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AN IMPROVED POISSON APPROXIMATION FOR BERNOULLI RANDOM SUMMANDS

K. Teerapabolarn

Department of Mathematics
Faculty of Science
Burapha University
Chonburi, 20131, THAILAND

Abstract: This paper gives a new bound for the total variation distance between the distribution of random sums of independent Bernoulli random variables and an appropriate Poisson distribution. The bound in this study is sharper than that reported in [2]. Two examples have been given to illustrate the result obtained.

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1. Introduction

Let $X_1, X_2, ...$ be a sequence of independent Bernoulli random variables, each with probability of success $p_i = 1 - q_i = P(X_i = 1) = 1 - P(X_i = 0)$. If N is a non-negative integer-valued random variable and independent of the X_i's, then $S_N = \sum_{i=1}^{N} X_i$ is the random sums of N independent Bernoulli random variables. Let $\lambda_N = \sum_{i=1}^{N} p_i$, $\lambda = E(\lambda_N)$ and U_λ a Poisson random variable with mean λ. For approximating the distribution of S_N by a Poisson distribution with mean λ, Yannaros [2] gave a bound for the total variation

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distance between the distributions of S_N and U_λ as follows:

$$d_{TV}(S_N, U_\lambda) \le E|\lambda_N - \lambda| + E\left(\frac{1-e^{-\lambda_N}}{\lambda_N}\sum_{i=1}^{N} p_i^2\right), \tag{1.1}$$

where $d_{TV}(S_N, U_\lambda) = \sup_{A\subseteq \mathbb{N}\cup\{0\}} |P(S_N \in A) - P(U_\lambda \in A)|$.

In this paper, we focus on giving a new bound for the total variation distance between the two such distributions, which is in Section 2. In Section 3, some examples are given to illustrate the main result, and the conclusion of this study is presented in the last section.

2. Result

The following theorem presents a bound for $d_{TV}(S_N, U_\lambda)$, which is the desired result.

Theorem 2.1. *Let* $\theta_N = \frac{\sum_{i=1}^{N} p_i^2}{\lambda_N}$, *then we have that*

$$\begin{aligned} d_{TV}(S_N, U_\lambda) \le \min&\left\{\min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\} E|\lambda_N - \lambda|, (1-e^{-\lambda})\frac{Var(\lambda_N)}{\lambda}\right\} \\ &+ \min\left\{E\left(\frac{1-e^{-\lambda_N}}{\lambda_N}\sum_{i=1}^{N} p_i^2\right), 0.46E\left(\frac{\theta_N}{(1-\theta_N)^{3/2}}\right)\right\}. \end{aligned} \tag{2.1}$$

Proof. It follows the fact that

$$\begin{aligned} d_{TV}(S_N, U_\lambda) &\le d_{TV}(S_N, U_{\lambda_N}) + d_{TV}(U_{\lambda_N}, U_\lambda) \\ &\le \sum_{n=0}^{\infty} P(N=n) d_{TV}(S_n, U_{\lambda_n}) + d_{TV}(U_{\lambda_N}, U_\lambda). \end{aligned} \tag{2.2}$$

The best results in [1] and [3] showed that

$$d_{TV}(S_n, U_{\lambda_n}) \le \frac{1-e^{-\lambda_n}}{\lambda_n}\sum_{i=1}^{n} p_i^2 \tag{2.3}$$

and

$$d_{TV}(S_n, U_{\lambda_n}) \le \frac{0.46\theta_n}{(1-\theta_n)^{3/2}}. \tag{2.4}$$

Also by applying two results in [1], we have that

$$d_{TV}(U_{\lambda_N}, U_\lambda) \leq \min\left\{\min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\} E|\lambda_N - \lambda|, (1-e^{-\lambda})\frac{Var(\lambda_N)}{\lambda}\right\}. \quad (2.5)$$

Taking the bounds in (2.3), (2.4) and (2.5) into (2.2), the result in (2.1) is obtained. □

When X_i's are identically distributed, the following corollary is an immediately consequence of the Theorem 2.1

Corollary 2.1. *Let $\widehat{n} = E(N)$, if $p_1 = p_2 = \cdots = p$, then we have the following:*

$$d_{TV}(S_N, U_\lambda) \leq \min\left\{\min\left\{1, \sqrt{\frac{2}{e\widehat{n}p}}\right\} E|N - \widehat{n}|, (1-e^{-\widehat{n}p})\frac{Var(N)}{\widehat{n}}\right\} p$$
$$+ \min\left\{E\left(1-e^{-Np}\right), \frac{0.46}{q^{3/2}}\right\} p. \quad (2.6)$$

Remark. Let us consider
1. $\min\left\{\min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\} E|\lambda_N - \lambda|, (1-e^{-\lambda})\frac{Var(\lambda_N)}{\lambda}\right\} \leq E|\lambda_N - \lambda|$ and
2. $\min\left\{E\left(\frac{1-e^{-\lambda_N}}{\lambda_N}\right), 0.46E\left(\frac{\theta_N}{(1-\theta_N)^{3/2}}\right)\right\} \leq E\left(\frac{1-e^{-\lambda_N}}{\lambda_N}\right)$.
Thus, the bound (2.1) are sharper than the bound in (1.1).

3. Examples

This section gives two examples to illustrate the result in the case of X_i's are identically distributed.

Example 3.1. For n ($n \in \mathbb{N}$) is fixed, let N be a positive integer-valued random variable with probability function

$$P(N = k) = \begin{cases} \frac{1}{2} & , k = n, \\ \frac{1}{2} & , k = 2n, \\ 0 & , \text{otherwise}. \end{cases}$$

Therefore $E(N) = \frac{3n}{2}$, $Var(N) = \frac{n^2}{4}$ and $E|N - E(N)| = \frac{n}{2}$. Let $p_1 = p_2 = \cdots = p$, then we have

$$d_{TV}(S_N, U_\lambda) \leq \min\left\{\min\left\{1, \sqrt{\frac{4}{3enp}}\right\} \frac{n}{2}, (1-e^{-\frac{3np}{2}})\frac{n}{6}\right\} p$$

$$+\min\left\{1-\frac{(e^{-np}+e^{-2np})}{2},\frac{0.46}{q^{3/2}}\right\}p.$$

Example 3.2. Let N be a positive integer-valued random variable with probability function

$$P(N=n)=\frac{1}{2^n},\ n=1,2,...,$$

then we have $E(N)=2$, $Var(N)=2$ and $E|N-E(N)|=1$. If $p_1=p_2=\cdots=p$, then we obtain

$$d_{TV}(S_N,U_\lambda)\leq \min\left\{\min\left\{1,\sqrt{\frac{1}{ep}}\right\},(1-e^{-2p})\right\}p+\min\left\{1,\frac{0.46}{q^{3/2}}\right\}p.$$

4. Conclusion

In this study, a new bound for the total variation distance between the distribution of random sums of independent Bernoulli random variables and an appropriate Poisson distribution was obtained. It is also sharper than that reported in [2]. Thus, the bound obtained in this study is more appropriate for measuring the accuracy of the approximation.

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