SUPERCYCLICITY OF SPECIAL DIRECT SUMS OF OPERATORS

Javad Izadi¹, Bahmann Yousefi²

¹,²Department of Mathematics
Payame Noor University
P.O. Box 19395-3697, Tehran, IRAN

Abstract: In this paper we state and prove equivalent conditions for a tuple of operators satisfying the supercyclicity criterion.

AMS Subject Classification: 47B37, 47B33
Key Words: tuple of operators, hypercyclic vector, supercyclic vector, supercyclicity criterion

1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X.

Definition 1.1. Let \( \mathcal{T} = (T_1, T_2, ..., T_n) \) be an n-tuple of operators acting on an infinite dimensional Banach space X. We will let

\[
\mathcal{F}_T = \{T_1^{k_1}T_2^{k_2}...T_n^{k_n} : k_i \geq 0, \ i = 1, ..., n\}
\]

be the semigroup generated by \( \mathcal{T} \). For \( x \in X \), the orbit of \( x \) under the tuple \( \mathcal{T} \) is the set

\[
\text{Orb}(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}_T\}.
\]
A vector \( x \) is called a hypercyclic vector for \( \mathcal{T} \) if \( \text{Orb}(\mathcal{T}, x) \) is dense in \( X \) and in this case the tuple \( \mathcal{T} \) is called hypercyclic. Also, a vector \( x \) is called a supercyclic vector for \( \mathcal{T} \) if \( \mathbb{C}\text{Orb}(\mathcal{T}, x) \) is dense in \( X \) and in this case the tuple \( \mathcal{T} \) is called supercyclic. By \( \mathcal{T}_d^{(k)} \) we will refer to the set of all \( k \) copies of an element of \( \mathcal{F}_\mathcal{T} \), i.e.

\[
\mathcal{T}_d^{(k)} = \{ S_1 \oplus ... \oplus S_k : S_1 = ... = S_k \in \mathcal{F}_\mathcal{T} \}.
\]

For any \( k \geq 2 \), we say that \( \mathcal{T}_d^{(k)} \) is hypercyclic provided there exist \( x_1, ..., x_k \in X \) such that

\[
\{ W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)} \}
\]

is dense in the \( k \) copies of \( X \), \( X \oplus ... \oplus X \), and similarly we say that \( \mathcal{T}_d^{(k)} \) is supercyclic provided there exist \( x_1, ..., x_k \in X \) such that

\[
\mathbb{C}\{ W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)} \}
\]

is dense in the \( k \) copies of \( X \).

The study of supercyclic operators has experimented a great development during the last years. A nice Criterion is an important tool in much recent works on supercyclic operators. Now we state the Supercyclicity Criterion for a tuple of operators.

**Theorem 1.1.** (The Supercyclicity Criterion for tuples). Suppose \( X \) is a separable infinite dimensional Banach space and \( \mathcal{T} = (T_1, T_2) \) is a pair of continuous linear mappings on \( X \). Suppose there exist two dense subsets \( Y \) and \( Z \) in \( X \), and a pair of strictly increasing sequences \( \{m_k\} \) and \( \{n_k\} \) and a sequence of mappings \( S_k : Z \to X \) such that:

1) \( T_1^{m_k} T_2^{n_k} S_k z \to z \) for every \( z \in Z \),

2) \( \|T_1^{m_k} T_2^{n_k} y\| \|S_k z\| \to 0 \) for every \( y \in Y \) and every \( z \in Z \).

Then \( \mathcal{T} \) is supercyclic.

If an operator \( T \) satisfies the hypothesis of Theorem 1.1, we say that \( T \) satisfies the Supercyclicity Criterion.

Here, we want to extend some properties of supercyclic operators to a tuple of commuting operators. For some other topics we refer to [1–15].
2. Main Results

In the present paper we prove that a tuple of operators satisfying the Supercyclicity Criterion if and only if $T_d^{(2)}$ is supercyclic. For simplicity we prove our results only for a pair of operators and the techniques work for any n-tuple of operators.

**Theorem 2.1.** Let $X$ be a separable infinite dimensional Banach space and $T = (T_1, T_2)$ be a pair of operators $T_1$, $T_2$. Then $T = (T_1, T_2)$ satisfies the Supercyclicity Criterion if and only if $T_d^{(2)}$ is supercyclic.

**Proof.** Let $T$ satisfies the Supercyclicity Criterion. Thus there exist two dense subsets $Y$ and $Z$ in $H$, a pair of sequences $\{n_k\}$ and $\{n_k\}$ of positive integers, and also there exist a sequence of mappings $S_k : Z \to X$ such that:

1) $T^m_1 T^n_2 S_k z \to z$ for every $z \in Z$,

2) $\|T^m_1 T^n_2 y\| \|S_k z\| \to 0$ for every $y \in Y$ and every $z \in Z$.

