$q$–RADIUS STABILITY OF MATRIX POLYNOMIALS

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Abstract: In this paper, the $q$–radius stability of a matrix polynomial $P(\lambda)$ relative to an open region $\Omega$ of the complex plane and its relation to the $q$–numerical range of $P(\lambda)$ are investigated. Also, we obtain a lower bound that involves the distance of $\Omega$ to the connected components of the $q$–numerical range of $P(\lambda)$.

Key Words: matrix polynomial, $q$–radius stability, $q$–numerical range

1. Introduction

Let $M_n$ be the algebra of all $n \times n$ complex matrices. Suppose that

$$P(\lambda) = A_m \lambda^m + A_{m-1} \lambda^{m-1} + ... + A_1 \lambda + A_0,$$

is a matrix polynomial, where $A_i \in M_n$ for $i = 0, 1, ..., m$, $A_m \neq 0$ and $\lambda$ is a complex variable. The numbers $m$ and $n$ are referred to as the degree and order of $P(\lambda)$, respectively. Matrix polynomials arise in many applications and their spectral analysis is very important when studying linear systems of ordinary differential equations with constant coefficients ([4]). If all the coefficients of $P(\lambda)$, as in (1), are Hermitian matrices, then $P(\lambda)$ is called selfadjoint. A scalar $\lambda_0 \in \mathbb{C}$ is an eigenvalue of $P(\lambda)$ if the system $P(\lambda_0)x = 0$ has a nonzero solution $x_0 \in \mathbb{C}^n$. This solution $x_0$ is known as an eigenvector of $P(\lambda)$ corresponding to $\lambda_0$, and the set of all eigenvalues of $P(\lambda)$ is said to be the spectrum of $P(\lambda)$, that

Received: July 22, 2014

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url: www.acadpubl.eu

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is denoted by $\sigma[P(\lambda)]$. So $\sigma[P(\lambda)] = \{\mu \in \mathbb{C} : detP(\mu) = 0\}$. The (classical) numerical range of $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$, is defined as follows:

$$W[P(\lambda)] = \{\mu \in \mathbb{C} : x^*P(\mu)x = 0 \text{ for some nonzero } x \in \mathbb{C}^n\}.$$  

It is closed and contains $\sigma[P(\lambda)]$ (see [7] for more information). The numerical range of matrix polynomials plays an important role in the study of overdamped vibration systems with a finite number of degree of freedom, and it is also related to the stability theory (see e.g., [4,6,7]). For a $q \in (0, 1]$, the $q$–numerical range of $P(\lambda)$ is defined by

$$W_q[P(\lambda)] = \{\mu \in \mathbb{C} : x^*P(\mu)x = 0, y^*x = 1, y^*y = 1 \text{ for some nonzero } x, y \in \mathbb{C}^n\}. \quad (2)$$

We also define $q$–spectrum of $P(\lambda)$ as follows:

$$\sigma_q[P(\lambda)] = \{\mu \in \mathbb{C} : detP(q^{-1}\mu) = 0\}. \quad (3)$$

Clearly, $W_q[P(\lambda)]$ is always closed and contains the spectrum $\sigma_q[P(\lambda)]$. When $P(\lambda) = q^{-1}I\lambda - A$, $W_q[P(\lambda)]$ coincides with the $q$–numerical range of matrix $A$, we get

$$W_q(A) = \{y^*Ax : x, y \in \mathbb{C}^n, x^*x = 1, y^*y = 1\}.$$  

(see [1,3]). Moreover, for $q = 1$, we obtain the numerical range of $P(\lambda)$. As shown in ([8, 9]), $W_q[P(\lambda)]$ is bounded if and only if $0 \notin W_q(A_m)$. One can find more about the geometry of $W_q[P(\lambda)]$ in [8, 9].

