ON THE DIOPHANTINE EQUATION $p^x + (p + 1)^y = z^2$

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Abstract: In this paper, we found that $(p, x, y, z) = (3, 1, 0, 2)$ is a unique solution of the Diophantine equation $p^x + (p + 1)^y = z^2$, where $p$ is an odd prime number and $x$, $y$ and $z$ are non-negative integers.

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1. Introduction

In 1844, Catalan [2] posed a conjecture that $(a, b, x, y) = (3, 2, 2, 3)$ is a unique solution of the Diophantine equation $a^x - b^y = 1$ where $a, b, x$ and $y$ are integers with $\min\{a, b, x, y\} > 1$. Then Mihăilescu [3] proved the Catalan’s conjecture in 2004. After that Acu [1] proved that $(3, 0, 3)$ and $(2, 1, 3)$ are only two solutions $(x, y, z)$ for the Diophantine equation $2^x + 5^y = z^2$ where $x$, $y$ and $z$ are non-negative integers.

In 2011, Suvarnamani, Singta and Chotchaisthit [9] proved that two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution. Then Suvarnamani [5] proved that two Diophantine equations $4^x + 13^y = z^2$ and $4^x + 17^y = z^2$ have no non-negative integer solution. After that Suvarnamani [6] proved that the Diophantine equation $2^x + p^y = z^2$ has some non-negative integer solutions where $p$ is a prime number.
In 2012, Suvarnamani [7] found that Diophantine equation \( A^x + B^y = C^z \) has some non-negative integer solutions. Then Suvarnamani [8] found that the Diophantine equation \( p^x + p^y = z^2 \) has some non-negative integer solutions where \( p \) is a prime number. After that Sroysang [4] proved that \((0, 1, 3)\) is a unique non-negative integer solution of the Diophantine equation \( 7^x + 8^y = z^2 \).

In this paper, we will use the Catalan’s conjecture to solving \( p^x + (p + 1)^y = z^2 \) where \( p \) is an odd prime number and \( x, y \) and \( z \) are non-negative integers.

**2. Preliminaries**

**Lemma 2.1.** \((a, b, x, y) = (3, 2, 2, 3)\) is a unique solution of the Diophantine equation \( a^x - b^y = 1 \) where \( a, b, x \) and \( y \) are integers with \( \min\{a, b, x, y\} > 1 \).

**Proof.** See in [4].

**Lemma 2.2.** If \( q \) is an odd prime number and \( y, z \) are non-negative integers. Then the Diophantine equation \( 1 + q^y = z^2 \) has no solution.

**Proof.** Let \( q \) is an odd prime number and \( y, z \) be non-negative integers such that \( 1 + (p + 1)^y = z^2 \). We consider in 3 cases.

Case 1: \( y = 0 \). Then \( z^2 = 2 \) which is impossible.

Case 2: \( y = 1 \). Thus \( z^2 = p + 2 \). That is \( z = 0 \) or \( 2 \). It is impossible.

Case 3: \( y > 1 \). Thus \( z^2 = (p + 1)^y + 1 > p + 2 \). Then \( z > 2 \). By Lemma 2.1, we have \( z = 3, p = 1 \) and \( y = 3 \). Contradiction.

**Lemma 2.3.** \((p, x, z) = (3, 1, 2)\) is a unique solution of the Diophantine equation \( p^x + 1 = z^2 \) where \( p \) is an odd prime number and \( x, z \) are non-negative integers.

**Proof.** Let \( p \) be an odd prime number and \( x, z \) be non-negative integers such that \( p^x + 1 = z^2 \). We consider in 3 cases.

Case 1: \( x = 0 \). Then \( z^2 = 2 \) which is impossible.

Case 2: \( x = 1 \). Thus \( z^2 = p + 1 \). That is \( z = 2 \). Then we get \( p = 3 \).

Case 3: \( x > 1 \). Thus \( z^2 = p^x + 1 > p + 1 \). Then \( z > 2 \). By Lemma 2.1, we have \( z = 3, p = 2 \) and \( x = 3 \). Contradiction.
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3. Main Theorem

Main Theorem 3.1. \((p, x, y, z) = (3, 1, 0, 2)\) is a unique solution of the Diophantine equation \(p^x + (p + 1)^y = z^2\) where \(p\) is an odd prime number and \(x, y, z\) are non-negative integers.

Proof. Let \(p\) is an odd prime number and \(x, y, z\) are non-negative integers such that \(p^x + (p + 1)^y = z^2\). By Lemma 2.2, we have \(x \geq 1\). Then we consider in 2 cases.

Case 1: \(y = 0\). Then \(p^x + 1 = z^2\). By Lemma 2.3, we get \((p, x, y, z) = (3, 1, 0, 2)\) is a solution.

Case 2: \(y \geq 1\). Then \(z^2 = p^x + (p + 1)^y\). So, \(z\) is odd. Then \(z^2 \equiv 1(mod 4)\). Next, we consider in 2 cases.

Subcase 1: \(x\) is even, i.e., \(x = 2k\) where \(k \in N\).
We get \((p + 1)^y = z^2 - p^{2k} = (z - p^k)(z + p^k)\), where \(z - p^k = (p + 1)^u\) and \(z + p^k = (p + 1)^{x - u}, y > 2u\). Then \((p + 1)^u((p + 1)^y-2u - 1) = 2 \cdot p^k\). That is \((p + 1)^u = 2 \cdot p^v\) and \((p + 1)^{y-2u - 1} = p^{k-v}\) where \(v \in N\). But it is impossible.

Subcase 2: \(x\) is odd, i.e., \(x = 2h + 1\) where \(h \in N\).
If \(y\) is odd, i.e., \(y = 2i + 1\) where \(i \in N\). We have \(z^2 = p^{2h+1} + (p + 1)^{2i+1}\). We get \(z = 0, 2\). But it is impossible.
If \(y\) is even, i.e., \(y = 2j\) where \(j \in N\). We have \(z^2 = p^{2h+1} + (p + 1)^{2j}\). We get \(z = 0, 2\). But it is impossible.

\(\square\)

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References


