a ternary hemiring H, then $\sqrt{P^{2n+1}} = P$ for all $n \in Z_0^+$.*

Proof. We prove this by induction on n. For $n = 0$, $\sqrt{P} = P$ is true, since P is a prime ideal of H. Assume that $\sqrt{P^{2n+1}} = P$ is true for $n = k$. Now we show that it is true for $n = k + 1$,

$$\sqrt{P^{2(k+1)+1}} = \sqrt{P^{2k+1}.P^2} = \sqrt{P^{2k+1}P.P} = \sqrt{P^{2k+1} \cap P \cap P}$$
$$= \sqrt{P^{2k+1}} \cap \sqrt{P} \cap \sqrt{P} = P \cap P \cap P = P.$$

Proposition 5.8. *An ideal A in a ternary hemiring H is semiprime if and only if $\sqrt{A} = A$.*

Proof. The proof is similar to that of proposition 5.6.

Proposition 5.9. *If A is an ideal in a ternary hemiring H then $\sqrt{A}$ is the intersection of all prime ideals in H that contain A.*

Proof. We show that $\sqrt{A} = \bigcap_{i \in \Omega} P_i$, $P_i's$ are prime ideals containing A.

Since $A \subseteq P_i$, for each $i \in \Omega$ $\sqrt{A} \subseteq \sqrt{P_i}$, for each $i \in \Omega$.

But P_i is a prime ideal, by proposition 5.6, $\sqrt{P_i} = P_i$.

Therefore $\sqrt{A} \subseteq P_i$, for each $i \in \Omega$. Hence $\sqrt{A} \subseteq \bigcap_{i \in \Omega} P_i \ldots\ldots (i)$.

Now let $a \in \bigcap_{i \in \Omega} P_i$ and $a \notin \sqrt{A}$. Then there exists an m-system $M(a)$ containing a such that $M(a) \cap A = \phi$. By Proposition 3.5, there exists a maximal prime ideal P containing A such that $P \cap M(a) = \phi$. This implies that $a \notin P$. This contradicts that a is an element of all prime ideals containing A. Therefore $a \in \sqrt{A}$. Therefore $\bigcap_{i \in \Omega} P_i \subseteq \sqrt{A} \ldots\ldots (ii)$. Thus by (i) and (ii), $\sqrt{A} = \bigcap_{i \in \Omega} P_i$.

Proposition 5.10. *If A is an ideal in a ternary hemiring H then $\sqrt{A}$ is the smallest semiprime ideal in H that contains A.*

Proof. By proposition 5.9, $\sqrt{A} = \bigcap_{i\in\Omega} P_i$, where P_i are prime ideals of H that contain A. Since intersection of prime ideals is semiprime, so $\sqrt{A}$ is a semiprime ideal. To show that $\sqrt{A}$ is the smallest semiprime ideal of H. Let B be a semiprime ideal of H containing A and $B \subseteq \sqrt{A}$. Now $A \subseteq B \subseteq \sqrt{A}$. By Proposition 5.5, $\sqrt{A} \subseteq \sqrt{B} \subseteq \sqrt{\sqrt{A}} = \sqrt{A}$. Thus $\sqrt{A} = \sqrt{B} = B$. Hence $\sqrt{A}$ is the smallest semiprime ideal that contains A.

6. Primary and Strongly Irreducible Ideals

In this section we introduce the notion of primary ideal and we prove some important results on primary ideals and strongly irreducible ideals of a ternary hemiring.

Definition 6.1. An ideal A in a ternary hemiring H is said to be primary if for $a, b, c \in H$, $abc \in A$ and $a \notin A$, $b \notin A$ then there exists positive odd integer n such that $c^n \in A$.

