International Journal of Pure and Applied Mathematics
**Volume 94 No. 5 2014, 737-740**
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
**url:** http://www.ijpam.eu
**doi:** http://dx.doi.org/10.12732/ijpam.v94i5.12
ijpam.eu

# SUFFICIENT CONDITIONS FOR CONTINUITY OF MAPS

Asha Gupta
Department of Applied Sciences
PEC University of Technology
Chandigarh, INDIA

**Abstract:** A function $f : X \to Y$ is said to have closed graph if graph of $f$, i.e. the set $(x, f(x))$ is a closed subset of the product space $X \times Y$. It is well established fact that a function with closed graph is KC as well as inversely KC. Moreover, a function with closed graph is continuous if $Y$ is compact. In the present paper, some conditions are investigated under which an inversely KC map or KC map becomes continuous.

**AMS Subject Classification:** 54C05
**Key Words:** continuous graph, closed graph, KC, inversely KC, Frechet space

## 1. Introduction

By a space, we shall mean a topological space. No separation axioms are assumed and no function is assumed to be continuous or onto unless mentioned explicitly; $cl(A)$ will denote the closure of the subset $A$ in the space $X$. If $A$ is a subset of $X$, we say that $X$ is $T_1$ at $A$ if each point of $A$ is closed in $X$. A point $x$ in $X$ is said to be a cluster point ($w$-limit point in the terminology of Thron [4])of a subset $A$ of $X$ if every neighborhood of $x$ contains infinite number of points of $A$. $X$ is said to be Frechet space (or closure sequential in

Received: June 19, 2014

the terminology of Wilansky [5]) if for each subset $A$ of $X$, $x \in cl(A)$ implies there exists a sequence $\{x_n\}$ in $A$ converging to $x$. $X$ is said to be a $k$-space if $O$ is open (equivalently:closed) in $X$ whenever $O \cap K$ is open (closed) in $K$ for every compact subset $K$ of $X$. Every space which is either locally compact or Frechet is a $k$-space.

A function $f : X \to Y$ is said to be compact preserving (compact) if image (inverse image) of each compact set is compact. $f : X \to Y$ will be called KC [5] (inversely KC) if image (inverse image) of every compact set is closed.

In 1968, Fuller [1] has proved the following:

**Theorem 1.1.** *Let $f : X \to Y$ have closed graph. Then $f$ is continuous if any one of the following conditions is satisfied.*

*(a) $Y$ is compact;*

*(b) $f$ is subcontinuous.*

In 1988, Piotrowski in [3] proved the following result.

**Theorem 1.2.** *Let $f : X \to Y$ have closed graph and be compact preserving where $X$ is a $k$-space. Then $f$ is continuous.*

A function with closed graph is KC as well as inversely KC. In the present paper, the condition of closed graph on the map $f$ in theorem 1.1 and theorem 1.2 is weakened by assuming the map KC or inversely KC and conditions are investigated under which a KC map (inversely KC map) becomes continuous.

## 2. Main Results

The following theorems 2.1 and 2.2 give conditions under which a KC map becomes continuous.

**Theorem 2.1.** *Let $f : X \to Y$ be KC, compact preserving and has closed point inverses where $X$ is a Frechet. Then $f$ is continuous.*

*Proof.* Let $F$ be a closed subset of $Y$ and let $x \in cl f^{-1}(F) - f^{-1}(F)$. Since $X$ is a Frechet space, there exists a sequence $\{x_n\}$ of points in $f^{-1}(F)$ such that $x_n \to x$. Let $H = \{x_n : n \in N\} \cup \{x\}$. Then $H$ is compact subset of $X$ and $f$ is compact-preserving, KC implies $f(H)$ is compact as well as closed subset of $Y$ and therefore $\{f(x_n) : n \in N\}$ being a sequence in compact set $f(H) \cap F$ has a cluster point $y$ in $f(H) \cap F$. Since $y \neq f(x)$, $U = X - \{f^{-1}(y)\}$ is an open set containing $x$. Then $x_n \to x$ implies there exists an integer $n_0$ such that $x_n \in U$ for all $n \geq n_0$. Let $K = \{x_n : n \geq n_0\} \cup \{x\}$.Then $K$ is compact subset of $X$ , but $f(K)$ is not closed as $y \in clf(K) - f(K)$ -a contradiction. Hence $f$ must be continuous.

