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SUPERCYCLICITY OF SPECIAL DIREC SUMS OF OPERATORS

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Abstract: In this paper we state and prove equivalent conditions for a tuple of operators satisfying the supercyclicity criterion.

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1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X.

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X. We will let

$$\mathcal{F}_{\mathbf{T}} = \{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n} : k_i \geq 0,\ i = 1, ..., n\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set

$$Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}_{\mathbf{T}}\}.$$

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A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic. Also, a vector x is called a supercyclic vector for $\mathcal{T}$ if $\mathbb{C}Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called supercyclic. By $\mathcal{T}_d^{(k)}$ we will refer to the set of all k copies of an element of $\mathcal{F}_{\mathbf{T}}$, i.e.

$$\mathcal{T}_d^{(k)} = \{S_1 \oplus ... \oplus S_k : S_1 = ... = S_k \in \mathcal{F}_{\mathbf{T}}\}.$$

For any $k \geq 2$, we say that $\mathcal{T}_d^{(k)}$ is hypercyclic provided there exist $x_1, ..., x_k \in X$ such that

$$\{W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)}\}$$

is dense in the k copies of X, $X \oplus ... \oplus X$, and similarly we say that $\mathcal{T}_d^{(k)}$ is supercyclic provided there exist $x_1, ..., x_k \in X$ such that

$$\mathbb{C}\{W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)}\}$$

is dense in the k copies of X.

The study of supercyclic operators has experimented a great development during the last years. A nice Criterion is an important tool in much recent works on supercyclic operators. Now we state the Supercyclicity Criterion for a tuple of operators.

Theorem 1.1. *(The Supercyclicity Criterion for tuples). Suppose X is a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ is a pair of continuous linear mappings on X. Suppose there exist two dense subsets Y and Z in X, and a pair of strictly increasing sequences $\{m_k\}$ and $\{n_k\}$ and a sequence of mappings $S_k : Z \to X$ such that:*

1) $T_1^{m_k} T_2^{n_k} S_k z \to z$ for every $z \in Z$,

2) $||T_1^{m_k} T_2^{n_k} y||\ ||S_k z|| \to 0$ for every $y \in Y$ and every $z \in Z$.

Then $\mathcal{T}$ is supercyclic.

If an operator T satisfies the hypothesis of Theorem 1.1, we say that T satisfies the Supercyclicity Criterion.

Here, we want to extend some properties of supercyclic operators to a tuple of commuting operators. For some other topics we refer to [1–15].

2. Main Results

In the present paper we prove that a tuple of operators satisfying the Supercyclicity Criterion if and only if $\mathcal{T}_d^{(2)}$ is supercyclic. For simplicity we prove our results only for a pair of operators and the techniques work for any n-tuple of operators.

Theorem 2.1. *Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ be a pair of operators T_1, T_2. Then $\mathcal{T} = (T_1, T_2)$ satisfies the Supercyclicity Criterion if and only if $\mathcal{T}_d^{(2)}$ is supercyclic.*

Proof. Let $\mathcal{T}$ satisfies the Supercyclicity Criterion. Thus there exist two dense subsets Y and Z in H, a pair of sequences $\{n_k\}$ and $\{n_k\}$ of positive integers, and also there exist a sequence of mappings $S_k : Z \to X$ such that:

1) $T_1^{m_k} T_2^{n_k} S_k z \to z$ for every $z \in Z$,

2) $||T_1^{m_k} T_2^{n_k} y||\ ||S_k z|| \to 0$ for every $y \in Y$ and every $z \in Z$.

Now let $\mathcal{Y}$ be the set of all sequences $(y_n)_n \in \oplus_{i=1}^{\infty} Y$ such that $y_n = 0$ for all but finitely many $n \in \mathbb{N}$. Similarly let $\mathcal{Z}$ be the set of all sequences $(z_n)_n \in \oplus_{i=1}^{\infty} Z$ such that $z_n = 0$ for all but finitely many $n \in \mathbb{N}$. Put $S_k^{\square} = \oplus_{i=1}^{\infty} S_k$ and consider it acting on $\mathcal{Z}$. Then both $\mathcal{Y}$ and $\mathcal{Z}$ are dense in $\oplus_{i=1}^{\infty} X$ and clearly the hypotheses of the Supercyclicity Criterion are satisfied. Thus $\mathcal{T}_d^{(\infty)}$ is supercyclic on $\oplus_{i=1}^{\infty} X$ from which we can conclude that clearly $\mathcal{T}_d^{(2)}$ is supercyclic on $X \oplus X$. Now let (x, y) be a supercyclic vector for $\mathcal{T}_d^{(2)}$. In particular x and y are supercyclic vectors for $\mathcal{T}$. For all $k \in \mathbb{N}$, put $U_k = B(0, \frac{1}{k})$. Then there exist m_k, $n_k \in \mathbb{N}$ and $\lambda_k \in \mathbb{C}$ such that

$$\lambda_k (T_1^{m_k} T_2^{n_k} \oplus T_1^{m_k} T_2^{n_k})(x, y) \in U_k \oplus (x + U_k).$$

Thus $\lambda_k T_1^{m_k} T_2^{n_k} x \in U_k$ and

$$\lambda_k T_1^{m_k} T_2^{n_k} y \in x + U_k$$

for all $k \in \mathbb{N}$. This implies that $\lambda_k T_1^{m_k} T_2^{n_k} x \to 0$ and

$$\lambda_k T_1^{m_k} T_2^{n_k} y \to x.$$

Let $Y = Z = \mathbb{C}Orb(\mathcal{T}, x)$ which is dense in X. Also for all $k \in \mathbb{N}, \lambda \in \mathbb{C}$ and $i, j \in \mathbb{N}$ define

$$S_k(\lambda T_1^i T_2^j x) = \lambda \lambda_k T_1^i T_2^j y.$$

Note that

$$T_1^{m_k}T_2^{n_k}S_k(\lambda T_1^iT_2^jx) = \lambda T_1^iT_2^j(\lambda_k T_1^{m_k}T_2^{n_k}y)$$

which tends to $\lambda T_1^iT_2^jx$ as $k \to \infty$. So $T_1^{m_k}T_2^{n_k}S_kz \to z$ for all $z \in Z$. Also for all $\lambda, w \in \mathbb{C}$ and $m, n, i, j \in \mathbb{N}$ we have

$$\begin{aligned} ||T_1^{m_k}T_2^{n_k}(\lambda T_1^mT_2^nx)|| \quad &. \quad ||S_k(wT_1^iT_2^jx)|| \\ &= |\lambda|\,|w|\,||T_1^mT_2^n(T_1^{m_k}T_2^{n_k}x)||\,||\lambda_kT_1^iT_2^jy|| \\ &\leq |\lambda|\,|w|\,|\lambda_k|\,||T_1^mT_2^n||\,||T_1^{m_k}T_2^{n_k}x||\,||T_1^iT_2^jy||. \end{aligned}$$

Since $|\lambda_k|\,||T_1^{m_k}T_2^{n_k}x|| \to 0$, hence

$$||T_1^{m_k}T_2^{n_k}(\lambda T_1^mT_2^nx)||\,||S_k(wT_1^iT_2^jx)|| \to 0$$

as $k \to \infty$. Thus for all $y \in Y$ and $z \in Z$, we get

$$||T_1^{m_k}T_2^{n_k}y||\,||S_kz|| \to 0$$

and so $\mathcal{T}$ satisfies the Supercyclicity Criterion. This completes the proof. □

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