### 2. $q$–Radius Stability

Consider an index set $J \subseteq \{0, 1, \ldots, m\}$. In this paper, we consider the $q$–spectrum of perturbations of the matrix polynomial:

$$P_J(\lambda) = (A_m + \Delta_m)\lambda^m + (A_{m-1} + \Delta_{m-1})\lambda^{m-1} + \cdots + (A_1 + \Delta_1)\lambda + A_0 + \Delta_0 \quad (3)$$

where $\Delta_s = 0$ for all $s \notin J$. With the perturbed polynomial in (3), we associate the $n \times n$ matrix polynomial $\Delta_J(\lambda) = \Delta_m\lambda^m + \cdots + \Delta_1\lambda + \Delta_0$ and the $n \times (m+1)$ complex matrix

$$D_J = [ \Delta_m \Delta_{m-1} \cdots \Delta_1 \Delta_0 ]. \quad (4)$$
Let $\Omega$ be an open region of $\mathbb{C}$ whose boundary, $\partial \Omega$, is a piecewise smooth curve. The matrix polynomial $P(\lambda)$ is said to be $\Omega_q$–stable if $\sigma_q[P(\lambda)] \subset \Omega$. In this case, we define the $J_q$–stability radius of $P(\lambda)$ relative to $\Omega$ as

$$R_{J_q}[P(\lambda), \Omega] = \inf_{D_J} \{ ||D_J||_2 : \sigma_q[P_J(\lambda)] \cap (\mathbb{C} \setminus \Omega) \neq \emptyset \}. $$

That is, $R_{J_q}[P(\lambda), \Omega]$ is the distance of $P(\lambda)$ to $\Omega_q$–instability, when the coefficients of $P(\lambda)$ indexed by $J$ are allowed to vary.

For the proof of the main result we will need the following lemma.

**Lemma 2.1.** Let $P(\lambda)$ be defined as in (1) and consider its perturbation $P_J(\lambda)$ as defined in (3). Also, let $\Omega$ be an open region of $\mathbb{C}$ such that $\sigma_q[P(\lambda)] \subset \Omega$. Then we have

$$R_{J_q}[P(\lambda), \Omega] = \inf_{\mu \in \partial \Omega} \{ \inf_{D_J} ||D_J||_2 : \det(I + \Delta_J(\mu)P(\mu)^{-1}) = 0 \}. $$

**Proof.** Let $\mu \in \partial \Omega$, and note that the matrix $P(\mu)$ is invertible. Also $\det P_J(\mu) = 0$ if and only if $\det(I + \Delta_J(\mu)P(\mu)^{-1}) = 0$. By definition of $J_q$–stability, it follows that

$$R_{J_q}[P(\lambda), \Omega] = \inf_{\mu \in \partial \Omega} \{ \inf_{D_J} ||D_J||_2 : \det P_J(\mu) = 0 \}$$

So the proof is complete. \qed

**Theorem 2.2.** Let $P(\lambda) = A_m \lambda^m + A_{m-1} \lambda^{m-1} + \ldots + A_1 \lambda + A_0$ be an $n \times n$ matrix polynomial with $\det A_m \neq 0$, and let $J \subseteq \{0, 1, \ldots, m\}$. If $\Omega$ is an open region of $\mathbb{C}$ such that $\sigma_q[P(\lambda)] \subset \Omega$, then we have

$$R_{J_q}[P(\lambda), \Omega] = \inf \left\{ \frac{1}{\sqrt{\sum_{k \in J} ||\lambda||^{2k} ||P(\mu)^{-1}||_2}} : \lambda \in \partial \Omega \right\}$$

**Proof.** Since $\det A_m \neq 0$, $P(\lambda)$ has $mn$ finite eigenvalues counting their multiplicities ([2,5]). Let $\mu \in \mathbb{C} \setminus \sigma_q[P(\lambda)]$, then $P(\mu)$ is invertible. Consider the matrix polynomial

$$\Delta_J(\mu) = \Delta_m \mu^m + \Delta_{m-1} \mu^{m-1} + \ldots + \Delta_1 \mu + \Delta_0$$
\[ D_J[ I\mu^m \cdots I\mu \ I ]^T, \]

where \( D_J \) is defined as in (4). Suppose that \( \det(I + \Delta_J(\mu)P(\mu)^{-1}) = 0 \). Then \(-1\) is an eigenvalue of the matrix \( \Delta_J(\mu)P(\mu)^{-1} \) and so, we have

\[ 1 \leq \|\Delta_J(\mu)P(\mu)^{-1}\|_2 \leq \|\Delta_J(\mu)\|_2 \|P(\mu)^{-1}\|_2. \]