In the next Theorem 2.2, the condition of compact preserving on the map $f$ is replaced by taking $Y$ as B-W compact.

**Theorem 2.2.** *Let $f : X \to Y$ be KC and have closed point inverses where $X$ is Frechet and $Y$ is B-W compact.Then $f$ is continuous.*

*Proof.* For the proof of this theorem, see [2].

The following theorems 2.3 and 2.4 give conditions under which an inversely KC map becomes continuous.

**Theorem 2.3.** *Let $f : X \to Y$ be inversely KC and compact preserving where $X$ is a $k$-space. Then $f$ is continuous.*

*Proof.* Let $F$ be a closed subset of $Y$. To prove $f^{-1}(F)$ is a closed subset of $X$ ,we prove $K \cap f^{-1}(F)$ is a closed subset of $K$ for every compact subset $K$ of $X$, since $X$ is a $k$-space. So let $K$ be a compact subset of $X$. Then $f(K)$ is a compact subset of $Y$, since $f$ is compact preserving. Now $F \cap f(K)$ being a closed subset of $f(K)$ is compact and therefore $f^{-1}(F \cap f(K)) = K \cap f^{-1}(F)$ is closed in $K$ as $f$ is inversely KC. This completes the proof.

In the next Theorem 2.4 the condition of compact preserving on the map $f$ is dropped and $Y$ is assumed as locally compact, regular, countable compact and $T_1$ at $f(X)$.

**Theorem 2.4.** *Let $f : X \to Y$ be inversely KC, where $X$ is Frechet and $Y$ is locally compact, regular, countable compact and $T_1$ at $f(X)$. Then $f$ is continuous.*

*Proof.* Let $F$ be a closed subset of $Y$ and let $x \in clf^{-1}(F) - f^{-1}(F)$. Since $X$ is a Frechet space, there exists a sequence $\{x_n\}$ of points in $f^{-1}(F)$ such that $x_n \to x$. Now $F$ being a closed subset of countable compact space $Y$ is countable compact ,the set $\{f(x_n) : n \in N\}$ has a cluster point $y$ in $F$, $y \neq f(x)$. Since $Y$ is $T_1$ at $f(X)$, $f(x)$ is closed .Now $Y$ is locally compact, regular implies there exists an open set $W$ containing $y$ such that $cl(W)$ is compact and $f(x) \notin cl(W)$. Since $f$ is inversely KC, $f^{-1}(clW)$ is closed which is a contradiction as $x \in clf^{-1}(clW) - f^{-1}(clW)$. Hence $f$ must be continuous.

The following example shows that none of the condition on the domain and range space in theorems 2.1 to 2.4 can be weakened.

**Example.** The function $f : X^+ \to X$ which is identity on $X$ with $f(\infty) = p$ where $X$ is any infinite set with discrete topology and $X^+ = X \cup \{\infty\}$ with one point compactification of $X$, is KC as well as inversely KC but not continuous. Here the domain space is Frechet and the range is $T_2$.

## References

[1] R.V. Fuller, Relations among continuous and various non continuous functions, *Pacific J. Math.*, **25** (1968), 495-509.

[2] Asha Gupta, Continuous and closed maps, *International Journal of Pure and Applied Mathematical Sciences*, **6**, No. 2 (2013), 193-195.

[3] Z. Piotrowsi, A. Szymanski, A closed graph theorem: Topological approach, *Rendiconti Del Circolo Matematico Di Palermo*, Serie II, **XXXVII** (1988), 88-99.

[4] W.J. Thron, *Topological Structures*, Holt, Rinehart and Winston (1966).

[5] A. Wilansky, *Topology for Analysis*, Xerox College Publishing Lexington, Massachusetts, Toronto (1970).