Proposition 6.2. *Let A be a primary ideal in a commutative ternary hemiring H. If B is a radical of A then B is prime ideal. Moreover, if $abc \in A$ and $a \notin A, b \notin A$, then $c \in B$.*

Proof. Given that A is a primary ideal and $B = \sqrt{A}$. By proposition 5.4, $\sqrt{A}$ is an ideal of H containing A. To prove that B is a prime ideal. Suppose $abc \in B$ and $a \notin B$, $b \notin B$. Now $abc \in B = \sqrt{A}$ implies that $(abc)^{2m+1} \in A$, for $m \in Z_0^+$,

$$(abc)^{2m+1} = a^{2m+1}.b^{2m+1}.c^{2m+1} \in A.$$

Now $a, b \notin B$. This implies that $a^{2m+1} \notin A$ and $b^{2m+1} \notin A$. Since A is primary ideal so there exists an odd positive integer k such that $(c^{2m+1})^k \in A$ i.e. $c^{(2m+1)k} \in A$. Now $(2m + 1)k$ is an odd positive integer. This implies that $c \in \sqrt{A} = B$ and hence B is a prime ideal of H. Given that $abc \in A$ and $a \notin A$, $b \notin A$, and A is a primary ideal,so $c^k \in A$ for some odd positive integer k.

This implies that $c \in \sqrt{A} = B$.

Proposition 6.3. *Let A and B be ideals in a commutative ternary hemiring H. Then A is primary and B is its radical if and only if the following conditions are satisfied.*

(1) $A \subseteq B$,

(2) If $b \in B$ then $b^k \in A$ for some positive odd integer k,

(3) If $abc \in A$ and $a \notin A$, $b \notin A$, then $c \in B$.

Proof. If A is a primary ideal and $B = \sqrt{A}$, then conditions (1), (2) and (3) are satisfied by proposition 6.2. Now we prove the converse part. Assume conditions (1), (2) and (3) hold. We show that A is primary and $B = \sqrt{A}$. Let $abc \in A$ and $a \notin A$, $b \notin A$, then by (3) $c \in B$. By (2), $c^k \in A$ for some positive odd integer k. This proves that A is a primary ideal. To show that $B = \sqrt{A}$. Let $b \in B$. Then by (2), $b^n \in A$ for some positive odd integer n i.e. $b \in \sqrt{A}$. This proves that $B \subseteq \sqrt{A} \ldots\ldots (i)$. Let $a \in \sqrt{A}$. This implies that $a^m \in A$ for some positive odd integer m. Let m_0 be the least such an odd exponent that $a^{m_0} \in A$

If $m_0 = 1$ then $a \in A \subseteq B$. Therefore $a \in B$.
If $m_0 = 3$, $a^3 = a.a.a \in A$, $a \notin A$, then by (3), $a \in B$.
If $m_0 = 5$, $a^5 = a^3.a.a \in A$, $a^3.a \notin A$, then by (3), $a \in B$.
If $m_0 > 5$, $a^{m_0} = a^{m_0-4}.a^3, a \in A$.But $c^{m_0-4} \notin A$, $a^3 \notin A$, by (3) $a \in B$.
This proves that $\sqrt{A} \subseteq B \ldots\ldots (ii)$. Therefore by (i) and (ii), $\sqrt{A} = B$

Proposition 6.4. *Let I be an ideal of a ternary hemiring H. Then the following conditions are equivalent.*

(1) I is strongly irreducible,

(2) If $a, b \in H$ satisfy that $< a > \cap < b > \subseteq I$ then $a \in I$ or $b \in I$,

(3) $H - I$ is an i-system.

Proof. The proof is analogous to that of Theorem 4.5 in [2].

Proposition 6.5. *Let H be a ternary hemiring. Let $0 \neq a \in H$ and let A be an ideal of H not containing a. Then there exists an irreducible ideal B of H containing A but not containing a.*

Proof. The proof is analogous to that of Proposition 7.34 in [3].

Proposition 6.6. *Any proper ideal I of a ternary semiring H is the intersection of all irreducible ideals containing it.*

Proof. The proof is analogous to that of Proposition 7.35 in [3].

Proposition 6.7. *An ideal I of a ternary hemiring H is prime if and only if it is semiprime and strongly irreducible.*

Proof. The proof is analogous to that of Theorem 4.8 in [2].

Proposition 6.8. *Let I be an ideal of a multiplicatively regular ternary hemiring H. Then the following conditions are equivalent.*

(1) I is prime,

(2) I is strongly irreducible.

