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$q-$RADIUS STABILITY OF MATRIX POLYNOMIALS

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Abstract: In this paper, the $q-$radius stability of a matrix polynomial $P(\lambda)$ relative to an open region Ω of the complex plane and its relation to the $q-$numerical range of $P(\lambda)$ are investigated. Also, we obtain a lower bound that involves the distance of Ω to the connected components of the $q-$numerical range of $P(\lambda)$.

Key Words: matrix polynomial, $q-$radius stability, $q-$numerical range

1. Introduction

Let M_n be the algebra of all $n \times n$ complex matrices. Suppose that

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + ... + A_1\lambda + A_0, \tag{1}$$

is a matrix polynomial, where $A_i \in M_n$ for $i = 0, 1, ..., m,\ A_m \neq 0$ and λ is a complex variable. The numbers m and n are referred to as the degree and order of $P(\lambda)$, respectively. Matrix polynomials arise in many applications and their spectral analysis is very important when studying linear systems of ordinary differential equations with constant coefficients ([4]). If all the coefficients of $P(\lambda)$, as in (1), are Hermitian matrices, then $P(\lambda)$ is called selfadjoint. A scalar $\lambda_0 \in \mathbb{C}$ is an eigenvalue of $P(\lambda)$ if the system $P(\lambda_0)x = 0$ has a nonzero solution $x_0 \in \mathbb{C}^n$. This solution x_0 is known as an eigenvector of $P(\lambda)$ corresponding to λ_0, and the set of all eigenvalues of $P(\lambda)$ is said to be the spectrum of $P(\lambda)$, that

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is denoted by $\sigma[P(\lambda)]$. So $\sigma[P(\lambda)] = \{\mu \in \mathbb{C} : detP(\mu) = 0\}$. The (classical) numerical range of $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + ... + A_1\lambda + A_0$, is defined as follows:

$$W[P(\lambda)] = \{\mu \in \mathbb{C} : x^*P(\mu)x = 0 \quad for\ some\ nonzero \quad x \in \mathbb{C}^n\}.$$

It is closed and contains $\sigma[P(\lambda)]$ (see [7] for more information). The numerical range of matrix polynomials plays an important role in the study of overdamped vibration systems with a finite number of degree of freedom, and it is also related to the stability theory (see e.g., [4,6,7]). For a $q \in (0, 1]$, the $q-$numerical range of $P(\lambda)$ is defined by

$$W_q[P(\lambda)] = \{\mu \in \mathbb{C} : y^*P(\mu)x = 0 \quad x, y \in \mathbb{C}^n,\ x^*x = y^*y = 1,\ y^*x = q\}. \quad (2)$$

We also define $q-$spectrum of $P(\lambda)$ as follows:

$$\sigma_q[P(\lambda)] = \{\mu \in \mathbb{C} : detP(q^{-1}\mu) = 0\}.$$

Clearly, $W_q[P(\lambda)]$ is always closed and contains the spectrum $\sigma_q[P(\lambda)]$. When $P(\lambda) = q^{-1}I\lambda - A$, $W_q[P(\lambda)]$ coincides with the $q-$numerical range of matrix A, we get

$$W_q(A) = \{y^*Ax : x, y \in \mathbb{C}^n, \quad x^*x = y^*y = 1,\ y^*x = q\}$$

(see [1,3]). Moreover, for $q = 1$, we obtain the numerical range of $P(\lambda)$. As shown in ([8, 9]), $W_q[P(\lambda)]$ is bounded if and only if $0 \notin W_q(A_m)$. One can find more about the geometry of $W_q[P(\lambda)]$ in [8, 9].

