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NEGATIVE BINOMIAL APPROXIMATION FOR GEOMETRIC RANDOM SUMMANDS

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Abstract: This paper determines a bound for the total variation distance between the distribution of random sums of independent geometric random variables and a negative binomial distribution. Two examples have been given to illustrate the result obtained.

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1. Introduction

Let $X_1, X_2, ...$ be a sequence of independent geometric random variables, each with $P(X_i = k) = p_i q_i^k$, $k = 0, 1, ...,$ where $q_i = 1 - p_i$. Let N be a non-negative integer-valued random variable and independent of the X_i's. The sum $S_N = \sum_{i=1}^N X_i$ is usually called *random sums.* Let $NB_{n,p}$ be a negative binomial random variable with parameters n and p. For $N = n \in \mathbb{N}$ is fixed, Vellaisamy and Upadhye [2] gave a bound for approximating the distribution of S_n by a negative binomial distribution in the form of

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$$d_{TV}(S_n, NB_{n,p}) \le \left(\sum_{i=1}^{n} \frac{q_i^2}{p_i} - \frac{nq^2}{p} \right) \min\left\{1, \frac{1}{\sqrt{2nqe}}\right\},$$

where $d_{TV}(S_n, NB_{n,p}) = \sup_{A \subseteq \mathbb{N} \cup \{0\}} |P(S_n \in A) - P(NB_{n,p} \in A)|$ is the total variation distance between the distribution of S_n and the negative binomial distribution with parameters n and $p = 1 - q = \frac{1}{n}\sum_{i=1}^{n} p_i$. Let $\widehat{n} = E(N)$ and $\widehat{q} = 1 - \widehat{p} = \frac{\lambda}{\widehat{n}}$ where $\lambda = E(\lambda_N) = E\left(\sum_{i=1}^{N} q_i\right)$. In this study, we are interested to determine a bound for $d_{TV}(S_N, NB_{\widehat{n},\widehat{p}})$, which is in Section 2. In Section 3, two examples have been given to illustrate the desired result. The conclusion of this study is presented in the last section.

2. Result

The following theorem presents a bound for the total variation distance between the distributions of S_N and $NB_{\widehat{n},\widehat{p}}$.

Theorem 2.1. *For $\lambda_N = \sum_{i=1}^{N} q_i$ and $\lambda = E(\lambda_N)$, then*

$$d_{TV}(S_N, NB_{\widehat{n},\widehat{p}}) \le \min\left\{1, \frac{0.42888}{\sqrt{\lambda}}\right\} \frac{\lambda^2}{\widehat{n} - \lambda} + \min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\} E|\lambda_N - \lambda|$$
$$+ \min\left\{ E\left(\frac{0.42888}{\sqrt{\lambda_N}} \sum_{i=1}^{N} \frac{q_i^2}{p_i} \right), E\left(\sum_{i=1}^{N} \frac{q_i^2}{p_i} \right) \right\}. \quad (2.1)$$

Proof. Let U_λ be the Poisson random variable with mean λ. It follows the fact that

$$d_{TV}(S_N, NB_{\widehat{n},\widehat{p}}) \le d_{TV}(S_N, U_\lambda) + d_{TV}(U_\lambda, NB_{\widehat{n},\widehat{p}}). \quad (2.2)$$

Teerapabolarn [1] and Vellaisamy and Upadhye [2] showed that

$$d_{TV}(S_N, U_\lambda) \le \min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\} E|\lambda_N - \lambda|$$
$$+ \min\left\{ E\left(\frac{0.42888}{\sqrt{\lambda_N}} \sum_{i=1}^{N} \frac{q_i^2}{p_i} \right), E\left(\sum_{i=1}^{N} \frac{q_i^2}{p_i} \right) \right\} \quad (2.3)$$

and

$$d_{TV}(U_\lambda, NB_{\widehat{n},\widehat{p}}) \le \min\left\{1, \frac{0.42888}{\sqrt{\widehat{n}\widehat{q}}}\right\} \frac{\widehat{n}\widehat{q}^2}{\widehat{p}} = \min\left\{1, \frac{0.42888}{\sqrt{\lambda}}\right\} \frac{\lambda^2}{\widehat{n} - \lambda}, \quad (2.4)$$

respectively. Hence, the inequality (2.1) is obtained by putting the right hand side of (2.3) and (2.4) to (2.2). □

If X_i's are identically distributed, then the following corollary is an immediately consequence of the Theorem 2.1

Corollary 2.1. *If $p_1 = p_2 = \cdots = p$, then we have the following:*

$$d_{TV}(S_N, NB_{\widehat{n},p}) \le \min\left\{1, \frac{0.42888}{\sqrt{\widehat{n}q}}\right\}\frac{\widehat{n}q^2}{p} + \min\left\{1, \sqrt{\frac{2}{e\widehat{n}q}}\right\} qE|N-\widehat{n}| + \min\left\{\frac{0.42888q^{\frac{3}{2}}E(\sqrt{N})}{p}, \frac{\widehat{n}q^2}{p}\right\}, \tag{2.5}$$

and by the fact that $E(\sqrt{N}) \le \sqrt{\widehat{n}}$, we also obtain

$$d_{TV}(S_N, NB_{\widehat{n},p}) \le \min\left\{1, \frac{0.42888}{\sqrt{\widehat{n}q}}\right\}\frac{2\widehat{n}q^2}{p} + \min\left\{1, \sqrt{\frac{2}{e\widehat{n}q}}\right\} qE|N-\widehat{n}|. \tag{2.6}$$

3. Examples

This section, we give two examples to illustrate the result in the case of X_i's are identically distributed, which is in the Corollary 2.1.

Example 3.1. For n ($n \in \mathbb{N}$) is fixed, let N be a positive integer-valued random variable with probability function

$$P(N=k) = \begin{cases} \frac{1}{2} & , k = n, \\ \frac{1}{2} & , k = 2n, \\ 0 & , \text{otherwise.} \end{cases}$$

Therefore $\widehat{n} = \frac{3n}{2}$ and $E|N-\widehat{n}| = \frac{n}{2}$. Let $p_1 = p_2 = \cdots = p$, then we have

$$d_{TV}(S_N, NB_{\frac{3n}{2},p}) \le \min\left\{1, \frac{0.35018}{\sqrt{nq}}\right\}\frac{3nq^2}{p} + \min\left\{1, \sqrt{\frac{0.49051}{nq}}\right\}\frac{nq}{2}.$$

Example 3.2. Let N be a positive integer-valued random variable with probability function

$$P(N=n) = \frac{1}{2^n}, \; n = 1, 2, ...,$$

then we have $\widehat{n} = 2$ and $E|N - \widehat{n}| = 1$. If $p_1 = p_2 = \cdots = p$, then we obtain

$$d_{TV}(S_N, NB_{2,\widehat{p}}) \leq \min\left\{1, \frac{0.30326}{\sqrt{q}}\right\} \frac{4q^2}{p} + \min\left\{1, \frac{0.36788}{\sqrt{q}}\right\} q.$$

4. Conclusion

In this study, a bound for the total variation distance between the distribution of random sums of independent geometric random variables and an appropriate negative binomial distribution with parameters $\widehat{n}$ and $\widehat{p}$ was obtained. With this bound, it is indicated that the negative binomial with parameters $\widehat{n}$ and $\widehat{p}$ can be used as an approximation of the distribution of random sums of independent geometric random variables when $\widehat{q} = 1 - \widehat{p}$ is small.

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