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TUPLE OF OPERATORS AND EPSILON SUPERCYCLICITY

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Abstract: In this paper we prove that if a tuple of operators is ϵ-supercyclic for all $\epsilon > 0$, then it is supercyclic.

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1. Introduction

From now on, let T_1, T_2, ..., T_n be commutative bounded linear operators on a Banach space X.

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X. We will let

$$\mathcal{F}_{\mathbf{T}} = \{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n} : k_i \geq 0,\ i = 1, ..., n\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set

$$Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}_{\mathbf{T}}\}.$$

A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic. Also, a vector x is called a supercyclic vector for $\mathcal{T}$ if $\mathbb{C}Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called supercyclic.

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Definition 1.2. Let $\epsilon \in (0,1)$ and $x \in X$. If for every non zero vector $y \in X$, there exist integers k_1, ..., k_n and $\lambda \in \mathbb{C}$ such that

$$\| \lambda T_1^{k_1} {T_2}^{k_2} ... {T_n}^{k_n} x - y \| < \epsilon \| y \|,$$

then the vector x is called ϵ-supercyclic for the n-tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$. A tuple of operators is ϵ-supercyclic if it admits an ϵ-supercyclic vector.

For some sources on these topics see [1–20].

2. Main Results

In this section we prove that if a tuple is ϵ-supercyclic for all $\epsilon > 0$, then it is supercyclic. We will use the idea of Theorem 2.1 in [20] for a tuple of operators. We will denote $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$. We will use $SC(\mathcal{T})$ for the collection of supercyclic vectors for a tuple of operators $\mathcal{T}$.

Theorem 2.1. *Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of operators T_1, T_2, ..., T_n. Then $\mathcal{T}$ is supercyclic, if and only if for any two non-void open sets U and V, there exist $m_i \geq 1$ for $i = 1, ..., n$ and and $\lambda \in \mathbb{C} \backslash \{0\}$ such that $\lambda T_1^{m_1} ... T_n^{m_n}(U) \cap V \neq \emptyset$.*

Proof. Suppose that x is a supercyclic vector for $\mathcal{T}$, hence $\mathbb{C}Orb(\mathcal{T}, x)$ is dense in X and so there exist $\lambda_1 \in \mathbb{C} \backslash \{0\}$ and $m_i \in \mathbb{N}$; $i = 1, ..., n$ such that $\lambda_1 T_1^{m_1} ... T_n^{m_n} x \in U$. Let $y = \lambda_1 T_1^{m_1} ... T_n^{m_n} x$. Since $\mathbb{C}Orb(\mathcal{T}, x)$ is also dense in X, thus $\lambda T_1^{m_1} ... T_n^{m_n} y \in V$ for some $\lambda \in \mathbb{C} \backslash \{0\}$ and $m_i \in \mathbb{N}$; $i = 1, ..., n$, hence $\lambda T_1^{m_1} ... T_n^{m_n}(U) \cap V \neq \emptyset$. Conversely, fix an enumeration $\{B_n : n \in \mathbb{N}\}$ of the open balls in X with rational radii, and centers in a countable dense subset of X. By the continuity of the operators T_1, T_2,..., T_n, the sets

$$G_k = \bigcup \{\lambda T_1^{-m_1} ... T_n^{-m_n}(B_k) : m_i \geq 0; i = 1, ..., n, \lambda \in \mathbb{C}\}$$

are open. Since for any two non-void open sets U and V, there exist $m_i \geq 1$; $i = 1, ..., n$, and $\lambda \in \mathbb{C} \backslash \{0\}$ such that $\lambda T_1^{m_1} ... T_n^{m_n}(U) \cap V \neq \emptyset$, thus one can see that $SC(\mathcal{T})$ is exactly equal to $\bigcap \{G_k : k \in \mathbb{N}\}$ that is dense in X. This completes the proof. □

Theorem 2.2. *Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be the n-tuple of operators T_1, T_2, ..., T_n. If for every $\epsilon > 0$, $\mathcal{T}$ is ϵ-supercyclic, then $\mathcal{T}$ is supercyclic.*