2. $q-$Radius Stability

Consider an index set $J \subseteq \{0, 1, ..., m\}$. In this paper, we consider the $q-$spectrum of perturbations of the matrix polynomial:

$$P_J(\lambda) = (A_m+\Delta_m)\lambda^m+(A_{m-1}+\Delta_{m-1})\lambda^{m-1}+...+(A_1+\Delta_1)\lambda+A_0+\Delta_0 \quad (3)$$

where $\Delta_s = 0$ for all $s \notin J$. With the perturbed polynomial in (3), we associate the $n\times n$ matrix polynomial $\Delta_J(\lambda) = \Delta_m\lambda^m+...+\Delta_1\lambda+\Delta_0$ and the $n\times n(m+1)$ complex matrix

$$D_J = [\ \Delta_m \quad \Delta_{m-1} \quad \cdots \quad \Delta_1 \quad \Delta_0\ ]. \quad (4)$$

Let Ω be an open region of $\mathbb{C}$ whose boundary , $\partial\Omega$, is a piecewise smooth curve. The matrix polynomial $P(\lambda)$ is said to be Ω_q- stable if $\sigma_q[P(\lambda)] \subset \Omega$. In this case, we define the J_q-stability radius of $P(\lambda)$ relative to Ω as

$$R_{J_q}[P(\lambda), \Omega] = \inf_{D_J}\{\| D_J \|_2 \colon \sigma_q[P_j(\lambda)] \bigcap (\mathbb{C} \setminus \Omega) \neq \emptyset\}.$$

That is, $R_{J_q}[P(\lambda), \Omega]$ is the distance of $P(\lambda)$ to Ω_q-instability, when the coefficients of $P(\lambda)$ indexed by J are allowed to vary.

For the proof of the main result we will need the following lemma.

Lemma 2.1. *Let $P(\lambda)$ be defined as in (1) and consider its perturbation $P_J(\lambda)$ as defined in (3). Also, let Ω be an open region of $\mathbb{C}$ such that $\sigma_q[P(\lambda)] \subset \Omega$. Then we have*

$$R_{J_q}[P(\lambda), \Omega] = \inf_{\mu \in \partial\Omega}\{\inf_{D_J}\{\|D_J\|_2 : \det(I + \Delta_J(\mu)P(\mu)^{-1}) = 0\}\}.$$

Proof. let $\mu \in \partial\Omega$, and note that the matrix $P(\mu)$ is invertible. Also $\det P_J(\mu) = 0$ if and only if $\det(I + \Delta_J(\mu)P(\mu)^{-1}) = 0$. By definition of J_q-stability, it follows that

$$\begin{aligned} R_{J_q}[P(\lambda), \Omega] &= \inf_{\mu \notin \Omega}\{\inf_{D_J}\{\|D_J\|_2 : \det P_J(\mu) = 0\}\} \\ &= \inf_{\mu \in \partial\Omega}\{\inf_{D_J}\{\|D_J\|_2 : \det P_J(\mu) = 0\}\} \\ &= \inf_{\mu \in \partial\Omega}\{\inf_{D_J}\{\|D_J\|_2 : \det(I + \Delta_J(\mu)P(\mu)^{-1}) = 0\}\}. \end{aligned}$$

So the proof is complete. □

Theorem 2.2. *Let $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + ... + A_1\lambda + A_0$ be an $n \times n$ matrix polynomial with $\det A_m \neq 0$, and let $J \subseteq \{0, 1, ..., m\}$. If Ω is an open region of $\mathbb{C}$ such that $\sigma_q[P(\lambda)] \subset \Omega$, then we have*

$$R_{J_q}[P(\lambda), \Omega] = \inf\{\frac{1}{\sqrt{\Sigma_{k\in J}\|\lambda|^{2k}\|P(\mu)^{-1}\|_2} : \lambda \in \partial\Omega\}}.$$

Proof. Since $\det A_m \neq 0$, $P(\lambda)$ has mn finite eigenvalues counting their multiplicities ([2,5]). Let $\mu \in \mathbb{C} \setminus \sigma_q[P(\lambda)]$, then $P(\mu)$ is invertible. Consider the matrix polynomial

$$\Delta_J(\mu) = \Delta_m\mu^m + \Delta_{m-1}\mu^{m-1} + ... + \Delta_1\mu + \Delta_0$$

$$= \; D_J[\; I\mu^m \;\; \cdots \;\; I\mu \;\; I \;]^T,$$

where D_J is defined as in (4). Suppose that $\det(I + \Delta_J(\mu)P(\mu)^{-1}) = 0$. Then -1 is an eigenvalue of the matrix $\Delta_J(\mu)P(\mu)^{-1}$ and so, we have

$$1 \leq \left\|\Delta_J(\mu)P(\mu)^{-1}\right\|_2 \leq \|\Delta_J(\mu)\|_2 \left\|P(\mu)^{-1}\right\|_2.$$