Now let $\mathcal{Y}$ be the set of all sequences $(y_n)_n \in \bigoplus_{i=1}^{\infty} Y$ such that $y_n = 0$ for all but finitely many $n \in \mathbb{N}$. Similarly let $\mathcal{Z}$ be the set of all sequences $(z_n)_n \in \bigoplus_{i=1}^{\infty} Z$ such that $z_n = 0$ for all but finitely many $n \in \mathbb{N}$. Put $S'_k = \bigoplus_{i=1}^{\infty} S_k$ and consider it acting on $Z$. Then both $\mathcal{Y}$ and $\mathcal{Z}$ are dense in $\bigoplus_{i=1}^{\infty} X$ and clearly the hypotheses of the Supercyclicity Criterion are satisfied. Thus $T_d^{(\infty)}$ is supercyclic on $\bigoplus_{i=1}^{\infty} X$ from which we can conclude that clearly $T_d^{(2)}$ is supercyclic on $X \oplus X$. Now let $(x, y)$ be a supercyclic vector for $T_d^{(2)}$. In particular $x$ and $y$ are supercyclic vectors for $T$. For all $k \in \mathbb{N}$, put $U_k = B(0, \frac{1}{k})$. Then there exist $m_k$, $n_k \in \mathbb{N}$ and $\lambda_k \in \mathbb{C}$ such that

$$\lambda_k (T^{m_k}_1 T^{n_k}_2 \oplus T^{m_k}_1 T^{n_k}_2)(x, y) \in U_k \oplus (x + U_k).$$

Thus $\lambda_k T^{m_k}_1 T^{n_k}_2 x \in U_k$ and

$$\lambda_k T^{m_k}_1 T^{n_k}_2 y \in x + U_k$$

for all $k \in \mathbb{N}$. This implies that $\lambda_k T^{m_k}_1 T^{n_k}_2 x \to 0$ and

$$\lambda_k T^{m_k}_1 T^{n_k}_2 y \to x.$$

Let $Y = Z = COrb(T, x)$ which is dense in $X$. Also for all $k \in \mathbb{N}, \lambda \in \mathbb{C}$ and $i, j \in \mathbb{N}$ define

$$S_k(\lambda T^i_1 T^j_2 x) = \lambda \lambda_k T^i_1 T^j_2 y.$$
Note that
\[ T_{1}^{m_{k}}T_{2}^{n_{k}}S_{k}(\lambda T_{1}^{i}T_{2}^{j}x) = \lambda T_{1}^{i}T_{2}^{j}(\lambda_{k}T_{1}^{m_{k}}T_{2}^{n_{k}}y) \]
which tends to \( \lambda T_{1}^{i}T_{2}^{j}x \) as \( k \to \infty \). So \( T_{1}^{m_{k}}T_{2}^{n_{k}}S_{k}z \to z \) for all \( z \in Z \). Also for all \( \lambda, w \in \mathbb{C} \) and \( m, n, i, j \in \mathbb{N} \) we have
\[
||T_{1}^{m_{k}}T_{2}^{n_{k}}(\lambda T_{1}^{m}T_{2}^{n}x)|| \cdot ||S_{k}(wT_{1}^{i}T_{2}^{j}x)|| \\
= ||\lambda||w||T_{1}^{m}T_{2}^{n}(T_{1}^{m_{k}}T_{1}^{n_{k}}x)|| ||\lambda_{k}T_{1}^{i}T_{2}^{j}y|| \\
\leq ||\lambda||w||\lambda_{k}||T_{1}^{m}T_{2}^{n}|| ||T_{1}^{m_{k}}T_{1}^{n_{k}}x|| ||T_{1}^{i}T_{2}^{j}y||.
\]
Since \( |\lambda_{k}||T_{1}^{m_{k}}T_{2}^{n_{k}}x| \to 0 \), hence
\[
||T_{1}^{m_{k}}T_{2}^{n_{k}}(\lambda T_{1}^{m}T_{2}^{n}x)|| \cdot ||S_{k}(wT_{1}^{i}T_{2}^{j}x)|| \to 0
\]
as \( k \to \infty \). Thus for all \( y \in Y \) and \( z \in Z \), we get
\[
||T_{1}^{m_{k}}T_{2}^{n_{k}}y|| ||S_{k}z|| \to 0
\]
and so \( \mathcal{T} \) satisfies the Supercyclicity Criterion. This completes the proof. \( \square \)

References