As a consequence, we get \( \|\Delta_J(\mu)\|_2 \geq \|P(\mu)^{-1}\|_2^{-1} \) which implies that

\[ \|\Delta_J(\mu)\|_2 \geq \frac{1}{\sqrt{\Sigma_{k\in J} |\mu|^{2k}} \|P(\mu)^{-1}\|_2}. \quad (5) \]

Furthermore, one can construct matrices \( \Delta_s (s = 0, 1, \ldots, m) \) for which \( D_J \) attains the above lower bound and let \( \det(I + \Delta_J(\mu)P(\mu)^{-1}) = 0 \), as follows. Now consider two vectors \( x, y \in \mathbb{C}^n \) such that \( \|x\|_2 = 1, \|P(\mu)^{-1}x\|_2 = \|P(\mu)^{-1}\|_2 \) and

\[ y_j = \frac{w_j}{\|P(\mu)^{-1}\|_2} \quad (j = 1, 2, \ldots, n) \]

where \( w = [ w_1 \ w_2 \ \cdots \ w_n ]^T := P(\mu)^{-1}x \). Define the matrix \( Q_0 \) by \( Q_0 = -xy^* \) and let \( \Delta_s = \frac{y^*}{\Sigma_{k\in J} |\mu|^{2k}} Q_0 \) for \( s \in J \) and \( \Delta_s = 0 \) if \( s \notin J \). Now, we get

\[ (I + \Delta_J(\mu)P(\mu)^{-1})x = x + Q_0P(\mu)^{-1}x = x + Q_0w. \]

Since \( y^*w = 1 \), we obtain

\[ (I + \Delta_J(\mu)P(\mu)^{-1})x = x - xy^*w = 0. \]

Thus, \( \det(I + \Delta_J(\mu)P(\mu)^{-1}) = 0 \). Also, we have

\[ \|D_J\|_2 = \sup \left\{ \frac{\|Q_0(\Sigma_{k\in J} \mu^k v_k)(\Sigma_{k\in J} |\mu|^{2k})^{-1}\|_2}{\sqrt{\Sigma_{k\in J} \|v_k\|_2^2}} : v_k \in \mathbb{C}^n \setminus 0 \right\} \leq \frac{1}{\Sigma_{k\in J} |\mu|^{2k}} \sup_{v_k \neq 0} \left\{ \|x\|_2 \|y\|_2 \|\Sigma_{k\in J} \mu^k v_k\|_2 \right\}. \]

Moreover, we can see that

\[ \|y\|_2 = \frac{\|w\|_2}{\|P(\mu)^{-1}\|_2} = \frac{\|P(\mu)^{-1}x\|_2}{\|P(\mu)^{-1}\|_2} = \frac{1}{\|P(\mu)^{-1}\|_2}. \]
and
\[ \| \Sigma_{k \in J} \mu^k v_k \|_2 \leq \| [ I \ I \mu \ \cdots \ I \mu^m ] \|_2 \| [ v_0^T \ v_1^T \ \cdots \ v_m^T ]^T \|_2 \]
where \( v_k = 0 \) whenever \( k \notin J \). Therefore,
\[ \| \Delta_J(\mu) \|_2 \leq \frac{1}{\sqrt{\sum_{k \in J} |\mu|^{2k} \|P(\mu)^{-1}\|_2}} \]
since \( \|x\|_2 = 1 \). So, for this special \( D_J \), equality holds in (5). Now the proof is complete in view of lemma 2.1, since \( \partial \Omega \cap \sigma_q[P(\lambda)] = \emptyset \).