As a consequence, we get $\|\Delta_J(\mu)\|_2 \geq \left\|P(\mu)^{-1}\right\|_2^{-1}$ which implies that

$$\|\Delta_J(\mu)\|_2 \geq \frac{1}{\sqrt{\Sigma_{k\in J}\,|\mu|^{2k}}\;\|P(\mu)^{-1}\|_2}. \tag{5}$$

Furthermore, one can construct matrices $\Delta_s\,(s = 0, 1, ..., m)$ for which D_J attains the above lower bound and let $\det(I+\Delta_J(\mu)P(\mu)^{-1}) = 0$, as follows. Now consider two vectors $x, y \in \mathbb{C}^n$ such that $\|x\|_2 = 1$, $\left\|P(\mu)^{-1}x\right\|_2 = \left\|P(\mu)^{-1}\right\|_2$ and

$$y_j = \frac{w_j}{\|P(\mu)^{-1}\|_2^2}, \quad (j = 1, 2, ..., n)$$

where $w = [\; w_1 \;\; w_2 \;\; \cdots \;\; w_n \;]^T := P(\mu)^{-1}x$. Define the matrix Q_0 by $Q_0 = -xy^*$ and let $\Delta_S = \frac{\bar{y}^s}{\Sigma_{k\in J}|\mu|^{2k}}Q_0$ for $s \in J$ and $\Delta_S = 0$ if $s \notin J$. Now, we get

$$(I + \Delta_J(\mu)P(\mu)^{-1})x = x + Q_0P(\mu)^{-1}x = x + Q_0w.$$

Since $y^*w = 1$, we obtain

$$(I + \Delta_J(\mu)P(\mu)^{-1})x = x - xy^*w = 0.$$

Thus, $\det(I + \Delta_J(\mu)P(\mu)^{-1}) = 0$. Also, we have

$$\begin{aligned} \|D_J\|_2 &= \sup\{\frac{\left\|Q_0(\Sigma_{k\in J}\bar{\mu}^k v_k)(\Sigma_{k\in J}\,|\mu|^{2k})^{-1}\right\|_2}{\sqrt{\Sigma_{k\in J}\,\|v_k\|_2^2} : v_k \in \mathbb{C}^n \setminus 0} \\ &\leq \frac{1}{\Sigma_{k\in J}\,|\mu|^{2k}} \sup_{v_k\neq 0}\{\frac{\|x\|_2\,\|y\|_2\,\left\|\Sigma_{k\in J}\bar{\mu}^k v_k\right\|_2}{\sqrt{\Sigma_{k\in J}\,\|v_k\|_2^2}}\}. \end{aligned}$$

Moreover, we can see that

$$\|y\|_2 = \frac{\|w\|_2}{\|P(\mu)^{-1}\|_2^2} = \frac{\left\|P(\mu)^{-1}x\right\|_2}{\|P(\mu)^{-1}\|_2^2} = \frac{1}{\|P(\mu)^{-1}\|_2}$$

and

$$\left\|\Sigma_{k\in J}\bar{\mu}^k v_k\right\|_2 \leq \left\|\begin{bmatrix} I & I\mu & \cdots & I\mu^m \end{bmatrix}\right\|_2 \left\|\begin{bmatrix} v_0^T & v_1^T & \cdots & v_m^T \end{bmatrix}^T\right\|_2$$

where $v_k = 0$ whenever $k \notin J$. Therefore,

$$\|\Delta_J(\mu)\|_2 \leq \frac{1}{\sqrt{\Sigma_{k\in J}|\mu|^{2k}}\,\|P(\mu)^{-1}\|_2}$$

since $\|x\|_2 = 1$. So, for this special D_J, equality holds in (5). Now the proof is complete in view of lemma 2.1, since $\partial\Omega \cap \sigma_q[P(\lambda)] = \emptyset$. □