Proof. (1) $\Longrightarrow$ (2) By proposition 6.7 if I is prime, then I is strongly irreducible (2) $\Rightarrow$ (1).

Suppose I is strongly irreducible and A, B, C be ideals of H such that $ABC \subseteq I$. Since H is multiplicatively regular, for any ideals of A, B, C of H

$$ABC = A \cap B \cap C.$$

Now $ABC = A \cap B \cap C = A \cap (B \cap C) \subseteq I$. This implies that $A \subseteq I$ or $B \cap C \subseteq I$.

Further $B \cap C \subseteq I$ implies that $B \subseteq I$ or $C \subseteq I$.

Therefore I is a prime ideal of H.

Proposition 6.9. *A commutative ternary hemiring H is multiplicatively regular if and only if every irreducible ideal of H is prime.*

Proof. If H is multiplicatively regular then by proposition 6.8 every irreducible ideal of H is prime. Conversely, assume that every irreducible ideal of H is prime. To show that H is multiplicatively regular. Let A, B, C be any ideals of H. By proposition 6.6, any ideal I of H is prime and hence $I = \sqrt{I}$. By proposition 5.5 we have

$$ABC = \sqrt{ABC} = \sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C} = A \cap B \cap C.$$

So, by proposition 6.35 in [3], H is multiplicatively regular.

Proposition 6.10. *A ternary hemiring H is multiplicatively regular if and only if every proper ideal of H is semiprime.*

Proof. The proof is anologous to that of Theorem 4.11 in [2].

7. The Prime Radical of a Ternary Hemiring

In this section we give some results on prime radical of a ternary hemiring.

Definition 7.1. *The radical of a ternary hemiring H is the radical of zero ideal of a ternary hemiring and is denoted by $P(H)$.*

Thus

$$P(H) = \{h \in H |\ \textit{every m-system of } H \textit{ containing } h \textit{ contains zero of } H \textit{ also}\}.$$

Proposition 7.2. *If $P(H)$ is the prime radical of a ternary hemiring H, then:*

(1) $P(H)$ is the intersection of all prime ideals in H,

(2) $P(H)$ is the semiprime ideal which is contained in every semiprime ideal in H.

Proof. (1) By proposition 5.9. $P(H)$ is the intersection of all prime ideals in H that contain (0) and every ideal of H contains 0. Therefore, $P(H)$ is the intersection of all prime ideals in H.

(2) By proposition 5.10, $P(H)$ is the smallest semiprime ideal in H that contains (0). Therefore $P(H)$ is the semiprime ideal which is contained in every semiprime ideal in H.

Definition 7.3. *An ideal A of a ternary hemiring H is said to be nilpotent if $A^n = (0)$, for some positive odd integer n.*

Definition 7.4. *An ideal A of a ternary hemiring H is said to be nil ideal if every element of H is nilpotent i.e. for every $a \in A$, $a^{n(a)} = 0$ for some positive odd integer $n(a)$ depending on a.*

Proposition 7.5. *If H is a ternary hemiring, then $P(H)$ is a nil ideal which contains every nilpotent ideal in H.*

Proof. We have to show that every element of $P(H)$ is nil-potent. Let $a \in P(H)$. Then by proposition 5.2, there exists a positive odd integer n such that $a^n = 0$. So every element of $P(H)$ is nilpotent. Hence $P(H)$ is a nil ideal.

Let A be any nilpotent ideal in H. Then $A^n = (0)$, for some fixed positive odd integer n. But $A^n = 0 \subseteq P(H)$. Since $P(H)$ is semiprime, $A^n \subseteq P(H)$ implies that $A \subseteq P(H)$. This proves that $P(H)$ contains every nilpotent ideal in H.

Proposition 7.6. *Let H be a ternary hemiring. If I is an ideal of H then $P(I) = I \cap P(H)$, where $P(I)$ denotes prime radical of I considering I as a ternary hemiring.*

Proof. The proof is analogous to that of Theorem 4.4 in [1].

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