Proof. Suppose that U and V are nonempty open subsets of X. Let $u \in U$ and $v \in V$ be two nonzero vectors, and consider

$$0 < \delta < min\{\| u \|, \| v \|\}$$

small enough such that $B(u, \delta) \subset U$ and $B(v, \delta) \subset V$. Choose

$$\epsilon < \delta/(6max\{\| u \|, \| v \|\})$$

and let $x \in X$ be an ϵ-supercyclic vector for $\mathcal{T}$. This implies that there exist nonnegative integers $k_1^{(0)}$, ..., $k_n^{(0)}$ and $\lambda \in \mathbf{C}$ such that

$$\| \lambda T_1^{k_1^{(0)}} T_2^{k_2^{(0)}} ... T_n^{k_n^{(0)}} x - u \| < \epsilon \| u \| < \delta.$$

Hence

$$\lambda T_1^{k_1^{(0)}} T_2^{k_2^{(0)}} ... T_n^{k_n^{(0)}} x \in B(u, \delta) \subset U.$$

We want to show that

$$V \cap \{\lambda {T_1}^{k_1} {T_2}^{k_2} ... {T_n}^{k_n} x : k_i \geq 0, i = 1, ..., n\}$$

contain infinitely many elements. Suppose on the contrary that it contains only the elements $\lambda T_1^{k_1^{(i)}} T_2^{k_2^{(i)}} ... T_n^{k_n^{(i)}} x$ for $i = 1, ..., k$. As we saw earlier, for each $v' \in B(v, \frac{2\delta}{3})$ there exist integers $k_1(v')$, ..., $k_n(v')$ which satisfies

$$\| \lambda T_1^{k_1(v')} T_2^{k_2(v')} ... T_n^{k_n(v')} x - v' \| \leq \epsilon |\lambda| \ \| v' \| \leq 2\epsilon |\lambda| \ \| v \| < \frac{\delta}{3}.$$

Hence

$$\lambda T_1^{k_1(v')} T_2^{k_2(v')} ... T_n^{k_n(v')} \in \{\lambda T_1^{k_1^{(i)}} T_2^{k_2^{(i)}} ... T_n^{k_n^{(i)}} : i = 1, ..., k\},$$

because

$$\| \lambda T_1^{k_1(v')} ... T_n^{k_n(v')} x - v \| \leq \| \lambda T_1^{k_1(v')} ... T_n^{k_n(v')} x - v' \| + \| v' - v \| < \delta.$$

Therefore

$$B(v, \frac{2\delta}{3}) \subset \bigcup_{i=1}^{k} B(\lambda T_1^{k_1^{(i)}} T_2^{k_2^{(i)}} ... T_n^{k_n^{(i)}} x, \frac{\delta}{3}),$$

that is a contradiction since X is infinite dimensional. Thus, indeed the set

$$V \cap \{\lambda {T_1}^{k_1} {T_2}^{k_2} ... {T_n}^{k_n} x : k_i \geq 0, i = 1, ..., n\}$$

contain infinitely many elements and so the set

$$B(v,\delta)\cap\{\lambda T_1^{k_1}...T_n^{k_n}x : k_i\geq 0, i=1,...,n\}$$

has infinite elements. In particular, there exist $k_i\in\mathbb{N}_0$ satisfying $k_i\geq {k_i}^{(0)}$ for $i=1,...,n$ such that $\lambda {T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n}x\in V$. Thus we get

$$\lambda T_1^{k_1-{k_1}^{(0)}}...T_n^{k_n-{k_n}^{(0)}}T_1^{{k_1}^{(0)}}...T_n^{{k_n}^{(0)}}x=\lambda {T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n}x,$$

which belongs to

$$\lambda T_1^{k_1-{k_1}^{(0)}}T_2^{k_2-{k_2}^{(0)}}...T_n^{k_n-{k_n}^{(0)}}(U)\cap V.$$

Now by Theorem 2.1 the proof is complete. □

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