The notion of \( J_q \)-stability radius of \( P(\lambda) \) is related to the \( (\epsilon, J_q) \)-pseudo \( q \)-spectrum of \( P(\lambda) \) which is defined by
\[ \sigma_{\epsilon, J_q}[P(\lambda)] = \{ \mu \in \mathbb{C} : \mu \in \sigma_q[P_J(\lambda)] \text{ for some } \Delta_J(\lambda) \text{ with } \| D_J \|_2 < \epsilon \} \]
for each \( \epsilon > 0 \), where \( P_J(\lambda) \) and \( D_J \) are defined as in (3) and (4), respectively. One can see that \( \sigma_{\epsilon, J_q}[P(\lambda)] \subset \Omega \) if and only if \( R_{J_q}[P(\lambda), \Omega] > \epsilon \).

**Theorem 2.3.** Let \( P(\lambda) = A_m \lambda^m + A_{m-1} \lambda^{m-1} + \cdots + A_1 \lambda + A_0 \) be an \( n \times n \) matrix polynomial with \( \det A_m \neq 0 \). Also, let \( J \subseteq \{0, 1, \ldots, m\} \) and \( \epsilon > 0 \) be given. Then
\[ \sigma_{\epsilon, J_q}[P(\lambda)] \setminus \sigma_q[P(\lambda)] = \{ \mu \in (\mathbb{C} \setminus \sigma_q[P(\lambda)]) : \frac{1}{\sqrt{\sum_{k \in J} |\mu|^{2k} \|P(\mu)^{-1}\|_2}} \leq \epsilon \}. \]

**Proof.** Consider a \( \mu \in (\mathbb{C} \setminus \sigma_q[P(\lambda)]) \). If \( \mu \in \sigma_{\epsilon, J_q}[P(\lambda)] \), then there is an \( n \times n \) matrix polynomial \( \Delta_J(\mu) = \Delta_m \mu^m + \cdots + \Delta_1 \mu + \Delta_0 \) such that \( \Delta_s = 0 \) for \( s \notin J \), \( \| [ \Delta_m \ \cdots \ \Delta_1 \ \Delta_0 ] \|_2 \leq \epsilon \), and \( \det (P(\mu) + \Delta_J(\mu)) = 0 \). Thus, by (5),
\[ \frac{1}{\sqrt{\sum_{k \in J} |\mu|^{2k} \|P(\mu)^{-1}\|_2}} \leq \| [ \Delta_m \ \cdots \ \Delta_1 \ \Delta_0 ] \|_2 \leq \epsilon. \]
Conversely, suppose that
\[ \frac{1}{\sqrt{\sum_{k \in J} |\mu|^{2k} \|P(\mu)^{-1}\|_2}} \leq \epsilon. \]
Then, as in the proof of Theorem 2.2, one can construct a matrix polynomial $\Delta_J(\mu) = \Delta_m \mu^m + \cdots + \Delta_1 \mu + \Delta_0$ such that $\Delta_s = 0$ for $s \notin J$.

$$\| [ \Delta_m \cdots \Delta_1 \Delta_0 ] \|_2 \leq \epsilon$$

and $\det (P(\mu) + \Delta_J(\mu))$. Thus, $\mu \in \sigma_{\epsilon,J}[P(\lambda)]$. \qed

**Corollary 2.4.** Let $P(\lambda) = A_m \lambda^m + A_{m-1} \lambda^{m-1} + \cdots + A_1 \lambda + A_0$ be an $n \times n$ matrix polynomial with $\det A_m \neq 0$, and let $J \subseteq \{0, 1, \ldots, m\}$ and $\epsilon > 0$. Then

$$\partial(\sigma_{\epsilon,J}[P(\lambda)] \setminus \sigma[q][P(\lambda)]) = \{ \mu \in (\mathbb{C} \setminus \sigma[q][P(\lambda)]) : \frac{1}{\sqrt{\sum_{k \in J} |\mu|^2 \| P(\mu)^{-1} \|_2}} = \epsilon \}.$$

**References**