The notion of J_q-stability radius of $P(\lambda)$ is related to the $(\epsilon, J_q)-$pseudo $q-$spectrum of $P(\lambda)$ which is defined by

$$\sigma_{\epsilon,J_q}[P(\lambda)] = \{\mu \in \mathbb{C} : \mu \in \sigma_q[P_J(\lambda)]\ for\ some\ \Delta_J(\lambda)\ with\ \|D_J\|_2 < \epsilon\}$$

for each $\epsilon > 0$, where $P_J(\lambda)$ and D_J are defined as in (3) and (4), respectively. One can see that $\sigma_{\epsilon,J_q}[P(\lambda)] \subset \Omega$ if and only if $R_{J_q}[P(\lambda), \Omega] > \epsilon$.

Theorem 2.3. *Let $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + ... + A_1\lambda + A_0$ be an $n \times n$ matrix polynomial with* $\det A_m \neq 0$. *Also, let $J \subseteq \{0, 1, ..., m\}$ and $\epsilon > 0$ be given. Then*

$$\sigma_{\epsilon,J_q}[P(\lambda)] \setminus \sigma_q[P(\lambda)] = \{\mu \in (\mathbb{C} \setminus \sigma_q[P(\lambda)]) : \frac{1}{\sqrt{\Sigma_{k\in J}|\mu|^{2k}}\,\|P(\mu)^{-1}\|_2} \leq \epsilon\}.$$

Proof. Consider a μ in $\mathbb{C} \setminus \sigma_q[P(\lambda)]$. If $\mu \in \sigma_{\epsilon,J_q}[P(\lambda)]$, then there is an $n \times n$ matrix polynomial $\Delta_J(\mu) = \Delta_m\mu^m + ... + \Delta_1\mu + \Delta_0$ such that $\Delta_s = 0$ for $s \notin J$, $\left\|\begin{bmatrix} \Delta_m & \cdots & \Delta_1 & \Delta_0 \end{bmatrix}\right\|_2 \leq \epsilon$, and $\det(P(\mu) + \Delta_J(\mu)) = 0$. Thus, by (5),

$$\frac{1}{\sqrt{\Sigma_{k\in J}|\mu|^{2k}}\,\|P(\mu)^{-1}\|_2} \leq \left\|\begin{bmatrix} \Delta_m & \cdots & \Delta_1 & \Delta_0 \end{bmatrix}\right\|_2 \leq \epsilon.$$

Conversely, suppose that

$$\frac{1}{\sqrt{\Sigma_{k\in J}|\mu|^{2k}}\,\|P(\mu)^{-1}\|_2} \leq \epsilon.$$

Then, as in the proof of Theorem 2.2, one can construct a matrix polynomial $\Delta_J(\mu) = \Delta_m\mu^m + ... + \Delta_1\mu + \Delta_0$ such that $\Delta_s = 0$ for $s \notin J$,

$$\left\| \begin{bmatrix} \Delta_m & \cdots & \Delta_1 & \Delta_0 \end{bmatrix} \right\|_2 \leq \epsilon$$

and $\det(P(\mu) + \Delta_J(\mu))$. Thus, $\mu \in \sigma_{\epsilon,J_q}[P(\lambda)]$. □

Corollary 2.4. *Let* $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + ... + A_1\lambda + A_0$ *be an* $n \times n$ *matrix polynomial with* $\det A_m \neq 0$, *and let* $J \subseteq \{0, 1, ..., m\}$ *and* $\epsilon > 0$. *Then*

$$\partial(\sigma_{\epsilon,J_q}[P(\lambda)] \setminus \sigma_q[P(\lambda)]) = \{\mu \in (\mathbb{C} \setminus \sigma_q[P(\lambda)]) : \frac{1}{\sqrt{\Sigma_{k\in J} \mid \mu \mid^{2k}} \parallel P(\mu)^{-1} \parallel_2} = \epsilon\}.$$

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